THE INFINITE FIBONACCI TREE AND OTHER TREES GENERATED BY RULES

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Abstract. Suppose that I is a subset of a set U and that C is a collection of operations f defined in U. Create a set S by these rules: every element of I is in S, and if x is in S, then $f(x)$ is in S for all f in C for which $f(x)$ is defined. Then S "grows" in successive generations. If I consists of a single number r then S can be regarded as a tree with root r. We examine several examples, including these: (1) $1 \in S$, and if $x \in S$ then $x+1 \in S$ and $1/x \in S$; (2) $1 \in S$, and if $x \in S$ then $x+1 \in S$ and $2x \in S$; (3) $1 \in S$, and if $x \in S$ then $x+1 \in S$, and if $x \neq 0$ then $-1/x \in S$; (4) $1 \in S$, and if $x \in S$ then $x+1 \in S$ and $\sqrt{-1}x \in S$, and if $x \neq 0$ then $1/x \in S$. The first of these examples is the infinite Fibonacci tree, in which every positive rational number occurs as a node.

1. Introduction

As early as 1619, Johannes Kepler created a tree of fractions using these rules: begin with $1/1$, and thereafter, each node i/j has two descendants, $(i+j)/i$ and $i/(i+j)$. Kepler's tree [1, 3] can be recast by saying that 1 is present, and if x is present, then so are $x+1$ and $1/(x+1)$. The tree starts with a single node which spawns 2 nodes (2 and $1/2$), which spawn 4 nodes $(3, 1/3, 3/2, 2/2)$, and so on, so that the nth generation has 2^n nodes. Moreover, every positive rational number occurs exactly once.

Now consider the set S defined by these rules: $1 \in S$, and if $x \in S$, then $x+1 \in S$ and $1/x \in S$. Deleting duplicates as they occur leaves the infinite Fibonacci tree, represented in Figure 1 (in Section 7) and discussed in Example 3.4. Another tree with Fibonacci connections is given by the rules $1 \in S$, and if $x \in S$, then $x+1 \in S$ and $2x \in S$, where duplicates are deleted as they occur. This tree, which includes every positive integer, is represented by Figure 2 and Corollary 2.2. A third tree, containing all the rational numbers, is given by the rules $1 \in S$, and if $x \in S$, then $x+1 \in S$ and $-1/x \in S$; a fourth tree, containing all the Gaussian rational numbers is given by the rules $1 \in S$, and if $x \in S$, then $x+1 \in S$ and $ix \in S$ and if $x \in S$ and $x \neq 0$, then $1/x \in S$.

The purpose of this paper is to discuss those four trees and others. Certain notations will be helpful; e.g., $a, b, c, d, e, f, g, h, k, m, n, r, s, t, u, v$ will denote integers, although f and g will also be used for functions. In particular, suppose that $f_1(x) = (ax+b)/(cx+d)$ and $f_2(x) = (ex+f)/(gx+h)$. For any initial x_0, we have a set S defined by the rules $x_0 \in S$, and if $x \in S$, then $f_1(x) \in S$ and $f_2(x) \in S$, and we shall refer to S not only as a set, but also as a tree determined by the rules, with deletion of duplicates as they occur. The set (and tree) is partitioned into generations $g(n)$ defined inductively by $g(1) = \{x_0\}$ and

$$g(n) = \{f_1(x) : x \in g(n-1)\} \cup \{f_2(x) : x \in g(n-1)\} \setminus \bigcup_{i=1}^{n-1} g(i). \tag{1.1}$$

for $n \geq 2$. Note that the generations are, by definition, pairwise disjoint. We are interested in cases in which S includes every positive integer, or every positive rational number, etc. Also of interest are the sizes $|g(n)|$ of the generations, recurrence relations for $|g(n)|$ and related sequences, and limits.

2. Multinacci Trees

Throughout this section, $m \geq 2$ is an integer. Let S be the set defined by these rules: $1 \in S$, and if $x \in S$, then $x+1 \in S$ and $mx \in S$. The generations are given by $g(1) = \{1\}$ and, following (1.1),

$$g(n) = \{x+1 : x \in g(n-1)\} \cup \{mx : x \in g(n-1)\} \setminus \bigcup_{i=1}^{n-1} g(i).$$

Let $x(n,i)$ be the number of numbers in $g(n)$ that are congruent to i mod m. Then clearly for all $n \geq 2$,

$$|g(n)| = x(n,0) + \cdots + x(n,m-1), \tag{2.1}$$

$$x(n,0) = |g(n-1)|, \tag{2.2}$$

$$x(n,i) = x(n-1,i-1) \quad \text{for } 1 \leq i \leq m-1. \tag{2.3}$$

Theorem 2.1. The number of numbers in generation $g(n)$ is given by

$$|g(n)| = \begin{cases} 2^{n-1} & \text{if } 1 \leq n \leq m-1 \\ 2^{m-1}-1 & \text{if } n = m \\ 2^m - 2 & \text{if } n = m+1 \\ |g(n-1)| + \cdots + |g(n-m)| & \text{if } n \geq m+2. \end{cases}$$

Proof: We begin with $m = 2$, for which $|g(n)|$ is clearly as asserted for $n = 1, 2, 3$. For $n \geq 4$, each x in $g(n-1)$ yields $2x$ in $g(n)$, each y in $g(n-2)$ yields $2y+1$ in $g(n)$, and the set of all such $2x$ and $2y+1$ contains no number in any $g(i)$ for $i \leq n-3$. Therefore, $|g(n)| = |g(n-1)| + |g(n-2)|$, so that the proof for $m = 2$ is complete.

Suppose now that $m \geq 3$ and that $1 \leq n \leq m-1$ and $2 \leq k \leq m-1$. Then $x(k-1, m-1) = 0$, so that by (2.2) and (2.3), $x(k,0) = |g(k-1)|$ and

$$x(k,1) + \cdots + x(k,m-1) = x(k-1,0) + \cdots + x(k-1,m-1).$$

Consequently, by (2.1), $|g(k)| = |g(k-1)| + |g(k-1)|$, which, starting with $|g(1)| = 1$, gives

$$|g(k)| = 2^{k-1}. \tag{2.4}$$

Next, suppose that $n = m$. Each of the 2^{n-2} numbers x in $g(n-1)$ yields $x+1$ and mx in $g(n)$. However, at least one of these is a duplicate of a number in a previous generation (i.e., $g(i)$ for some $i \leq m-2$). In order to account for all possible duplicates, we consider cases:

Case 1: $x+1 = y+1$ or $mx = my$ for some y in a previous generation $g(i)$. In this case, $x = y$, contrary to $g(m-1) \cap g(i) = \varnothing$.

