SOME BINOMIAL IDENTITIES ARISING FROM A PARTITION OF AN n-DIMENSIONAL CUBE

ANDRZEJ P. KISIELEWICZ

Abstract. Using a partition of the cube $[0,2)^n$ into boxes obtained from a cube tiling of $\mathbb{R}^n$ constructed by Lagarias and Shor, proofs of three well-known binomial identities related to the Lucas cube are given.

1. Introduction

A cube tiling of $\mathbb{R}^n$ is a family of cubes $[0,2)^n + T = \{[0,2)^n + t : t \in T\}$, where $T \subset \mathbb{R}^n$, which fill in the whole space without gaps and overlaps. In [4] Lagarias and Shor constructed a cube tiling code of $\mathbb{R}^n$. In this note we show how it can be used to prove the following well-known identities:

$$\sum_{k\geq 0} \binom{n-k}{k} \frac{n}{n-k} 2^k = 2^n + (-1)^n, \tag{1.1}$$

$$\sum_{k\geq 0} \binom{n-k}{k} 2^k = \frac{2^{n+1} + (-1)^n}{3} \tag{1.2}$$

and

$$\sum_{k\geq 0} \binom{n-k}{k} \frac{k}{n-k} 2^k = \frac{2^n + (-1)^n 2}{3}. \tag{1.3}$$

The code of Lagarias and Shor is constructed as follows. Let $n \geq 3$ be an odd positive integer, and let A be an $n \times n$ circulant matrix of the form $\mathrm{A} = \mathrm{A}(n) = \mathrm{circ}(1,2,0,\ldots,0)$. Let A^T be the transpose of A. By $V(\mathrm{A})$ and $V(\mathrm{A}^T)$ we denote the sets of all possible sums of distinct rows in A and A^T, respectively. Moreover, we add to these sets the vector $(0,\ldots,0)$. Let

$$V = V_e(\mathrm{A}) \cup (V_o(\mathrm{A}^T) + (2,\ldots,2)) \mod 4,$$

where $V_e(\mathrm{A})$ denotes the set of all vectors in $V(\mathrm{A})$ with an even number of 3's, and $V_o(\mathrm{A}^T)$ is the set of all vectors in $V(\mathrm{A}^T)$ with an odd number of 0's. We will refer to the code V as the *Lagarias-Shor cube tiling code*. This code has very interesting applications. Originally in [4] it was used to design a certain cube tiling of $\mathbb{R}^n$ that was the basis for estimating distances between cubes in cube tilings of $\mathbb{R}^n$. Recently in [3] the Lagarias-Shor cube tiling code was used to construct interesting partitions and matchings of an n-dimensional cube.

To obtain a cube tiling of $\mathbb{R}^n$ from the code V, let $T = V - \mathbf{1} + 4\mathbb{Z}^n$, where $\mathbf{1} = (1,\ldots,1)$. It follows from [4, Proposition 3.1 and Theorem 4.1] that $[0,2)^n + T$ is a cube tiling of $\mathbb{R}^n$. (To be precise, a tiling considered in [4] is of the form $[0,1)^n + T'$, where $T' = \frac{1}{2}V + 2\mathbb{Z}^n$, but $[0,2)^n + T = [0,2)^n + 2T' - \mathbf{1}$.) In the proofs of the identities (1.1)–(1.3) we do not need the entire tiling $[0,2)^n + T$ but only the cubes from it that intersect the cube $[0,2)^n$. These cubes induce a partition of the cube $[0,2)^n$ of the form $\mathcal{F} = \{[0,2)^n \cap ([0,2)^n + t) : t \in T\}$. Our proofs of the identities (1.1)–(1.3) are based on the properties of the partition $\mathcal{F}$.

All three identities are related to the *Lucas cube* Λ_n. This is a graph whose vertices are all elements of the box $\{0,1\}^n$ which do not contain two consecutive 1's in the cyclic order (i.e., in $(v_1,\dots,v_n) \in \{0,1\}^n$ the coordinates v_1 and v_n are consecutive) and in which two vertices are adjacent if they differ in exactly one position. It is known that $\binom{n-k}{k}\frac{n}{n-k}$ is the number of all vertices in the Lucas cube Λ_n of *weight* k, i.e., containing k 1's. This is also the number of all k-element subsets of the set $[n] = \{1,\dots,n\}$ without two consecutive integers in the cyclic order ([5]). The number of all vertices in Λ_n of weight k which have 1 at the ith position, $i \in [n]$, is equal to $\binom{n-k}{k}\frac{k}{n-k}$, while $\binom{n-k}{k}$ is the number of all vertices in Λ_n of weight k which have 0 at the ith position.

