ON THE $q-$SEIDEL MATRIX

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Abstract. Clarke and et. al recently introduced the q-Seidel matrix, and obtained some properties. In this article, we define a different form of q-Seidel matrix by $a_n^k(x,q) = xq^{n+2k-3}a_n^{k-1}(x,q)+a_{n+1}^{k-1}(x,q)$ with $k \geq 1,\ n \geq 0$ for an initial sequence $a_n^0(x,q) = a_n(x,q)$. By using our definition, we obtain several properties of the q-analogues of generalized Fibonacci and Lucas polynomials.

1. Introduction

The q-analogues of generalized Fibonacci and Lucas polynomials were investigated by many authors [3, 5, 7]. Carlitz [10] defined the q-Fibonacci polynomials by

$$\phi_{n+1}(a) - a\phi_n(a) = q^{n-1}\phi_{n-1}(a) \qquad (n > 1), \tag{1.1}$$

where $\phi_1(a) = 1,\ \phi_2(a) = a$.

The sequence of polynomials $S_n(x,q)$ is defined by the recurrence relation

$$S_{n+1}(x,q) = S_n(x,q) + xq^{n-2}S_{n-1}(x,q) \qquad (n \geq 1), \tag{1.2}$$

where $S_0(x,q) = a$ and $S_1(x,q) = b$. For $a = 0$ and $b = 1$, $S_n(x;q) = U_{n-1}\left(1; 0, -xq^{-1}\right)$, $S_n(x;q)$ is a special case Al-Salam and Ismail polynomials $U_n(x;a,b)$ introduced in [13]. Also the sequence of polynomials $S_n(x,q)$ is a special case $F_n(x;s,q)$ which is studied by Cigler in [7]. In particular, if we take $x = 1$, $q \to 1^-$ in (1.2), we get the classical Fibonacci and Lucas numbers for initial values $a = 0, b = 1$ and $a = 2, b = 1$ respectively.

q-Calculus started with L. Euler in the eighteenth century. q-Analogue of the binomial coefficients play important role in number theory, combinatorics, linear algebra and finite geometry. Now we mention some definitions of q-calculus [1]. Given value of $q > 0$, the q-integer $[n]_q$ is defined by

$$[n]_q = \begin{cases} \dfrac{1-q^n}{1-q} & \text{if } q \neq 1 \\ n & \text{if } q = 1, \end{cases}$$

and the q-factorial $[n]_q!$ is defined by

$$[n]_q! = \begin{cases} [n]_q \cdot [n-1]_q \cdots [1]_q & \text{if } n = 1, 2, ... \\ 1 & \text{if } n = 0 \end{cases}$$

for $n \in \mathbb{N}$. The $q-$binomial coefficients are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[n-k]_q!\ [k]_q!}, \qquad n \geq k \geq 0$$

with $\begin{bmatrix} n \\ 0 \end{bmatrix}_q = 1$ and $\begin{bmatrix} n \\ k \end{bmatrix} = 0$ for $n < k$. Note that the q-binomial coefficient satisfies the recurrence equations

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_q = q^k \begin{bmatrix} n \\ k \end{bmatrix}_q + \begin{bmatrix} n \\ k-1 \end{bmatrix}_q \tag{1.3}$$

and

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ k \end{bmatrix}_q + q^{n-k+1} \begin{bmatrix} n+1 \\ k-1 \end{bmatrix}_q . \tag{1.4}$$

In [9] Clarke and et. al give a kind of the generalization of a Seidel matrix, and obtain some properties by using the following relation:

$$\begin{aligned} a_n^0(x,q) &= a_n(x,q) && (n \geq 0), \\ a_n^k(x,q) &= xq^n a_n^{k-1}(x,q) + a_{n+1}^{k-1}(x,q) && (k \geq 1,\ n \geq 0). \end{aligned} \tag{1.5}$$

Here $(a_n(x,q))$ is a sequence of elements in a commutative ring. We can write $a_n^k(x,q)$ in terms of the initial sequence as

$$a_n^k(x,q) = \sum_{i=0}^{k} (xq^n)^{k-i} \begin{bmatrix} k \\ i \end{bmatrix}_q a_{n+i}^0(x,q). \tag{1.6}$$

