FIBONACCI NUMBERS OF THE FORM $x^a \pm x^b \pm 1$

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Abstract. In this paper, we show that the Diophantine equation $F_n = x^a \pm x^b \pm 1$ has only finitely many positive integer solutions (n, x, a, b) with $n \geq 3$, $\max\{a, b\} \geq 2$ and x with exactly two distinct prime factors.

1. Introduction

In this paper, we consider the Diophantine equation

$$F_n = x^a \pm x^b \pm 1 \tag{1.1}$$

in positive integer variables n, x, a, b with $\max\{a, b\} \geq 2$ and $n \geq 3$. Luca and Szalay [3] showed that equation (1.1) has only finitely many positive integer solutions (n, x, a, b) with prime x. We extend this result to the case when x has exactly two distinct prime factors.

Theorem 1.1. *Equation* (1.1) *has only finitely many positive integer solutions* (n, x, a, b) *with* $n \geq 3$, $\max\{a, b\} \geq 2$ *and* x *having exactly two distinct prime factors. All such solutions have* $\max\{a, b\} < 4 \times 10^{14}$ *and*

$$x < \exp\left(\exp\left(\exp\left(\exp\left(5 \times 10^{45}\right)\right)\right)\right).$$

We point out that Bennett and Bugeaud [2] treated the similar equation (1.1) with F_n replaced by some perfect power y^q of integer exponent $q \geq 2$.

2. Preliminary Results

For the proof of Theorem 1.1, we need the following explicit lower bound for a linear form in logarithms of real algebraic numbers due to Matveev [4]. But first, we need to remind the reader of the definition of the height of an algebraic number. Let η be an algebraic number of degree d over $\mathbb{Q}$ with minimal primitive polynomial over the integers

$$f(X) = a_0 \prod_{i=1}^{d} (X - \eta^{(i)}) \in \mathbb{Z}[X],$$

where the leading coefficient a_0 is positive. The *logarithmic height* of η is given by

$$h(\eta) = \frac{1}{d}\left(\log a_0 + \sum_{i=1}^{d} \log \max\{|\eta^{(i)}|, 1\}\right).$$

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Lemma 2.1. *(Matveev). Let $\mathbb{L}$ be a real number field of degree D, $\alpha_1, \alpha_2, \ldots, \alpha_t$ be non-zero elements of $\mathbb{L}$ and $b_1, b_2, \ldots, b_t$ be nonzero integers. Set $B = \max\{b_1, \ldots, b_t\}$ and*

$$\Lambda = \alpha_1^{b_1} \cdots \alpha_t^{b_t} - 1.$$

Let $A_1, \ldots, A_t$ be real numbers with

$$A_j \geq \max\{Dh(\alpha_j), |\log \alpha_j|, 0.16\} \quad \text{for all } 1 \leq j \leq t.$$

Assume that $\Lambda \neq 0$. Then

$$\log|\Lambda| \geq -1.4 \cdot 30^{t+3} t^{4.5} D^2 (1 + \log D)(1 + \log B) A_1 \cdots A_t.$$

We also recall the following result of Baker from 1964 (see [1]).

Lemma 2.2. *(Baker). Let $f(X) = a_0 X^d + a_1 X^{d-1} + \cdots + a_d \in \mathbb{Z}[X]$ be a polynomial of degree d. Let (x, y) be an integer solution to the equation*

$$y^2 = f(x).$$

If $f(X)$ has at least three simple roots, then

$$\max\{|x|, |y|\} \leq \exp(\exp(\exp((d^{10d} H)^{d^2}))), \tag{2.1}$$

where $H = \max\{|a_0|, \ldots, |a_d|\}$.

In order to be able to apply Lemma 2.2, we need the following result.

Lemma 2.3. *Let $a > b \geq 1$ be fixed integers and*

$$f(X) = X^a + \varepsilon_1 X^b + \varepsilon_2 \quad \textit{and} \quad g(X) = 5f(X)^2 + 4\varepsilon_3, \quad \textit{where} \quad \varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{\pm 1\}.$$

Then $g(X)$ has only simple roots.