In the last section we show that for $n \geq 3$ odd, the set of all vertices of the Lucas cube Λ_n is a selector of a partition of the box $\{0,1\}^n$ into boxes, which is a discrete analogue of the above mentioned partition $\mathcal{F}$ of the cube $[0,2)^n$.

There are many tiling proofs that rely on counting the number of 1-dimensional tilings of a $1\times n$ board by polyominoes (squares, dominoes, etc.) (see [1, 2]). In our case we examine just one partition of the n-dimensional cube $[0,2)^n$ into boxes and the structure of that partition which reflects the local structure of the tiling $[0,2)^n + T$ is exploited.

2. Proofs

Since the Lagarias-Shor cube tiling code is defined for odd numbers, we prove identities (1.1)-(1.3) for odd and even positive integers separately.

Proof of (1.1) for $n \geq 3$ odd. We intersect the cube $[0,2)^n$ with the cubes from the tiling $[0,2)^n + T$. Let $\mathcal{F}(n) = \mathcal{F} = \{[0,2)^n \cap ([0,2)^n + t) : t \in T\}$. Since $[0,2)^n + T$ is a tiling, $\mathcal{F}$ is a partition of the cube $[0,2)^n$ into boxes. Let $m(K)$ denote the volume of the box $K \in \mathcal{F}$, and let $m(\mathcal{F}) = \sum_{K\in\mathcal{F}} m(K)$. For every $K \in \mathcal{F}$ we have $m(K) = 2^k$, where k is the number of 1's in the vector $v \in V$ such that $K = [0,2)^n \cap ([0,2)^n + v - \mathbf{1})$. Let $M_k = |\{K \in \mathcal{F} : m(K) = 2^k\}|$. The family $\mathcal{F}$ is a partition of $[0,2)^n$ and therefore $m(\mathcal{F}) = 2^n$ and

$$m(\mathcal{F}) = \sum_{k\geq 0} M_k 2^k.$$

Note now that if $v \in V$ contains 3 at some position $i \in [n]$, then the cubes $[0,2)^n + v - \mathbf{1}$ and $[0,2)^n$ are disjoint. This means that these two cubes intersect if and only if $v \in U \cup \{(0,\dots,0),(2,\dots,2)\}$, where U consists of all sums of non-adjacent rows of the matrix A, and the row numbers are in the cyclic order (thus, the first and last rows are adjacent). Therefore, for $k \geq 1$ the number M_k is equal to the number of all k-element subsets of the set $\{1,\dots,n\}$ without two consecutive integers in the cyclic order, and $M_0 = 2$ because $[0,1)^n$ and $[1,2)^n$ are the only cubes in $\mathcal{F}$ with volume 1. Hence, $M_k = \binom{n-k}{k}\frac{n}{n-k}$ for $k \geq 1$. This completes the proof of (1.1) for $n \geq 3$ odd. □

This proof needs only the portion $U = U(n)$ of the Lagarias-Shor cube tiling code, where the code U consists of all sums of non-adjacent rows of the matrix A(n), and the row numbers of A(n) are in the cyclic order. For example,

$$\mathrm{A}(5) = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 & 2 \\ 2 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathrm{U}(5) = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 & 2 \\ 2 & 0 & 0 & 0 & 1 \\ 1 & 2 & 1 & 2 & 0 \\ 1 & 2 & 0 & 1 & 2 \\ 0 & 1 & 2 & 1 & 2 \\ 2 & 1 & 2 & 0 & 1 \\ 2 & 0 & 1 & 2 & 1 \end{bmatrix},$$

where the rows of the matrix U(5) are the vectors of the family $U(5)$.

Observe that if we replace 2 by 0 in every vector $v \in U \cup \{(0,\dots,0)\}$, then we obtain the set of all vertices in the Lucas cube Λ_n.

To prove (1.2) and (1.3) for $n \geq 3$ odd let $\mathcal{F}_0^i, \mathcal{F}_2^i$ and $\mathcal{F}_1^i$, $i \in [n]$, denote the sets of all boxes in $\mathcal{F}$ which have the factors $[0,1), [1,2)$ and $[0,2)$ at the ith position, respectively. Since $\mathcal{F} = \{[0,2)^n \cap ([0,2)^n + v - \mathbf{1}) : v \in U \cup \{(0,\dots,0),(2,\dots,2)\}\}$, for every $k \in \{0,1,2\}$ the set $\mathcal{F}_k^i$ consists of all boxes in $\mathcal{F}$ which are determined by the vectors $v \in U \cup \{(0,\dots,0),(2,\dots,2)\}$ such that $v_i = k$. Let

$$m(\mathcal{F}_{02}^i) = \sum_{K \in \mathcal{F}_{02}^i} m(K) \ \text{ and } \ m(\mathcal{F}_1^i) = \sum_{K \in \mathcal{F}_1^i} m(K),$$

where $\mathcal{F}_{02}^i = \mathcal{F}_0^i \cup \mathcal{F}_2^i$.