Moreover $\left(a_n^0(x,q)\right)$ is called the initial sequence and $(a_0^n(x,q))$ the final sequence of the q-Seidel matrix. By using the Gauss inversion formula, we obtain relations between the initial sequence and final sequence:

$$a_0^n(x,q) = \sum_{i=0}^{n} x^{n-i} \begin{bmatrix} n \\ i \end{bmatrix}_q a_i^0(x,q), \tag{1.7}$$

$$a_n^0(x,q) = \sum_{i=0}^{n} (-x)^{n-i} q^{\binom{n-i}{2}} \begin{bmatrix} n \\ i \end{bmatrix}_q a_0^i(x,q). \tag{1.8}$$

Define the generating functions as follows:

$$a(t) = \sum_{n\geq 0} a_n^0(x,q)\, t^n, \qquad \overline{a}(t) = \sum_{n\geq 0} a_0^n(x,q)\, t^n \tag{1.9}$$

and

$$A(t) = \sum_{n\geq 0} a_n^0(x,q) \frac{t^n}{[n]_q!}, \qquad \overline{A}(t) = \sum_{n\geq 0} a_0^n(x,q) \frac{t^n}{[n]_q!}. \tag{1.10}$$

Thus the generating functions of the initial and final sequences are related by following equations:

$$\overline{a}(t) = \sum_{n\geq 0} a_n^0(x,q) \frac{t^n}{(xt;q)_{n+1}}, \tag{1.11}$$

$$\overline{A}(t) = e_q(xt)\, A(t). \tag{1.12}$$

Define $(t;q)_n = (1-t)(1-qt)\dots\left(1-q^{n-1}t\right)$ and $(t;q)_\infty = \lim_{n\to\infty}(t;q)_n$. Then

$$e_q(t) = \sum_{n\geq 0} \frac{t^n}{[n]_q!} = \frac{1}{((1-q)\,t;q)_\infty}. \tag{1.13}$$

Also

$$\frac{1}{(t;q)_{n+1}} = \sum_{k=0}^{\infty} \begin{bmatrix} n+k \\ k \end{bmatrix}_q t^k. \tag{1.14}$$

In this paper, we define a generalization of the q-Seidel matrix and obtain some properties for the generalized q-Seidel matrix. Furthermore we consider the q-analogues of generalized Fibonacci and Lucas polynomials $S_n(t,q)$ and give several properties of the sequence of polynomials $S_n(t,q)$ by using the generalized q-Seidel matrix method.

2. The Generalized q-Seidel Matrix

Let $(a_n(x,q))$ be a given real or complex sequence. The generalized $q-$Seidel matrix associated with $\left(a_n^0(x,q)\right)$ is defined recursively by the formula

$$\begin{aligned} a_n^0(x,q) &= a_n(x,q) \qquad (n\geq 0),\\ a_n^k(x,q) &= xq^{n+2k-3}a_n^{k-1}(x,q)+a_{n+1}^{k-1}(x,q) \qquad (n\geq 0,\ k\geq 1), \end{aligned} \tag{2.1}$$

where $a_n^k(x,q)$ represent the entry in the kth row and nth column.

We note that for $q\to 1^-$ and $x=1$, the q-Seidel matrix turns into the usual Euler-Seidel matrix [2, 4, 6].

Lemma 2.1. *Let $\left(a_n^k(x,q)\right)$ satisfy equation (2.1) with initial sequence $\left(a_n^0(x,q)\right)$. Then*

$$a_n^k(x,q)=\sum_{i=0}^{k}x^{k-i}q^{(n+k-2)(k-i)}\begin{bmatrix}k\\ i\end{bmatrix}_q a_{n+i}^0(x,q). \tag{2.2}$$

Proof. We use induction to prove the proposition. The equation clearly holds for $k=1$. Now, suppose that the equation is true for k. By (1.3) and (2.1) we have