Case 2: $mx = y+1$ for some y in a previous generation $g(i)$, but clearly, no such $g(i)$ contains a number y congruent to $m-1$ mod m.

Case 3: $x+1 = my$ for some y in a previous generation. Here, x is congruent to $m-1$ mod m, which holds for exactly one number x in $g(m-1)$, namely $m-1$. Therefore

$$|g(m)| = 2 \cdot 2^{m-2} - 1 = 2^{m-1} - 1. \tag{2.5}$$

Next, suppose that $n = m+1$. Each x in $g(m)$ yields $x+1$ and mx in $g(m+1)$, unless some such $x+1$ is also my for some y in some $g(i)$ for $i \leq m$, but we shall show that this is not possible. The residues mod m of the numbers in $g(2)$ are 2 and 0, so that the greatest and next-to-greatest residues in successive generations from $g(2)$ to $g(m)$ is as shown here:

$$2 \to 3 \to 4 \to \cdots \to m-1$$

$$0 \to 1 \to 2 \to \cdots \to m-3 \to m-2.$$

Thus, $g(m)$ contains no number x congruent to $m-1 \mod m$. Consequently, no $x+1$, for x in $g(m)$, is a duplicate in the counting of numbers in $g(m+1)$, so that

$$|g(m+1)| = 2^m - 2. \tag{2.6}$$

Finally, suppose that $n \geq m+2 \geq 5$. As an induction hypothesis, assume that

$$|g(n-1)| = |g(n-2)| + \cdots + |g(n-m-1)|, \tag{2.7}$$

where the initial case $(n = m+2)$ follows directly from (2.4)-(2.6). Every y in $g(n-m-1)$ yields ym in $g(n-m)$, which yields $ym+m-1$ in $g(n-1)$; conversely, every number congruent to $m-1$ in $g(n-1)$ descends from a number in $g(n-m-1)$. Therefore,

$$|g(n-m-1)| = x(n-1, m-1).$$

Accordingly,

$$|g(n-1)| = x(n-1, m-1) + |g(n-1)| - |g(n-m-1)|,$$

so that by (2.7),

$$|g(n-1)| = x(n-1, m-1) + |g(n-2)| + \cdots + |g(n-m)|,$$

and

$$2|g(n-1)| - x(n-1, m-1) = |g(n-1)| + |g(n-2)| + \cdots + |g(n-m)|. \tag{2.8}$$

Equations (2.1)-(2.3) imply

$$\begin{aligned} |g(n)| &= x(n,0) + \cdots + x(n, m-1) \\ &= |g(n-1)| + x(n-1, m-1) + x(k-1, 0) + \cdots + x(k-1, m-1) \\ &= |g(n-1)| + |g(n-1)| - x(n-1, m-1), \end{aligned}$$

so that by (2.8),

$$|g(n)| = |g(n-1)| + \cdots + |g(n-m)|. \qquad \square$$

Corollary 2.2. If $m = 2$, then $|g(n)| = F(n)$, the nth Fibonacci number.

This corollary is simply a special case of Theorem 2.1, and the proof of the theorem shows more: that $g(n)$ consists of $F(n-1)$ even numbers and $F(n-2)$ odd numbers. See Figure 2 in Section 7.

To summarize, Theorem 2.1 shows that for $m \geq 3$, the generation sizes satisfy the m-multinacci recurrence relation $a_n = a_{n-1} + a_{n-2} + \cdots + a_{n-m}$, with initial values $1, 2, \ldots, 2^{m-2}, 2^{m-1} - 1$.

We thank the referee for noting that $g(n)$ consists of the numbers whose base m representation has

$$\text{(number of digits)} + \text{(sum of digits)} = n + 1. \tag{2.9}$$

For example, for $m = 4$, the numbers in $g(4)$ are $12, 9, 32, 6, 20, 17, 64$; in base 4 these are $30, 21, 200, 12, 110, 101, 1000$, for which the compositions of 5 indicated by (2.9) are

$$2+3, \quad 2+3, \quad 3+2, \quad 2+3, \quad 3+2, \quad 3+2, \quad 4+1.$$

3. More Fibonacci-Related Trees

Corollary 2.2 describes a tree of integers satisfying $|g(n)| = F(n)$; in this section, we consider other trees of fractions whose generations have sizes that are Fibonacci numbers. We begin with a lemma.

Lemma 3.1. Suppose that $m \geq 1$. The greatest k for which $k^2 + 4km$ is a square is $(m-1)^2$.

Proof: If $k = (m-1)^2$, then $k^2 + 4km = (m-1)^4 + 4(m-1)^2 m = (m^2-1)^2$. Now suppose that $k > (m-1)^2$. Then

$$(k + 2m - 2)^2 < k^2 + 4km < (k + 2m)^2,$$

so that if $k^2 + 4km$ is a square, then $k^2 + 4km = (k + 2m - 1)^2$. However, this implies $2k = (2m-1)^2$, contrary to the fact that $2m - 1$ is odd. □

Theorem 3.2. Suppose that k is a positive integer. Let S be the set defined by these rules: $1 \in S$, and if $x \in S$, then $x + k \in S$ and $k/x \in S$. Partition S into generations $g(n)$ inductively: $g(1) = \{1\}$, and for $n \geq 2$,

$$g(n) = \{x + k : x \in g(n-1)\} \cup \{k/x : x \in g(n-1)\} \setminus \bigcup_{i=1}^{n-1} g(i).$$

If $k = 1$, then $|g(n)| = F(n)$ for $n \geq 1$, and if $k > 1$, then $|g(n)| = F(n+1)$ for $n \geq 1$.

Proof: First, suppose that $k = 1$. Clearly $|g(n)| = F(n)$ for $n \leq 2$. Assume for arbitrary $n \geq 2$ that $g(n)$ consists of $F(n-1)$ numbers $x > 1$ together with $F(n-2)$ numbers $x \leq 1$. Each of the former spawns $x + 1$ and $1/x$ in $g(n+1)$, and each of the others spawns the single number $x + 1$ in $g(n+1)$. These numbers are distinct because the equation $x + 1 = 1/x$ has no integer solution. Therefore, $g(n+1)$ consists of $2F(n-1) + F(n-2) = F(n+1)$ numbers.