The partition $\mathcal{F}$ (Figure 1) has the structure which will be utilized below. Note that for every $i \in [n]$ the set $\bigcup \mathcal{F}_{02}^i$, is an *i-cylinder*, i.e., for every line segment $l_i \subset [0,2)^n$ of length 2 which is parallel to the ith edge of the cube $[0,2)^n$ we have

$$l_i \subset \bigcup \mathcal{F}_{02}^i \ \text{ or } \ l_i \cap \bigcup \mathcal{F}_{02}^i = \emptyset. \tag{2.1}$$

Obviously, the set $\bigcup \mathcal{F}_1^i$ is an i-cylinder too.

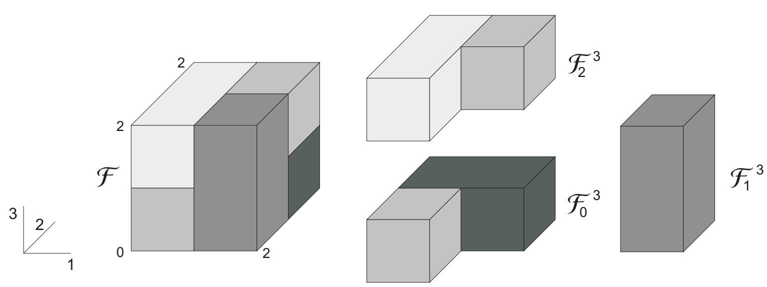

FIGURE 1. The boxes in $\mathcal{F}$ are determined by the vectors $U = \{(1,2,0),(0,1,2),(2,0,1)\}$ (the three "long" boxes) and $\{(0,0,0),(2,2,2)\}$ (the two unit cubes).

Proof of (1.2) and (1.3) for $n \geq 3$ odd. We will calculate $m(\mathcal{F}_{02}^i)$ and $m(\mathcal{F}_1^i)$. For every $v \in U$ we have $v_i = 1$ if and only if $v_{i+1} = 2$. Thus, $\mathcal{F}_2^{i+1} = \mathcal{F}_1^i \cup \{[1,2)^n\}$ and then $m(\mathcal{F}_2^{i+1}) = m(\mathcal{F}_1^i) + 1$ (clearly, $n+1$ is taken modulo n). It follows from (2.1) that $m(\mathcal{F}_0^i) = m(\mathcal{F}_2^i)$ (see Figure 1). Since A is a circulant matrix, we have $m(\mathcal{F}_1^i) = m(\mathcal{F}_1^j)$ for $i,j \in [n]$, and

$m(\mathcal{F}) = m(\mathcal{F}_0^i) + m(\mathcal{F}_2^i) + m(\mathcal{F}_1^i)$ because $\mathcal{F}$ is a partition. Thus, $2^n = 3m(\mathcal{F}_1^i) + 2$ and consequently

$$m(\mathcal{F}_{02}^i) = \frac{2(2^n+1)}{3} \quad \text{and} \quad m(\mathcal{F}_1^i) = \frac{2^n-2}{3}. \tag{2.2}$$

As it was noted before the proof, the code $U \cup \{(0,\ldots,0)\}$ can be identified with the set of all vertices in the Lucas cube Λ_n. Since $\binom{n-k}{k}\frac{k}{n-k}$ is the number of all vertices of weight k in Λ_n with 1 at the first position, we have $|\mathcal{F}_1^1| = \sum_{k\geq 1}\binom{n-k}{k}\frac{k}{n-k}$, and consequently

$$m(\mathcal{F}_1^1) = \sum_{k\geq 1}\binom{n-k}{k}\frac{k}{n-k}2^k,$$

which, by (2.2), gives (1.3) for $n \geq 3$ odd. Since

$$\sum_{k\geq 0}\binom{n-k}{k}\frac{n}{n-k}2^k = \sum_{k\geq 0}\binom{n-k}{k}2^k + \sum_{k\geq 0}^{n}\binom{n-k}{k}\frac{k}{n-k}2^k,$$