$$\begin{aligned} a_n^{k+1}(x,q) &= xq^{n+2k-1}a_n^k(x,q)+a_{n+1}^k(x,q)\\ &= xq^{n+2k-1}\sum_{i=0}^{k}x^{k-i}q^{(n+k-2)(k-i)}\begin{bmatrix}k\\ i\end{bmatrix}_q a_{n+i}^0(x,q)\\ &\quad+\sum_{i=0}^{k}x^{k-i}q^{(n+k-1)(k-i)}\begin{bmatrix}k\\ i\end{bmatrix}_q a_{n+1+i}^0(x,q)\\ &= \sum_{i=0}^{k}x^{k+1-i}q^{(n+k-1)(k+1-i)}\begin{bmatrix}k\\ i\end{bmatrix}_q a_{n+i}^0(x,q)\\ &\quad+\sum_{i=1}^{k+1}x^{k+1-i}q^{(n+k-1)(k+1-i)}\begin{bmatrix}k\\ i-1\end{bmatrix}_q a_{n+i}^0(x,q)\\ &= x^{k+1}q^{(n+k-1)(k+1)}a_n^0(x,q)\\ &\quad+\sum_{i=1}^{k}x^{k+1-i}q^{(n+k-1)(k+1-i)}\left\{q^i\begin{bmatrix}k\\ i\end{bmatrix}_q+\begin{bmatrix}k\\ i-1\end{bmatrix}_q\right\}a_{n+i}^0(x,q)+a_{n+k+1}^0(x,q)\\ &= \sum_{i=0}^{k+1}x^{k+1-i}q^{(n+k-1)(k+1-i)}\begin{bmatrix}k+1\\ i\end{bmatrix}_q a_{n+i}^0(x,q). \end{aligned}$$

Hence, the equation is true for $n=k+1$, which completes the proof. □

If we take $q\to 1^-$, $x=1$ for (2.2), we get the well-known formula for the classical Euler-Seidel matrix [4].

The first row and column of the generalized q-Seidel matrix are defined by the inverse relation as in following corollary.

Corollary 2.2. *Let $a_n^0(x,q)$ and $a_0^n(x,q)$ be the first row and column in the generalized q-Seidel matrix. Then $a_n^0(x,q)$ and $a_0^n(x,q)$ have the inverse relation*

$$a_0^n(x,q)=\sum_{i=0}^{n}x^{n-i}q^{(n-2)(n-i)}\begin{bmatrix}n\\ i\end{bmatrix}_q a_i^0(x,q) \tag{2.3}$$

and

$$a_n^0(x,q)=\sum_{i=0}^{n}(-x)^{n-i}\,q^{\frac{(n-i)(n-3+i)}{2}}\begin{bmatrix}n\\ i\end{bmatrix}_q a_0^i(x,q)\,. \tag{2.4}$$

Proposition 2.3. *Let $a_n^0(x,q)$ and $a_0^n(x,q)$ be the first row and column in the generalized q-Seidel matrix. Then $a_n^0(x,q)$ and $a_0^n(x,q)$ have the orthogonality relation*

$$\sum_{j=i}^{n}(-1)^{j-i}\,q^{(n-2)(n-j)}q^{\frac{(j-i)(j-3+i)}{2}}\begin{bmatrix}n\\ j\end{bmatrix}_q\begin{bmatrix}j\\ i\end{bmatrix}_q=\delta_{ni}. \tag{2.5}$$

Proof. We prove this by induction on n. A similar proof can be seen in [8, 11]. □

2.1. **Generating Functions.**

Proposition 2.4. *Let*

$$a(t)=\sum_{n=0}^{\infty}a_n^0(x,q)\,t^n$$

be the generating function of the initial sequence $\left(a_n^0(x,q)\right)$. Then the generating function of the sequence $\left(a_0^n(x,q)\right)$ is

$$\overline{a(t)}=\sum_{n=0}^{\infty}a_n^0(x,q)\,t^n\sum_{k=0}^{\infty}\begin{bmatrix}n+k\\ k\end{bmatrix}_q(xt)^k\,q^{k(k-2+n)}. \tag{2.6}$$

Proof. Considering (2.3) we write

$$\begin{aligned}\overline{a(t)}&=\sum_{n=0}^{\infty}\left(\sum_{i=0}^{n}x^{n-i}q^{(n-2)(n-i)}\begin{bmatrix}n\\ i\end{bmatrix}_q a_i^0(x,q)\right)t^n\\&=\sum_{n,k=0}^{\infty}\begin{bmatrix}n+k\\ k\end{bmatrix}_q x^k t^{n+k}q^{k(k-2+n)}a_n^0(x,q)\,.\end{aligned}$$

Hence we obtain

$$\overline{a(t)}=\sum_{n=0}^{\infty}a_n^0(x,q)\,t^n\sum_{k=0}^{\infty}\begin{bmatrix}n+k\\ k\end{bmatrix}_q(xt)^k\,q^{k(k-2+n)}.$$