Next, suppose that $k > 1$. Clearly $g(n) = F(n)$ for $n \leq 2$. Assume for arbitrary $n \geq 2$ that $g(n)$ consists of $F(n)$ numbers $x > k$ together with $F(n-1)$ numbers $x \leq k$. Each of the former spawns $x + k$ and k/x in $g(n+1)$, and each of the others spawns the single number $x + k$ in $g(n+1)$. To confirm that these numbers are distinct, suppose that $x + k = k/x$ for some x. Then $x^2 + kx - k = 0$, so that $k^2 + 4k$ must be a square, contrary to Lemma 1. Therefore, $g(n+1)$ consists of $2F(n) + F(n-1) = F(n+2)$ numbers. □

Consider the rule "if $x \in S$, then $x + k \in S$" in the statement of Theorem 3.2. If this rule is changed to "if $x \in S$, then $x + 1 \in S$" and the other rule remains "if $x \in S$, then $k/x \in S$", then the resulting tree, for $k > 1$, has generation sizes $|g(n)|$ which form a sequence not closely related to the Fibonacci sequence; indeed, the sequence appears to be not linearly recurrent. Nevertheless, the tree S contains every positive rational number, in accord with the following theorem.

Theorem 3.3. Suppose that k is a positive integer. Let S be the set defined by these rules: $1 \in S$, and if $x \in S$, then $x + 1 \in S$ and $k/x \in S$. Then S is the set of positive rational numbers.

Proof: Clearly, every positive rational $b/1 \in S$. For arbitrary $d \geq 1$, assume that if u/v is a reduced positive rational with $v \leq d$, then $u/v \in S$. Suppose that $b/(d+1)$ is a reduced positive rational. As a first case, suppose that $b \leq d$. By the induction hypothesis, $(d+1)/b \in S$ and, by the same hypothesis, the number $x = k(d+1)/b \in S$. Consequently, $k/x \in S$; i.e., $b/(d+1) \in S$. To cover all remaining cases, suppose that $b > d + 1$, so that

$b = (d+1)q + r$, where $0 \le r < d+1$. Then $b/(d+1) = q + r/(d+1)$. As in the first case, $r/(d+1) \in S$. Now q applications of $x \to x+1$ show that $b/(d+1) \in S$. $\square$

Example 3.4 Taking $k = 1$ in Theorem 3.2 and Theorem 3.3 gives the infinite Fibonacci tree represented by Figure 1. In the following array, row n shows the numbers in generation $g(n)$ arranged in decreasing order:

1												
2												
3	1/2											
4	3/2	1/3										
5	5/2	4/3	2/3	1/4								
6	7/2	7/3	5/3	5/4	3/4	2/5	1/5					
7	9/2	10/3	8/3	9/4	7/4	7/5	6/5	4/5	3/5	3/7	2/7	1/6

Note that the $F(n)$ numbers in row $n \ge 3$, taken in order, consist of $F(n-1)$ numbers $x+1$ from x in row $n-1$, followed by $F(n-2)$ numbers $1/(x+1)$ from x in row $n-2$.

Not every tree having $|g(n)| = F(n)$ for $n \ge 1$ is given by Corollary 2.2 and Example 3.4, as indicated by the following example.

Example 3.5. Let S be the tree defined by these rules: $1 \in S$, and if $x \in S$, then $1/x \in S$ and $1/(x+1) \in S$. Inductively, for $n \ge 2$, $g(n)$ consists of $F(n-2)$ numbers ≥ 1, each of the form $x+1$ for x in $g(n-2)$, together with $F(n-1)$ numbers < 1, each of the form $1/(x+1)$ for x in $g(n-1)$. Hence, $|g(n)| = F(n)$. It is easy to prove by induction that every fraction u/v in S is reduced to lowest terms and that if $v = 1$, then u is the only integer in $g(2u-1)$. Next, assume for arbitrary $v \ge 1$ that every fraction a/b with $b \le v$ is in S, and suppose that $u/(v+1)$ is a fraction. If $u < v+1$, then by the induction hypothesis, $(v+1)/u \in S$, so that the rule $x \to 1/x$ applies, and $u/(v+1) \in S$. On the other hand, if $u > v+1$, write $u = (v+1)q + r$ with $0 \le r < v+1$, so that $(v+1)/r \in S$. Then $r/(v+1) \in S$. Let $g(n)$ be the generation containing $r/(v+1)$. Then $u/(v+1) = r/(v+1) + q \in g(n+2q)$. Therefore, S contains every positive rational number.

Example 3.6. We have already seen examples of trees in which all the positive rational numbers occur. Consider next the tree S_1 given by the rules $1 \in S_1$, and if $x \in S_1$, then $x + 4 \in S_1$ and $12/x \in S_1$. It is easy to see that the numbers 2 and 3 are missing from S_1. Starting another tree, S_3, with 3 and the same iterative membership requirements leads to a tree that includes 1 (in $g(5)$) and hence contains S_1 as a subtree, as in Figure 3. Regarding S_3, we observe that all positive integers not congruent to 2 mod 4 occur, that $|g(n)| = F(n+1)$ for $n \ge 1$, and that all fractions, as generated, are in reduced form. Since 2 is missing, it is natural to examine the tree S_2 having 2 as root, where, again, if $x \in S_2$, then $x + 4 \in S_2$ and $12/x \in S_2$, as in Figure 4. The method of proof for Example 3.5 can be used to prove that $S_2 \cup S_3$ includes every positive rational number.

Example 3.7. Let S be the tree generated by these rules: $1 \in S$, and if $x \in S$, then $2x \in S$ and $1 - x \in S$. To see that every integer h is in S, note first that this holds for $|h| \le 2$, and assume for arbitrary $h \ge 2$ that if $|m| < h$, then $m \in S$. Now suppose that m satisfies $|m| = h > 2$. If m is even, write $m = 2k$, so that $k = m/2$, whence $|k| < |m| = h$, so that $k \in S$, whence $m \in S$. On the other hand, if $m = 2k + 1$, then $k = (m-1)/2$, whence $|k| < |m| = h$, so that $-k \in S$; therefore $-2k \in S$, so that $1-(-2k)$, which is m, is in S. Thus, S contains every integer. Moreover, $|g(n)| = F(n)$ for $n \ge 3$. Conjecture: every generation $g(n)$ contains $\pm F$ for some Fibonacci number F.