the proof of the identity (1.2) for $n \geq 3$ odd is also completed. □

For $n \geq 3$ odd all three identities are strongly related to the partition $\mathcal{F}$. The sums $\sum_{k\geq 1}\binom{n-k}{k}2^k + 2$ and $\sum_{k\geq 1}\binom{n-k}{k}\frac{k}{n-k}2^k$ are the total volumes of the boxes from the partition $\mathcal{F}$ which belong to the sets $\mathcal{F}_{02}^i$ and $\mathcal{F}_1^i$, respectively (the number 2 in the first sum is the sum of the volumes of the boxes $[0,1)^n$ and $[1,2)^n$). The summands $\binom{n-k}{k}2^k$ and $\binom{n-k}{k}\frac{k}{n-k}2^k$ for $k = 1,\ldots,\lfloor\frac{n}{2}\rfloor$ are the total volumes of the boxes in $\mathcal{F}_{02}^i$ and $\mathcal{F}_1^i$, respectively which have exactly k factors $[0,2)$.

From now on we assume that $n \geq 3$ is an odd number. The identities (1.1)–(1.3) for $n-1 \geq 2$ even can be derived from the partitions $\mathcal{F}(n)$ and $\mathcal{F}(n-2)$, where $\mathcal{F}(1) = \{[0,1),[1,2)\}$.

Proofs of (1.1)-(1.3) for $n-1 \geq 2$ even. Denote by $r_1,\ldots,r_n$ the rows of the matrix $\mathrm{A}(n)$, and let $\mathcal{G} = \mathcal{G}(n-1) \subset \mathcal{F}(n)$ be the set of all boxes which are determined by the vectors $v \in U(n)$ which are sums of non-adjacent rows from the set $\{r_1,\ldots,r_{n-1}\}$, where r_1 and r_{n-1} are treated as adjacent. Thus, the number $|\mathcal{G}|$ is the same as the number of all vertices in the Lucas cube Λ_{n-1}. Consequently

$$m(\mathcal{G}) = \sum_{k\geq 1}\binom{n-1-k}{k}\frac{n-1}{n-1-k}2^k.$$

Since $\mathcal{G}_0^n = \mathcal{F}_0^n \setminus \{[0,1)^n\}$, it follows that $m(\mathcal{G}_0^n) = m(\mathcal{F}_0^n) - 1$, and by (2.2) and the fact that $m(\mathcal{F}_0^n) = m(\mathcal{F}_2^n)$ we get

$$m(\mathcal{G}_0^n) = \frac{2(2^{n-1}-1)}{3}.$$

We now calculate $m(\mathcal{G}_2^n)$. Every box in $\mathcal{G}_2^n$ is generated by a vector $v \in U$ which has 2 at the nth position. Therefore, $v = r_{n-1} + \sum_{i\in I} r_i$ for some $I \subset \{2,\ldots,n-3\}$. Let R be the set of all such sums $\sum_{i\in I} r_i$. Every vector in R is a sum of non-adjacent rows from the set $\{r_2,\ldots,r_{n-3}\}$, where r_2 and r_{n-3} are not treated as adjacent. Let $U_0^{n-2}(n-2)$ be the set of all vectors in $U(n-2)$ having 0 at the last position. Observe now that the function $b : R \to U_0^{n-2}(n-2)$ defined by the formula $b(u) = \sum_{i\in I-1} h_i$, where $h_1,\ldots,h_{n-2}$ are rows of the matrix $\mathrm{A}(n-2)$

and $I-1=\{i-1 : i\in I\}$, is a bijection. Therefore, $m(\mathcal{G}_2^n)=2m(\mathcal{F}_0^{n-2}(n-2))$ (recall that we add r_{n-1} to $\sum_{i\in I} r_i$). By (2.1), $m(\mathcal{F}_0^{n-2}(n-2))=m(\mathcal{F}_2^{n-2}(n-2))$, and by (2.2),

$$m(\mathcal{G}_2^n)=\frac{2^{n-1}+2}{3}.$$

Thus, $m(\mathcal{G})=m(\mathcal{G}_0^n)+m(\mathcal{G}_2^n)=2^{n-1}$ because $\mathcal{G}_1^n=\emptyset$. This completes the proof of (1.1) for $n-1\geq 2$ even.