□

Proposition 2.5. *Let*

$$A(t)=\sum_{n=0}^{\infty}a_n^0(x,q)\,\frac{t^n}{[n]_q!}$$

be the exponential generating function of the initial sequence $\left(a_n^0(x,q)\right)$. *Then the exponential generating function of the sequence* $\left(a_0^n(x,q)\right)$ *is*

$$\overline{A(t)} = \sum_{n=0}^{\infty} a_n^0(x,q) \frac{t^n}{[n]_q!} \sum_{k=0}^{\infty} q^{k(k-2+n)} \frac{(xt)^k}{[k]_q!}. \tag{2.7}$$

Proof. The proof follows from equation (2.3). □

3. Applications of Generalized $q-$Seidel Matrices

In this section, we show that the generalized q-Seidel matrix is quite applicable for the q-analogues of generalized Fibonacci and Lucas polynomials. First we give the relationship between $S_{n+2k}(x,q)$ and the initial sequence $S_n(x,q)$ by using the generalized q-Seidel matrix.

Corollary 3.1. *The q-analogues of generalized Fibonacci and Lucas polynomials satisfy the following relation:*

$$S_{n+2k}(x,q) = \sum_{i=0}^{k} x^{k-i} q^{(n+k-2)(k-i)} \begin{bmatrix} k \\ i \end{bmatrix}_q S_{n+i}(x,q). \tag{3.1}$$

Proof. Let $a_n^0 = S_n(x,q)$, $n \geq 0$ be initial sequence. By using induction on k, (1.2) and (2.1), we have

$$a_n^k = S_{n+2k}(x,q).$$

Using (2.2) and applying $a_n^0 = S_n(x,q)$, we obtain

$$a_n^k = \sum_{i=0}^{k} x^{k-i} q^{(n+k-2)(k-i)} \begin{bmatrix} k \\ i \end{bmatrix}_q S_{n+i}(x,q).$$

This completes the proof. □

Corollary 3.2. *We have*

$$S_{2n}(x,q) = \sum_{i=0}^{n} x^{n-i} q^{(n-2)(n-i)} \begin{bmatrix} n \\ i \end{bmatrix}_q S_i(x,q), \tag{3.2}$$

$$S_n(x,q) = \sum_{i=0}^{n} (-x)^{n-i} q^{\frac{(n-i)(n-3+i)}{2}} \begin{bmatrix} n \\ i \end{bmatrix}_q S_{2i}(x,q) \tag{3.3}$$

and

$$S_{2n+1}(x,q) = \sum_{i=0}^{n} x^{n-i} q^{(n-1)(n-i)} \begin{bmatrix} n \\ i \end{bmatrix}_q S_{i+1}(x,q). \tag{3.4}$$

The following remark show that the well-known formulas [12] of Fibonacci numbers can be easily seen by using the properties of q-analogues of generalized Fibonacci and Lucas polynomials.

Remark 3.3. *Setting $a = 0, b = 1$ and $x = 1$, $q \to 1^-$ in (3.1), we get the following equation of the Fibonacci numbers*

$$F_{n+2k} = \sum_{i=0}^{k} \binom{k}{i} F_{n+i}.$$

By taking $a=0,b=1$ and $x=1$, $q\to 1^-$ as a special case of the equations (3.2), (3.3) and (3.4) we have the following identities for Fibonacci numbers:

$$F_{2n}=\sum_{i=0}^{n}\binom{n}{i}F_i,$$

$$F_n=\sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}F_{2i},$$

$$F_{2n+1}=\sum_{i=0}^{n}\binom{n}{i}F_{i+1}$$

respectively. Also it is easily obtain similar formulas for the Lucas numbers.

Proposition 3.4. *The generating function of the polynomials $S_n(t,q)$ is*

$$\sum_{n=0}^{\infty}S_n(x,q)\,t^n=\frac{a+(b-a)\,t}{1-t-xq^{-1}t^2\mu_t}, \tag{3.5}$$

where μ_t is the Fibonacci operator which is $\mu_t f(t)=f(tq)$ for any given function $f(t)$.