Proof. Let x_0 be a double zero of $g(X)$. Then

$$g(x_0) = 5f(x_0)^2 + 4\varepsilon_3 = 0 \quad \text{and} \quad g'(x_0) = 5f(x_0)f'(x_0) = 0. \tag{2.2}$$

From the second equation (2.2), we get that either $f(x_0) = 0$ or $f'(x_0) = 0$. If $f(x_0) = 0$, the the first equation (2.2) gives $4 = 0$, which is false. Thus,

$$0 = f'(x_0) = a x_0^{a-1} + \varepsilon_1 b x_0^{b-1} = x_0^{b-1}(a x_0^{a-b} + \varepsilon_1 b).$$

If $x_0 = 0$, then the first equation (2.2) gives $5 + 4\varepsilon_3 = 0$, which is false. So $x_0^{a-b} = -\varepsilon_1 b/a$. Returning to $g(x_0) = 0$, we get

$$x_0^b(x_0^{a-b} + \varepsilon_1) + \varepsilon_2 = f(x_0) = \varepsilon_4 \sqrt{-4\varepsilon_3/5}, \quad (\varepsilon_4 \in \{\pm 1\})$$

and

$$x_0^b = \frac{-\varepsilon_2 + \varepsilon_4 \sqrt{-4\varepsilon_3/5}}{\varepsilon_1 (a-b)/a}. \tag{2.3}$$

Raising equation (2.3) to the power $a - b$, we get

$$\left(\frac{-\varepsilon_2 + \varepsilon_4 \sqrt{-4\varepsilon_3/5}}{\varepsilon_1 (a-b)/a} \right)^{a-b} = (x_0^{a-b})^b = (-\varepsilon_1 b/a)^b,$$

which leads to the conclusion that $(-\varepsilon_2 + \varepsilon_4 \sqrt{-4\varepsilon_3/5})^{a-b} \in \mathbb{Q}$. Analyzing this situation over all the possibilities $\varepsilon_2,\ \varepsilon_3,\ \varepsilon_4 \in \{\pm 1\}$, we get to the conclusion that one of the numbers $2 \pm \sqrt{5}$ or $2 \pm \sqrt{-5}$ raised to some nonzero integer exponent is an integer, which is false. $\square$

3. Proof of Theorem 1.1

Without loss of generality, we may assume that $n \geq 500$, $a \geq b$ and $x \geq 6$ since x has exactly two distinct prime factors. We rewrite equation (1.1) as

$$F_n \mp 1 = x^b(x^{a-b} \pm 1). \tag{3.1}$$

From [3, Lemma 2], we know that

$$F_n + \varepsilon = F_{\frac{n-\delta}{2}} L_{\frac{n+\delta}{2}} \tag{3.2}$$

where

$$\delta = \begin{cases} -\varepsilon & \text{if } n \equiv 1 \pmod 4 \\ \varepsilon & \text{if } n \equiv -1 \pmod 4 \\ -2\varepsilon & \text{if } n \equiv 2 \pmod 4 \\ 2\varepsilon & \text{if } n \equiv 0 \pmod 4 \end{cases} \qquad (\varepsilon \in \{\pm 1\}).$$

Here and in what follows, L_m is the mth Lucas number. Since

$$F_{\frac{n-\delta}{2}} \mid F_{n-\delta}, \quad L_{\frac{n+\delta}{2}} \mid F_{n+\delta} \quad \text{and} \quad \gcd(F_u, F_v) = F_{(u,v)},$$

we get that

$$\gcd(F_{\frac{n-\delta}{2}}, L_{\frac{n+\delta}{2}}) \mid \gcd(F_{n-\delta}, F_{n+\delta}) \mid F_{2|\delta|} \mid F_4 = 3,$$

therefore,

$$\gcd(F_{\frac{n-\delta}{2}}, L_{\frac{n+\delta}{2}}) = 1 \text{ or } 3 \text{ and it is 3 exactly when } n \text{ is even and } n \equiv \delta \pmod 8.$$

From equations (3.1) and (3.2), we get

$$x^b(x^{a-b} \pm 1) = F_{\frac{n-\delta}{2}} L_{\frac{n+\delta}{2}}.$$

Note that $x^a \pm x^b \pm 1$ is always odd. So, F_n is odd, therefore $3 \nmid n$. A case by case analysis shows that either $3 \mid (n-\delta)/2$ or $3 \mid (n+\delta)/2$. We then write $(n+\eta\delta)/2 = 3k$ for some $\eta \in \{\pm 1\}$. Recall that

$$F_{3k} = F_k(5F_k^2 + 3(-1)^k) \ \text{ and } \ L_{3k} = L_k(L_k^2 - 3(-1)^k).$$