Example 3.8. Let S be the tree generated by these rules: $1 \in S$, and if $x \in S$ then $1 + 1/x \in S$ and $1/x \in S$. An easy induction argument shows that in arbitrary $g(n)$, for $n > 2$, each node x greater than 1 begets a new node in $(0, 1)$ and a new node in $(1, \infty)$, and each node x less than 1 begets a single new node in $(1, \infty)$. Thus, $g(n)$ consists of $F(n-1)$ nodes in $(1, \infty)$ and $F(n-2)$ nodes in $(0, 1)$, leading to $|g(n)| = F(n)$ for $n \geq 1$. To see that every positive rational number is in S, the following lemma is useful: if $x \in S$ and $x > 1$ then $x - 1 \in S$; to prove this, write $x = 1 + 1/u$, $u \in S$; then $x - 1 = 1/u$, which is in S. Clearly every $b/1$ is in S; suppose that b/d is an arbitrary fraction in reduced terms, with $d > 1$. By the lemma, we may assume that $b < d$, so that by induction hypothesis, $d/b \in S$. Consequently, $b/d \in S$. A final observation is that $F(n+1)/F(n) \in g(n)$, and that the numerator and denominator of this fraction are maximal for fractions in $g(n)$.

Example 3.9. Let S be the tree generated by these rules: $1 \in S$, and if $x \in S$, then $x/(x+1) \in S$ and $1/x \in S$. For every n, the set $g(n)$ has $F(n-1)$ numbers < 1 and $F(n-2)$ numbers ≥ 1, so that $|g(n)| = F(n)$ for all n. An induction proof on the size of denominators establishes that S contains every positive rational number. Another way to obtain this tree is to apply the reciprocation mapping $x \to 1/x$ to each node in the tree at Example 3.4.

4. All the Rational Numbers

Previous examples include trees which contain every positive integer, or every integer, or every positive rational number. We turn now to trees which contain every rational number.

Example 4.1. Decree that $0 \in S$ and that if $x \in S$, then $x + 1 \in S$ and if $x + 1 \neq 0$ then $-1/(x+1) \in S$. Then $g(1) = \{0\}$ and for all other generations, $g(n+1)$ consists of $F(n)$ negative numbers and $F(n)$ positive numbers, so that $|g(n+1)| = 2F(n)$. A proof that S contains every rational number depends on the method for Example 3.8: first, clearly every positive integer is in S; then inductively, every $1/n$ and $-n-1$ are in S, because $1/n = f_1(f_2(1/n))$ and $-n-1 = f_2(f_2(1/n))$, where $f_1(x) = x+1$ and $f_2(x) = -1/(x+1)$. The rest of the proof follows by induction on the size of denominators, together with reciprocation and the fact that if $x \in S$, then $x - 1 = f_2(f_2(f_1(f_2(f_2(x))))) \in S$. Every negative integer is a terminal node in S. The $F(n)$ positive numbers in $g(n+1)$ consist of $F(n-1)$ numbers $x+1$ from x in $g(n)$, together with $F(n-2)$ numbers $x/(x+1)$ from x in $g(n-1)$; the $F(n)$ negative numbers in $g(n+1)$ are the negative reciprocals of the positive numbers in $g(n+1)$.

Example 4.2. Let S be the tree generated by these rules: $1 \in S$, and if $x \in S$ then $x + 1 \in S$, and if $x \in S$ and $x \neq 0$, then $-1/x \in S$. A proof that S contains every rational number is similar to the proof for Example 4.1; here, the corresponding lemma, that if $x \in S$ then $x - 1 \in S$, stems from the fact that if $f_1(x) = x + 1$ and $f_2(x) = -1/x$, then $x - 1 = f_2(f_1(f_2(f_1(f_2(x)))))$. For $n \geq 1$, let $S(n, i)$ be the set of nodes in $g(n)$ that have i offspring in $\bigcup_{h=1}^{n-1} g(h)$; e.g., $S(n, 0)$ counts terminal nodes, and $S(n, 2)$ counts nodes that beget 2 new nodes. The sequence $(S(n, i))$ satisfies the recurrence $a(n) = a(n-1) + a(n-3)$ for $n \geq 7$, so that the sequence $(|g(n)|) = (1, 2, 3, 3, 5, 7, 10, 15, 22, \ldots)$ satisfies $|g(n)| = |g(n-1)| + |g(n-3)|$ for $n \geq 7$.

Example 4.3. Here, we show another way to generate the set S of Example 4.2, but in a more general manner. Suppose that $m \geq 3$, and define $h_m(n) = \{n\}$ for $n = 1, 2, \ldots, m$ and

$$h_m(n) = \{x + 1 : x \in h_m(n-1)\} \cup \{x/(x+1) : x \in h_m(n-m)\}, \; S_m = \bigcup_{n=1}^{\infty} h_m(n)$$

for $n \geq 4$. The now familiar proof by "denominator induction" shows that S_m is the set of positive rational numbers, and clearly, $|h_m(n)| = |h_m(n-1)| + |h_m(n-3)|$ for $n \geq m+1$, with $|h_m(n)| = 1$ for $n \leq m$. To obtain the numbers in the set S of Example 4.2, let $g(1) = h_3(1) = \{1\}$, $g(2) = \{-1, 2\}$, $g(3) = \{-1/2, 0, 3\}$, and for $n \geq 4$, let $g(n)$ be the set of numbers in $h_3(n)$ together with $-1/x$ for each x in $h_3(n-1)$. The array having $g(n)$ as row n, consisting of all the rational numbers, has these first six rows:

1						
−1	2					
−1/2	0	3				
−1/3	1/2	4				
−2	−1/4	2/3	3/2	5		
−3/2	−2/3	−1/5	3/4	5/3	5/2	6

Example 4.4. As a generalization of Example 4.2, suppose that $m \geq 2$ and that "$-1/x \in S$" is replaced by "$-m/x \in S$". Then S contains every rational number.