Since $m(\mathcal{G}_2^{i+1})=m(\mathcal{G}_1^i)$, $m(\mathcal{G}_1^i)=m(\mathcal{G}_1^j)$ and $m(\mathcal{G})=m(\mathcal{G}_1^i)+m(\mathcal{G}_{02}^i)$ for $i,j\in[n-1]$, it follows that

$$m(\mathcal{G}_1^i)=\frac{2^{n-1}+2}{3} \text{ and } m(\mathcal{G}_{02}^i)=\frac{2(2^{n-1}-1)}{3}$$

for $i\in[n-1]$. By the definition of the set $\mathcal{G}$, we have $|\mathcal{G}_1^1|=\sum_{k\geq 1}\binom{n-1-k}{k}\frac{k}{n-1-k}$, and thus

$$m(\mathcal{G}_1^1)=\sum_{k\geq 1}\binom{n-1-k}{k}\frac{k}{n-1-k}2^k$$

which proves (1.3) for $n-1\geq 2$ even. Having this in the same manner as for $n\geq 3$ odd, we prove (1.2) for $n-1\geq 2$ even. □

3. Vertices of the Lucas Cube as a Selector

Let $L=L(n)$ be the code that arises from $U=U(n)$ by making in every vector $v\in U$ the following substitutions: $0\to 0$, $2\to 1$ and $1\to *$. For example,

$$\mathrm{L}(5)=\begin{bmatrix} * & 1 & 0 & 0 & 0\\ 0 & * & 1 & 0 & 0\\ 0 & 0 & * & 1 & 0\\ 0 & 0 & 0 & * & 1\\ 1 & 0 & 0 & 0 & *\\ * & 1 & * & 1 & 0\\ * & 1 & 0 & * & 1\\ 0 & * & 1 & * & 1\\ 1 & * & 1 & 0 & *\\ 1 & 0 & * & 1 & * \end{bmatrix},$$

where the rows of the matrix L(5) are the vectors of the family $L(5)$.

The set L consists of all sums of non-adjacent rows of the matrix $\operatorname{circ}(*,1,0,\dots,0)$, where the row numbers of this matrix are in the cyclic order. Therefore, if we replace $*$ by 0 in every vector of $L\cup\{(0,\dots,0)\}$, then we obtain the set $V(\Lambda_n)$ of all vertices in the Lucas cube Λ_n.

The code $L\cup\{(0,\dots,0),(1,\dots,1)\}$ induces a partition $\mathcal{L}$ of the discrete box $\{0,1\}^n$ into boxes which is a discrete analogue of the partition $\mathcal{F}$ from the previous section. The boxes $K(l)=K_1(l)\times\cdots\times K_n(l)\in\mathcal{L}$, where $l\in L\cup\{(0,\dots,0),(1,\dots,1)\}$, are of the form

$$K_i(l)=\begin{cases}\{0\} & \text{if } l_i=0,\\ \{1\} & \text{if } l_i=1,\\ \{0,1\} & \text{if } l_i=*\end{cases}$$

for $i\in[n]$, and $|K(l)|=2^k$ for every box $K(l)\in\mathcal{L}$ having k factors $\{0,1\}$. Therefore, the proofs from the previous section can be repeated, but this time we consider the partition $\mathcal{L}$ instead of $\mathcal{F}$.

Observe now that for every $n \geq 3$ odd the set $V(\Lambda_n)$ of the vertices of the Lucas cube is a *selector* of the family of boxes $\mathcal{L} \setminus \{\{1\} \times \cdots \times \{1\}\}$: for every $v \in V(\Lambda_n)$ there is exactly one $K(l) \in \mathcal{L} \setminus \{\{1\} \times \cdots \times \{1\}\}$ such that

$$v \in K(l).$$

Indeed, let $K(l) \in \mathcal{L} \setminus \{\{1\} \times \cdots \times \{1\}\}$ and pick $v = (v_1, \ldots, v_n) \in K(l)$ in the following way:

$$v_i = \begin{cases} 0 & \text{if } K_i(l) = \{0\}, \\ 1 & \text{if } K_i(l) = \{1\}, \\ 0 & \text{if } K_i(l) = \{0,1\}. \end{cases}$$

Since l does not contain two consecutive 1's in the cyclic order and if $K_i = \{0,1\}$, then $K_{i+1} = \{1\}$, it follows that $v \in V(\Lambda_n)$ and for every $w \in K(l)$, $w \neq v$, there is $i \in [n]$ such that $w_i = 1$ while $v_i = 0$. Thus, $K(l) \cap V(\Lambda_n) = \{v\}$.

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Wydział Matematyki, Informatyki i Ekonometrii, Uniwersytet Zielonogórski, ul. Z. Szafrana 4a, 65-516 Zielona Góra, Poland

E-mail address: A.Kisielewicz@wmie.uz.zgora.pl