In each of the two cases, the above two factors are either coprime or their greatest common divisor is exactly 3. Hence, we have from (3.2) that

$$x^b(x^{a-b} \pm 1) = \begin{cases} F_{3k}L_{3k+\delta} = F_k(5F_k^2 + 3(-1)^k)L_{3k+\delta}, & \text{if } \frac{n-\delta}{2} = 3k; \\ F_{3k-\delta}L_{3k} = F_{3k-\delta}L_k(L_k^2 - 3(-1)^k), & \text{if } \frac{n+\delta}{2} = 3k. \end{cases} \tag{3.3}$$

Hence, we can write $x^b(x^{a-b} \pm 1) = G_1G_2G_3$, where the pairwise greatest common divisor of G_1, G_2 and G_3 is either 1 or 3 (note that $G_1G_2G_3$ is positive since otherwise it would be zero and we would get that $F_n = \pm 1$, which is impossible since we are assuming that $n \geq 500$). We label the G_i's such that $G_1 = \min\{G_1, G_2, G_3\}$. From formula (3.3) and the fact that $n \geq 500$ (so $k \geq 50$), it is easy to see that $G_1 = F_k$ or L_k according to whether $n+\delta = 6k$ or $n-\delta = 6k$, respectively.

We now let $x = p_1^{e_1}p_2^{e_2}$, where p_1 and p_2 are distinct primes and e_1 and e_2 are positive integer exponents. Suppose first that $a = b$. Then $G_1G_2G_3 = 2x^a = 2p^{ae_1}q^{ae_2}$. The greatest common divisor conditions imply $G_1 \leq 6$, so $k \leq 5$, which is not possible since $n \geq 500$.

Assume next that $a > b$. From (3.3), we get $x^b = p_1^{e_1 b} p_2^{e_2 b}$ divides either $9G_1G_2$, or $9G_2G_3$, or $9G_3G_1$. Therefore,

$$x^b \le 9G_2G_3 = \frac{9(F_n \pm 1)}{G_1} \le \frac{9(F_n+1)}{F_k} \le \frac{\alpha^5 \cdot \alpha^{n-1}}{\alpha^{k-2}} = \alpha^{n-k+6} < \alpha^{\frac{5n}{6}+7}$$

where $\alpha = (1+\sqrt{5})/2$. Here, we used the fact that $9 < \alpha^5$, $F_k \ge \alpha^{k-2}$ for all $k \ge 1$, and $F_n \le \alpha^{n-1} - 1$ for $n \ge 500$. These inequalities are consequences of the Binet formula

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \qquad \text{where} \qquad \beta = (1-\sqrt{5})/2. \tag{3.4}$$

On the other hand,

$$2x^a + 1 \ge x^a \pm x^b \pm 1 = F_n > \alpha^{n-2} + 1 \qquad (n \ge 500),$$

giving

$$x^a > \frac{\alpha^{n-2}}{2} > \alpha^{n-4}.$$

Thus,

$$x^b < \alpha^{\frac{5n}{6}+7} < \left(\alpha^{n-4}\right)^{\frac{6}{7}} < x^{\frac{6a}{7}}, \quad \text{so} \quad b < \frac{6a}{7}$$

where in the above we used the fact that

$$\frac{5n}{6} + 7 < \frac{6(n-4)}{7}$$

which holds because $n \ge 500$. Hence, $a - b > a/7$. This inequality together with (3.2) and the Binet formula for the Fibonacci numbers (3.4) implies

$$\left| \frac{\alpha^n}{\sqrt{5}} - x^a \right| = \left| \pm x^b + \frac{\beta^n}{\sqrt{5}} \pm 1 \right| < 1.2x^b,$$

where the right-most inequality holds because $x \ge 6$ and $b \ge 1$, giving

$$\left| \frac{\alpha^n x^{-a}}{\sqrt{5}} - 1 \right| < 1.2x^{-(a-b)}. \tag{3.5}$$

The above inequality (3.5) implies that the left-hand side is $\le 1/2$ since $x \ge 6$ and $a - b \ge 1$. Hence,

$$\left| \frac{\alpha^n x^{-a}}{\sqrt{5}} - 1 \right| \le \min\left\{ \frac{1}{2}, \frac{1.2}{x^{a-b}} \right\} \le \min\left\{ \frac{1}{2}, \frac{1.2}{x^{a/7}} \right\} \le \min\left\{ \frac{1}{2}, \frac{1}{x^{(a-1)/7}} \right\}. \tag{3.6}$$