Proof. Let $g(x)=\sum\limits_{n=0}^{\infty}S_n(x,q)\,t^n$. We need to show the following equation:

$$g(x)\left(1-t-xq^{-1}t^2\mu_t\right)=a+(b-a)\,t.$$

We have

$$\begin{aligned} g(x)\left(1-t-xq^{-1}t^2\mu_t\right)&=a+bt+\sum_{n=2}^{\infty}S_n(x,q)\,t^n-\sum_{n=0}^{\infty}S_n(x,q)\,t^{n+1}-\sum_{n=0}^{\infty}S_n(x,q)\,xq^{n-1}t^{n+2}\\ &=a+bt-at+\sum_{n=2}^{\infty}\left\{S_n(x,q)-S_{n-1}(x,q)-xq^{n-3}S_{n-3}(x,q)\right\}t^n. \end{aligned}$$

This completes the proof. □

Corollary 3.5. *The generating function of $S_{2n}(x,q)$ is*

$$\sum_{n=0}^{\infty}S_{2n}(x,q)\,t^n=\frac{a+(b-a)\,t}{1-t-xq^{-1}t^2\mu_t}\sum_{k=0}^{\infty}\begin{bmatrix}n+k\\ n\end{bmatrix}_q(xt)^k\,q^{k(k-2+n)}. \tag{3.6}$$

Proof. If we want to obtain the generating function of $S_{2n}(x,q)$ by using equation (2.6), we realize that by setting $a_n^0(x,q)=S_n(x,q)$ in (2.1). We obtain $a_0^n(x,q)=S_{2n}(x,q)$. By considering (2.6), we find

$$\overline{a(t)}=\sum_{n=0}^{\infty}a_0^n(x,q)\,t^n=\sum_{n=0}^{\infty}a_n^0(x,q)\,t^n\sum_{k=0}^{\infty}\begin{bmatrix}n+k\\ n\end{bmatrix}_q(xt)^k\,q^{k(k-2+n)}.$$

Therefore

$$\sum_{n=0}^{\infty}S_{2n}(x,q)\,t^n=\sum_{n=0}^{\infty}S_n(x,q)\,t^n\sum_{k=0}^{\infty}\begin{bmatrix}n+k\\ n\end{bmatrix}_q(xt)^k\,q^{k(k-2+n)}.$$

From (3.5) we have

$$\sum_{n=0}^{\infty}S_{2n}(x,q)\,t^n=\frac{a+(b-a)\,t}{1-t-xq^{-1}t^2\mu_t}\sum_{k=0}^{\infty}\begin{bmatrix}n+k\\ n\end{bmatrix}_q(xt)^k\,q^{k(k-2+n)}.$$

□

This corollary points out that the generating functions of the first row and column of the generalized q-Seidel matrix are useful to obtain the generating function of $S_{2n}(x,q)$.

References

[1] A. Aral, V. Gupta and R.P. Agarwal, Applications of q-calculus in operator theory, New York, Springer, 2013.

[2] A. Dil and I. Mező, *A symmetric algorithm for hyperharmonic and Fibonacci numbers*, Appl. Math. Comput., **206** (2008), 942–951.

[3] C. Z. Jia , H. M. Liu and T. M. Wang, *q-Analogs of generalized Fibonacci and Lucas polynomials*, Fibonacci Quart., **45** (2007), 26–34.

[4] D. Dumont, *Matrices d'Euler-Seidel*, Séminaire Lotharingien de Combinatorie, 1981.

[5] G. E. Andrews, *Fibonacci numbers and the Rogers-Ramanujan identities*, Fibonacci Quart., **42** (2004), 3–19.

[6] I. Mező and A. Dil, *Euler-Seidel method for certain combinatorial numbers and a new characterization of Fibonacci sequence*, Cent. Eur. J. Math., **7** (2009), 310–321.

[7] J. Cigler, *q-Fibonacci polynomials*, Fibonacci Quart., **41** (2003), 31–40.

[8] J. Riordan, *Inverse relations and combinatorial identities*, Amer. Math. Monthly, **71** (1964), 485–498.

[9] R. J. Clarke, G. N. Han, J. Zeng, *A combinatorial interpretation of the Seidel generation of q-derangement numbers*, Ann. Comb., **4** (1997), 313–327.

[10] L. Carlitz, *Fibonacci notes 4: q-Fibonacci Polynomials,* Fibonacci Quart., **13** (1975), 97-102.

[11] R. B. Corcino, *On p,q-binomial coefficients*, Integers, **8** (2008), 1–16.

[12] S. Vajda, Fibonacci and Lucas numbers, and the Golden Section: Theory and Applications, Halsted Press, 1989.

[13] W. A. Al-Salam and M. E. H. Ismail, *Orthogonal Polynomials Associated with the Rogers-Ramanujan Continued Fraction*, Pacific J. Math., 104 (1983), 269–283.

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