5. Limits

Suppose that S is given by these rules: $1 \in S$, and if $x \in S$, then

$$\frac{ax+b}{cx+d} \in S \quad \text{and} \quad \frac{ex+f}{gx+h} \in S.$$

All the previously mentioned trees are special cases of S; e.g., the infinite Fibonacci tree (as in Example 3.4) is given by $(a, b, c, d) = (1, 1, 0, 1)$ and $(e, f, g, h) = (0, 1, 1, 0)$. When S includes all the positive rationals, every convergent sequence of rationals can be identified with a sequence of nodes in S. If the nodes lie in a single path, their limit is of interest. In order to study such paths, call an edge of the form $x \to \dfrac{ax+b}{cx+d}$ an *up-edge,* denoted by U, and an edge of the form $x \to \dfrac{ex+f}{gx+h}$ a *down-edge,* denoted by D. An up-edge followed by a down-edge corresponds to

$$x \to \frac{ax+b}{cx+d} \to \frac{(ae+cf)x+be+df}{(ag+ch)x+bg+dh} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix}, \tag{5.1}$$

where

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} e & f \\ g & h \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and a down edge followed by an up-edge corresponds to

$$x \to \frac{ex+f}{gx+h} \to \frac{(ae+bg)x+af+bh}{(ce+dg)x+cf+dh} = \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix}, \tag{5.2}$$

where

$$\begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix}.$$

In (5.1) and (5.2), the matrix product notation has the usual meaning but also serves as a useful way to represent the indicated fraction. An infinite path of the form $UDUDUD\ldots$ is a *zigzag path.* Call nodes of the form (5.1) *upper nodes* and those of the form (5.2) *lower* *nodes*. We shall see that under suitable conditions, the upper nodes converge and the lower nodes converge. In order to state the conditions, let

$$\Delta = (ae - dh + cf - bg)^2 + 4(be + df)(ag + ch),$$

and deem S *regular* if $\Delta \neq 0$, $ag + ch \neq 0$ and $ce + dg \neq 0$, where a, b, c, d, e, f, g are all nonnegative. A first theorem about convergence along paths in S follows.

Theorem 5.1: Suppose that path p is a zigzag graph in a regular tree S and that the limits of the upper nodes and lower nodes on p exist. The limits are, respectively,

$$\frac{ae - dh + cf - bg + \sqrt{\Delta}}{2(ag + ch)} \quad \text{and} \quad \frac{ae - dh - cf + bg + \sqrt{\Delta}}{2(ce + dg)}. \tag{5.3}$$

Proof: We begin with upper nodes, for which the limit, if it exists, is given by iterating the mapping (5.1). Let p be a zigzag path $UDUDUD\ldots$. Then the node given by $x(UD)^n$ has the form

$$\begin{pmatrix} \alpha_n & \beta_n \\ \gamma_n & \delta_n \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^n \begin{pmatrix} x \\ 1 \end{pmatrix}, \tag{5.4}$$

so that

$$\begin{pmatrix} \alpha_{n+1} & \beta_{n+1} \\ \gamma_{n+1} & \delta_{n+1} \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^n \begin{pmatrix} x \\ 1 \end{pmatrix},$$

or equivalently,

$$\frac{\alpha_{n+1}x + \beta_{n+1}}{\gamma_{n+1}x + \delta_{n+1}} = \frac{(\alpha\alpha_n x + \beta\gamma_n)x + \alpha\beta_n + \beta\delta_n}{(\gamma\alpha_n x + \delta\gamma_n)x + \gamma\beta_n + \delta\delta_n}. \tag{5.5}$$

Let $u = \lim\limits_{n\to\infty} \alpha_n/\gamma_n$, $v = \lim\limits_{n\to\infty} \beta_n/\gamma_n$, $w = \lim\limits_{n\to\infty} \delta_n/\gamma_n$. Taking limits in (5.5) gives

$$\frac{ux + v}{x + w} = \frac{(\alpha u + \beta)x + \alpha v + \beta w}{(\gamma u + \delta)x + \gamma v + \delta w}.$$

Cross-multiplying, collecting coefficients of $x^2, x, 1$, and regarding x as an indeterminate, we find $u^2\gamma + u(\delta - \alpha) - \beta = 0$ and $v^2\gamma + vw(\delta - \alpha) - \beta w^2 = 0$, so that

$$u = v/w = (\alpha - \delta \pm \sqrt{\Delta})/2\gamma. \tag{5.6}$$

The hypothesis that a, b, c, d, e, f, g are all nonnegative forces u to be the greater of the two possibilities, that is,

$$u = \frac{v}{w} = \frac{\alpha - \delta + \sqrt{(\alpha - \delta)^2 + 4\beta\gamma}}{2\gamma}. \tag{5.7}$$

The limit is then simply u, since, from $uw = v$, we have $ux + uw = ux + v$, so that $(ux + v)/(x + w) = u$. Now substituting $\alpha = ea + fc$, $\beta = eb + fd$, $\gamma = ga + hc$, $\delta = gh + hd$ into (5.7) gives (5.3). A proof for lower nodes following the same steps finds a discriminant $(\alpha' - \delta')^2 + 4\beta'\gamma' = (\alpha - \delta)^2 + 4\beta\gamma$. Of course, by (5.2), the second limit in (5.3) is $(eu + f)/(gu + h)$. □

Limits for selected choices of a, b, c, d, e, f, g, h are shown below. The first two rows match the infinite Fibonacci tree (Example 3.4) and the Kepler tree of fractions (Section 1).

a,b,c,d	e,f,g,h	$(UD)^\infty$	$(DU)^\infty$
$1,1,0,1$	$0,1,1,0$	$(-1+\sqrt{5})/2$	$(1+\sqrt{5})/2$
$1,1,0,1$	$0,1,1,1$	$-1+\sqrt{2}$	$\sqrt{2}$
$1,-1,0,1$	$1,0,1,5$	$(-3+\sqrt{5})/2$	$(-5+\sqrt{5})/2$
$1,-1,0,1$	$1,0,1,6$	$-2+\sqrt{3}$	$-3+\sqrt{3}$
$2,1,0,1$	$0,1,1,0$	$1/2$	2
$3,1,0,1$	$0,1,1,0$	$(-1+\sqrt{13})/6$	$(1+\sqrt{13})/2$
$1,2,1,1$	$0,1,1,0$	$(-1+\sqrt{5})/2$	$(1+\sqrt{5})/2$
$1,3,1,1$	$0,1,1,0$	$-1+\sqrt{2}$	$1+\sqrt{2}$
$1,3,2,1$	$0,1,1,0$	$(-1+\sqrt{5})/2$	$(1+\sqrt{5})/2$
$2,5,3,1$	$0,1,1,0$	$(-1+\sqrt{3})/2$	$1+\sqrt{3}$

Limits other than those indicated by $(UD)^\infty$ and $(DU)^\infty$ are also of interest. Consider an infinite path p of the form $U^{k_1}DU^{k_2}D\cdots U^{k_m}D\cdots$ in the infinite Fibonacci tree. Clearly, the nodes of p converge if and only if the sequence (k_i) is bounded. Assuming (k_i) bounded, we now study limits along periodic paths—where a period is a finite branch of the form $B = U^{k_1}DU^{k_2}D\cdots U^{k_m}D$, and the periodic path is the infinite concatenation $BBB\cdots$, denoted by B^∞.