In the above chain of inequalities we used the fact that $x \ge 6 > 1.2^7$. An argument of Shorey and Stewart [5] implies that a is bounded. Let us recall their argument and use it to compute an explicit bound for a. Write $n = aq + r$ with $0 \le r < a$. Then inequality (3.6) is

$$\left| \frac{\alpha^r}{\sqrt{5}} \left(\frac{\alpha^q}{x} \right)^a - 1 \right| \le \min\left\{ \frac{1}{2}, \frac{1}{x^{(a-1)/7}} \right\}. \tag{3.7}$$

We apply Lemma 2.1 to the left-hand side above with the parameters $\mathbb{L} = \mathbb{Q}(\sqrt{5})$, $t = 3$, $\alpha_1 = \alpha$, $\alpha_2 = \sqrt{5}$, $\alpha_3 = \alpha^q/x$, $b_1 = r$, $b_2 = 1$, $b_3 = a$. Hence,

$$\Gamma = \frac{\alpha^r}{\sqrt{5}} \left(\frac{\alpha^q}{x} \right)^a - 1. \tag{3.8}$$

Clearly $D = 2$ and $B = a$. We can take $A_1 = 0.5 \geq \max\{2h(\alpha_1), \log \alpha_1, 0.16\}$. Also, we can take $A_2 = 1.7 > \max\{2h(\alpha_2), \log \alpha_2, 0.16\}$. We need to compute A_3. For this, we note that the minimal polynomial of α^q/x over $\mathbb{Z}$ is

$$f(Y) = x^2Y^2 - (\alpha^q + \beta^q)xY + (-1)^q.$$

The conjugate of α^q/x is β^q/x whose absolute value is clearly smaller than 1. Further, by (3.7), we have

$$\frac{\alpha^r}{\sqrt{5}}\left(\frac{\alpha^q}{x}\right)^a \leq \frac{3}{2}, \quad \text{therefore} \quad \frac{a^q}{x} \leq \alpha^{-r/a}\left(\frac{3\sqrt{5}}{2}\right)^{1/a} < 2.$$

Hence,

$$h(\alpha_3) = \frac{1}{2}\left(\log x^2 + \log \max\left\{1, \frac{\alpha^q}{x}\right\}\right) \leq \log x + \frac{\log 2}{2} < 1.5 \log x$$

since $x \geq 6$. Thus, we can take $A_3 = 1.5 \log x$. We verify that $\Gamma \neq 0$. Indeed, if this were not so, then we would get that

$$\frac{\alpha^n x^{-a}}{\sqrt{5}} = 1.$$

After squaring and manipulating the above relation, we get $\alpha^{2n} \in \mathbb{Q}$, implying $n = 0$, which is false. So, we may apply Matveev's Theorem Lemma 2.1 to the left-hand side of inequality (3.7), getting

$$|\Gamma| > \exp\left(-1.4 \times 30^6 \times 3^{4.5} \times 2^2(1 + \log 2)(1 + \log a) \times 0.5 \times 1.7 \times 1.5 \log x\right). \tag{3.9}$$

Hence,

$$|\Gamma| > \exp(-1.3 \times 10^{12}(1 + \log a)\log x). \tag{3.10}$$

Combining the above inequality (3.10) with inequality (3.7), we get

$$((a-1)/7)\log x < 1.3 \times 10^{12}(1 + \log a)\log x$$

giving $a < 4 \times 10^{14}$. This proves the assertion about a. Assume now that both $a > b \geq 1$ are fixed and let

$$f(X) = X^a \pm X^b \pm 1.$$

Inserting the relation $F_n = f(x)$ into the formula

$$L_n^2 - 5F_n^2 = 4(-1)^n,$$

we get, with $y = L_n$, that

$$y^2 = g(x), \tag{3.11}$$

where $g(X) = 5f(X)^2 \pm 4 \in \mathbb{Z}[X]$. We shall apply Lemma 2.2 to bound the solutions of equation (3.11). The condition that $g(X)$ has at least three simple zeros is satisfied since $\deg(g(X)) = 2a \geq 4$ and by Lemma 2.3, the roots of $g(X)$ are simple. Further, one checks easily that $H(g) \leq 15$. Now (2.1) implies that

$$x < \exp\left(\exp\left(\exp\left(((2a)^{20a} \times 15)^{4a^2}\right)\right)\right).$$

Inserting $a < 4 \times 10^{14}$, we get the desired inequality for x. □

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