Theorem 5.2. Suppose that $f(x) = (\alpha x+\beta)/(\gamma x+\delta)$, where $\alpha \neq 0$, $\gamma \neq 0$, and $(\alpha-\delta)^2+4\beta\gamma > 0$. Define $f_1(x) = f(x)$ and $(f_n(x) = f(f_{n-1}(x))$ for $n \geq 2$. Then $\lim_{n\to\infty} f_n(x) = (v-\delta)/\gamma$, where v is an eigenvalue of $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$.

Proof: Define $(\alpha_n, \beta_n, \gamma_n, \delta_n)$ as in (5.4), so that (7) and (5.6) hold, which is to say that the number $u = \lim_{n\to\infty} f_n(x)$ is one of the two numbers

$$u_1 = \frac{\alpha-\delta+\sqrt{(\alpha-\delta)^2+4\beta\gamma}}{2\gamma}, \quad u_2 = \frac{\alpha-\delta-\sqrt{(\alpha-\delta)^2+4\beta\gamma}}{2\gamma}.$$

The eigenvalues of $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ are

$$v_1 = \frac{\alpha+\delta+\sqrt{(\alpha-\delta)^2-4(\alpha\delta-\beta\gamma)}}{2\gamma}, \quad v_2 = \frac{\alpha+\delta-\sqrt{(\alpha-\delta)^2-4(\alpha\delta-\beta\gamma)}}{2\gamma}.$$

Thus, if $u = u_1$, then $u = (v_1-\delta)/\gamma$, and if $u = u_2$, then $u = (v_2-\delta)/\gamma$. □

Returning to any suitable choice of a,b,c,d,e,f,g,h, let $U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $D = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$; that is, $f_1(x) = (ax+b)/(cx+d)$ and $f_2(x) = (ex+f)/(gx+h)$. Then

$$f_1^{k_1} f_2 f_1^{k_2} f_2 \cdots f_1^{k_m} f_2(x) = (\alpha x+\beta)/(\gamma x+\delta), \tag{5.8}$$

where

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = B = U^{k_1}DU^{k_2}D\cdots U^{k_m}D, \tag{5.9}$$

Therefore, Theorem 5.2 applies, so that the limit along B^∞, starting at any node x in S, is $(v-\delta)/\gamma$, where v is an eigenvalue of B. Next, we show the connection between such a limit and its continued fraction as determined by B.

Corollary 5.3. Let S be the infinite Fibonacci tree given by $(a, b, c, d, e, f, g, h) = (1, 1, 0, 1, 0, 1, 1, 0)$. Let B^∞ be the infinite path formed by concatenating the finite path $B = U^{k_1}DU^{k_2}D \cdots U^{k_m}D$, represented by the matrix in (5.9), and let $u = \lim_{n\to\infty} f_n(x)$. Then $u = [0, \overline{k_m, k_{m-1}, \ldots, k_1}]$.

Proof: The assertion follows from the fact that left multiplication $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^n$ matches attaching $[k_m, k_{m-1}, \ldots, k_1]$ at the end of the continued fraction consisting of n copies of $[k_m, k_{m-1}, \ldots, k_1]$.

Example 5.4. In the infinite Fibonacci tree of Corollary 5.3, let $B = UUDUUUDUD$, so that $(k_1, k_2, k_3) = (2, 3, 1)$. By Corollary 5.3, $\lim_{n\to\infty} f_n(x) = [0, \overline{1, 3, 2}] = (-3 + \sqrt{37})/4$.

6. Gaussian Fractions

In this section, the set (or tree) S is given by these rules: $1 \in S$, and if $x \in S$, then $x + 1 \in S$ and $ix \in S$, and if $x \neq 0$, then $1/x \in S$. We shall prove that S contains every Gaussian rational number; that is, every number $(a + bi)/(c + di)$, where $c^2 + d^2 > 0$.

Lemma 6.1. Suppose that b, c, d are integers. If any one of the numbers $(b + ci)/d$, $(bi - c)/d$, $(-b - ci)/d$, $(bi + c)/d$ is in S, then the other three are also in S.

Proof: Iterating the rule that if $x \in S$, then $ix \in S$ shows that ix, $-x$, and $-ix$ are in S. □

Lemma 6.2. If $x \in S$, then $x - 1 \in S$.

Proof: If $x \in S$, then by Lemma 6.1, $-x \in S$. Consequently, $-x + 1 \in S$, so that $x - 1 \in S$ by Lemma 6.1. □

Lemma 6.3. If $x \in S$ and $a + bi$ is a Gaussian integer, then $x + a + bi \in S$.

Proof: Suppose that $x \in S$ and that a is a real integer. If $a > 0$, then $x \to x + 1 \to x + 2 \to \cdots \to x + a$ are in S; if $a < 0$, then $x \to x - 1 \to x - 2 \to \cdots \to x - a$ are in S, by Lemma 6.2. So, we have $x + a$ in S for every integer a. Suppose now that b is an integer. Then $-i(x + a) \in S$, by Lemma 6.1, whence $-ix - ia + b \in S$, by Lemma 6.3. Then $i(-ix - ia + b) \in S$, which is to say that $x + a + bi \in S$. □

Theorem 6.4. Every Gaussian rational number $(a + bi)/(c + di)$ is in S.

Proof: $1 \in S$, so that $-1 \in S$ by Lemma 6.1. Then $0 \in S$, by rule 1, whence $a + bi \in S$ for every Gaussian integer $a + bi$. Now suppose that w/z is an arbitrary Gaussian rational, where w and z are Gaussian integers and $|z|$ is least possible. If $|z| = 1$, then w is a Gaussian integer, so that $w/z \in S$. Assume then, that $w/z \notin S$. Then there is a least integer $\delta > 1$ for which there is a Gaussian rational w'/z' such that $|z'| = \delta$ and $w'/z' \notin S$. We may and do assume that $w' = w$ and $z' = z$. By the division algorithm [2], there exist Gaussian integers q and r such that $w = qz + r$, where $|r| < |z|$. Then $w/z = q + r/z$. If $r/z \in S$, then $w/z \in S$, by Lemma 6.3, a contradiction. On the other hand, if $r/z \notin S$, then $z/r \in S$ since $|r| < |z|$. But then $r/z \in S$, another contradiction. Therefore, $w/z \in S$. □

We conclude this section with a tree of Gaussian integers.

Example 6.5. Let S be the tree generated by these rules: $0 \in S$, and if $x \in S$ then $x+1 \in S$ and $ix \in S$. Iterating the mapping $x \to x+1$ shows that every positive integer n is in S. Then in, $-n$, and $-in$ are in S, so that $0 = -1+1 \in S$, and $b'i+1, b'i+2, b'i+3, \ldots$ are in S for every integer b'. For each of these numbers $b'i+c$, the number $-b'i-c$ is in S, so that, in conclusion, S includes every Gaussian integer $a+bi$. For $n \geq 1$, let $S(n,i)$ be the set of nodes in $g(n)$ that have i offspring in $\bigcup_{h=1}^{n-1} g(h)$; e.g., $S(n,0)$ counts terminal nodes, and $S(n,2)$ counts nodes that beget 2 new nodes. We conjecture that $S(n,0) = n-5$ for $n \geq 5$, that $S(n,1) = 2n-7$ for $n \geq 4$, and that $S(n,2) = n-1$ for $n \geq 1$, so that, if the conjectures are true, then the sequence $(|g(n)|) = (1,1,2,4,7,11,15,19,23,\ldots)$ satisfies $|g(n)| = 4n-13$ for $n \geq 5$.

7. Figures

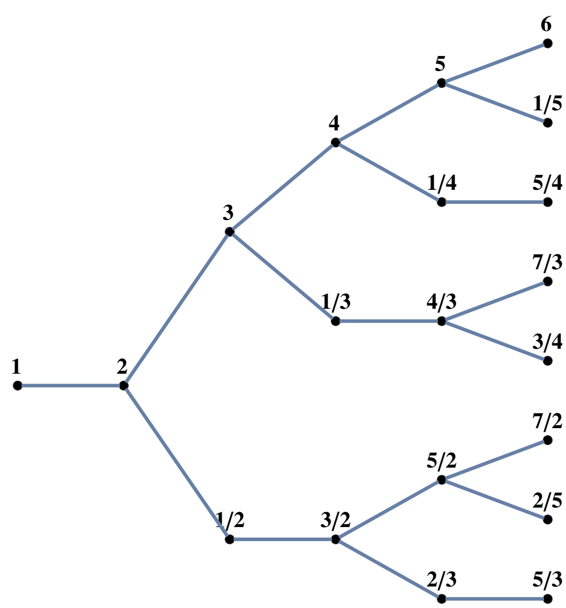

Figure 1. $x \to x+1, x \to 1/x$; Example 3.4

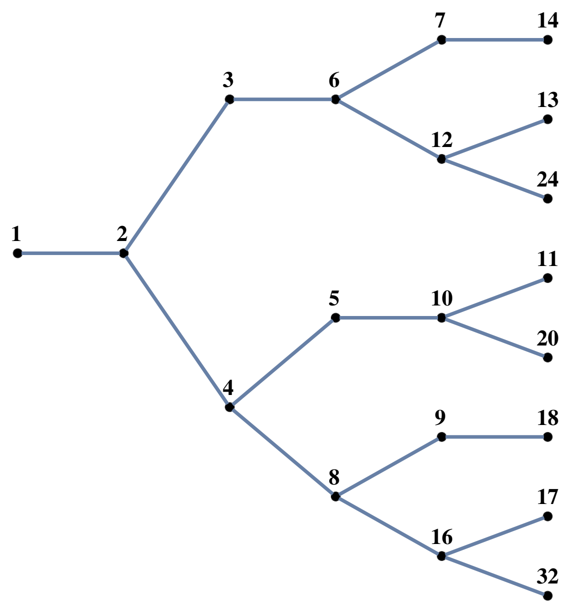

FIGURE 2. $x \to x + 1, x \to 2x$; Theorem 2.1

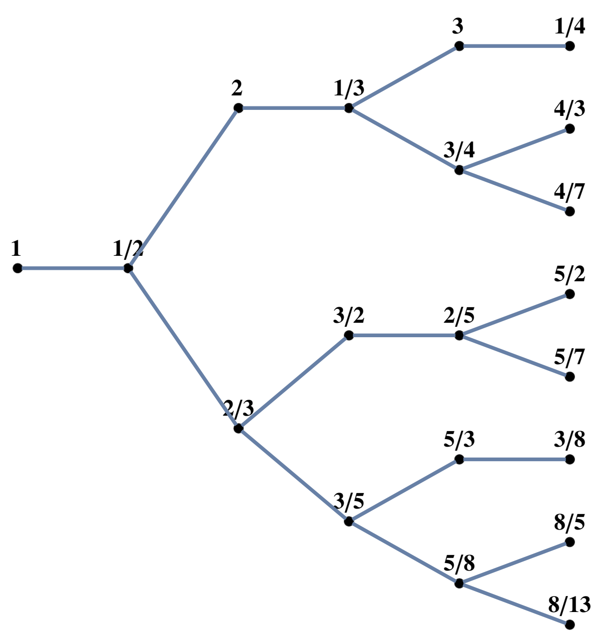

FIGURE 3. $x \to 1/x, x \to 1/(x + 1)$; Example 3.5

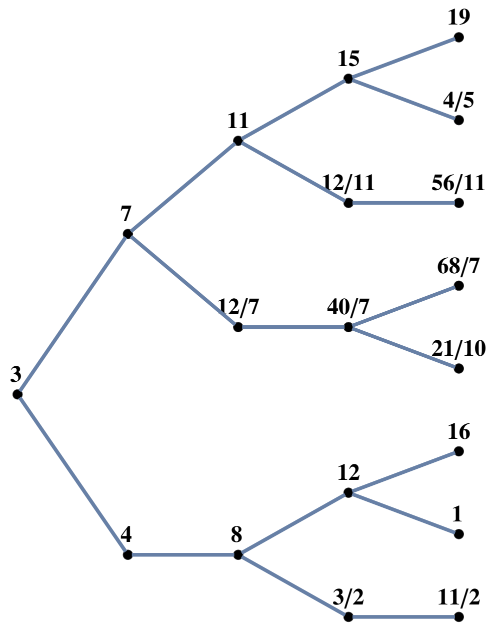

FIGURE 4. $x \to x + 4, x \to 12/x$, from 3; Example 3.6

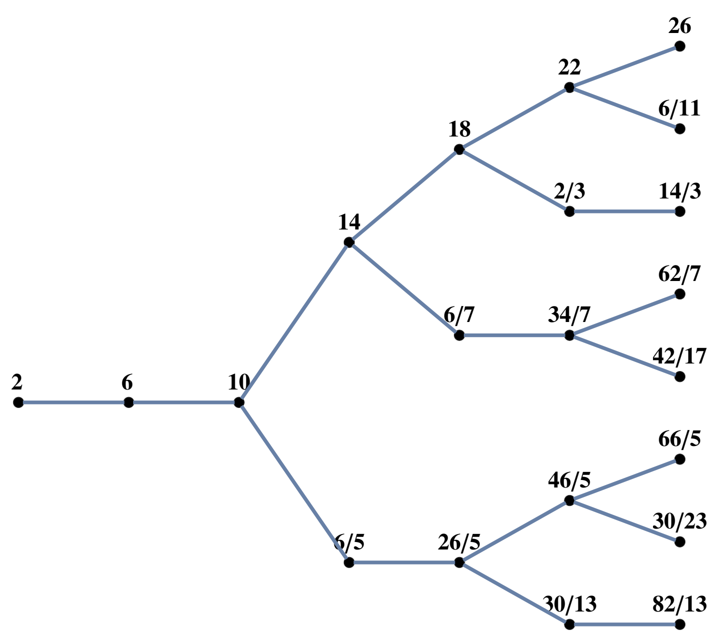

FIGURE 5. $x \to x + 4, x \to 12/x$; from 2; Example 3.6

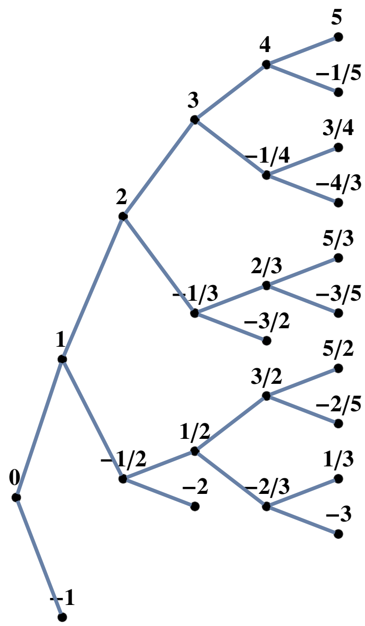

FIGURE 6. $x \to x+1, x \to -1/(x+1)$; Example 4.1

8. Concluding Remarks

In many of the foregoing trees of rational numbers, the numbers in $g(n)$ occur as "already reduced" fractions This observation leads to the question of conditions on a, b, c, d under which the numbers in $g(n)$ given by (1.1) are reduced fractions; i.e., for each $x = u/v$ in $g(n-1)$, the integers $au + bv$ and $cu + dv$ in the fraction $(au + bv)/(cu + dv)$ are relatively prime. It is easy to prove that one such condition (which holds for many of the trees considered in this paper) is that

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \pm 1.$$

Examples in which the fractions are not automatically reduced are given by Theorem 3.3 with $k > 1$. A number of the trees are represented in the Online Encyclopedia of Integer Sequences [3]; for Kepler's tree, see A020651, and for a list of others, see the Comments section at A226080. We conclude with three representative Mathematica (version ≥ 7) programs which may be useful for further research.

Program 1. All the positive rational numbers, generated as in Figure 1, Example 3.4, and A226080

```
z=10;g[1]={1};g[2]={2};g[3]={3,1/2};
d[s_,t_]:=Part[s,Sort[Flatten[Map[Position[s,#]&,Complement[s,t]]]]];
n=3;While[n<=z,n++;g[n]=d[Riffle[g[n-1]+1,1/g[n-1]],g[n-2]]];
Table[g[n],{n,z}]
```

Program 2. All the rational numbers, generated as in Example 4.1, with a ListPlot of the 20th generation

```
g[1]= {0};f1[x_]:=x+1;f2[x_]:=-1/(x+1);h[1]=g[1];
b[n_]:=b[n]=Union[f1[g[n-1]],f2[g[n-1]]];
h[n_]:=h[n]=Union[h[n-1],g[n-1]];
g[n_]:=g[n]=Complement[b[n],Intersection[b[n],h[n]]]
Table[g[n], {n,12}]
ListPlot[g[20]]
```

Program 3. All the Gaussian rationals, generated as in Theorem 6.4, with positions of real integers

```
Off[Power::infy];x= {0};
Do[x=DeleteDuplicates[
    Flatten[Transpose[{x,x+1,1/x,I*x}/.ComplexInfinity-> 0]]
                    ], {6}];x
On[Power::infy];
t1=Flatten[Position[x, _?(IntegerQ[#] && NonNegative[#]&)]] (*A233694*)
t2=Flatten[Position[x, _?(IntegerQ[#] && Negative[#]&)]] (*A233695*)
Union[t1,t2] (*A233696*)
```

References

[1] Johannes Kepler, [excerpt from Book III of *Harmonices Mundi*, 1619], translated with an Introduction and Notes by E. J. Aiton, A. M. Duncan, and J. V. Field, *Memoirs of the American Philosophical Society*, 209 (1997), p. 163 (in Book III, Chapter II, "On the Harmonic Division of the String").

[2] William Judson LeVeque, *Topics in Number Theory,* v. 1, Addison Wesley, 1956, 130-131.

[3] Online Encyclopedia of Integer Sequences, **https://oeis.org/**

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