Proceedings of the Sixteenth International Conference on Fibonacci Numbers and Their Applications

Rochester Institute of Technology,
Rochester, New York,
2014 July 20–27

edited by

Peter G. Anderson
Rochester Institute of Technology
Rochester, New York

Christian Ballot
Université of Caen-Basse Normandie
Caen, France

William Webb
Washington State University
Pullman, Washington

The Sixteenth Conference

The 16th International Conference on Fibonacci Numbers and their Applications was held on the large campus of the Rochester Institute of Technology, situated several miles off from downtown. It hosted about 65 participants from at least a dozen countries and all continents, northern Americans being most represented. Besides regular and occasional participants, there were a number of people who attended this conference for the first time. For instance, Márton, 24, from Hungary, took three flights to reach Rochester; it was his first flying experiences, and we believe many appreciated his presence, and he himself enjoyed the whole package of the conference. These conferences are very congenial, being both scientific, social, and cultural events.

This one had the peculiarity of having three exceptional presentations open to the public held on the Wednesday morning in a large auditorium filled with local young people and students, in addition to the conference participants. The Édouard Lucas invited lecturer, Jeffrey Lagarias, gave a broad well-applauded historic talk which ran from antiquity to present; Larry Ericksen, painter and mathematician, also had us travel through time and space, commenting on often famous artwork—okay, maybe the Golden ratio appeared a few too many times—and Arthur Benjamin, mathemagician (and mathematician) who, for some of his magics, managed the feat of both performing and explaining without loosing the audience a second.

Peter Anderson was the grand host and organizer—at least two of the editors wish to express their thanks to him—but many among the 65 participants will also remember fondly Ginny Gross-Abbey and Kim Shearer, for their uncompromising week-long help performed with competence and joy.

We also offer our sincere thank you to the Rochester Institute of Technology, the B. Thomas Golisano College of Computing and Information Sciences, and its Dean Andrew Sears for generously hosting our meeting and providing us their superb facilities.

A wine and cheese reception was held on the Sunday evening, which for many was their arrival date, an optional, memorable cruise and dinner on the Erie Canal was organized on the Tuesday; Wednesday afternoon was off with the possibility of visiting George Eastman's house, now an impressive museum. The conference banquet was held on the Thursday at a most original place, Artisan Works, surrounded by fine art. A day-trip to Niagara Falls took place on the last Saturday.

Forword

The sixteenth International Conference on Fibonacci Numbers and their Applications was held in Rochester, New York, USA, July 20-26, on the campus of the Rochester Institute of Technology.

This book contains 23 items which include a beautiful paper of Marjorie Bicknell-Johnson on the 50+ years of the existence of the Fibonacci Association, a compendium of problems posed, with occasional solutions, or partial solutions, found during the problem sessions and put together by Clark Kimberling, and 21 research articles from among the 49 papers and abstracts presented at the conference. These articles were selected and criticized by expert referees, whose time and care brought added value to this volume. Even though Fibonacci numbers and recurrences are the common bond to them all, the variety of topics and creativity of the papers compiled herein, is a testimony to the liveliness of this area of mathematics.

It is our belief that the investigations reported in these proceedings, the 15th such proceedings emanating from this international conference, will stir up the curiosity of a number of researchers.

Peter G. Anderson
Christian Ballot
William Webb

Contents

THE FIBONACCI ASSOCIATION: HISTORICAL SNAPSHOTS

MARJORIE BICKNELL-JOHNSON

Abstract. The *Fibonacci Quarterly* is now in its 52nd year of publication. V. E. Hoggatt, Jr., was editor for 18 years; G. E. Bergum, also editor for 18 years; and current editor, Curtis Cooper, 1998 to date. Marjorie Bicknell-Johnson was secretary of the Fibonacci Association from its beginning until 2010.

This article gives a short history of the Fibonacci Association and some vignettes to bring that history to life.

1. Historical Snapshots and Rambles

The first *Fibonacci Quarterly*, published in February 1963, had a subscription rate of $4.00 per year, and its Editor, Verner E. Hoggatt, Jr., held that position for L_7 years. Vern's friends told him the *Quarterly* wouldn't last three years. Undaunted, he kept a mental list of "backsliders" who had not renewed their subscriptions and contacted each one personally. He persuaded, cajoled, and implored them so much that, in the end, it was impossible to say no.

Vern corresponded and made friends with mathematicians from all over the world. He once hosted the world-famous mathematician Paul Erdös for a month. Erdös arrived with one small suitcase, filled with silk underwear (because of allergies he was said to have). I met him, but he much preferred to talk to Vern. I found him strange; he had no home, no family, and all of his possessions were in his suitcase. I think he found me strange as well, visiting a college professor with my two little "epsilons," the Erdös word for children. Erdös and Vern worked on dozens of problems during that time and wrote one paper [6] together, which I typed. As many of you may know, it was considered a great honor to have published a paper with Erdös, so much so that on Google, one can find the Erdös number of those who wrote with Erdös: 1; those who wrote with someone who wrote with Erdös: 2; and so on. So, Vern has Erdös number 1, while Jerry Bergum and I each have Erdös number 2.

The Managing Editor, Brother Alfred Brousseau, who was equally positive and enthusiastic about anything dealing with Fibonacci numbers, typed the first issue of the *Quarterly* and kept track of subscriptions and the bank account. He played the accordion and loved to lead group singing. In tune with his personality, he wrote the ballad, "Do What Comes Fibonaturally," to the melody of "The Blue-Tail Fly." Additionally, he compiled a bibliography of 700 Fibonacci references ranging from recreational to serious research, quite a feat in pre-computer times, and collected cones from every species of pine found in California to illustrate the Fibonacci patterns found in spirals of their scales. He inspired Vern to grow a large sunflower, his so-called Lucas sunflower, which had 76 clockwise spirals and 47 counterclockwise ones.

As one of Professor Hoggatt's students, I came on board in 1962. He was well liked by students at San Jose State College; they nicknamed him "Professor Fibonacci," and I soon found out why: he took any and every opportunity to lecture on the Fibonacci sequence. We all loved it.

Dr. Hoggatt—I called him that as his student—brought unusual problems to us for homework. A curiosa in *Scripta Mathematica* [7] claimed that, if the nine positive digits are arranged in a square array, the non-negative determinant values range from 0 through 512 but with a couple dozen missing values. So we were each assigned to write and evaluate 20 such determinants—find those missing values, or show that they were indeed impossible. One student found a "missing value" and another programmed a computer to do the job, but computers were very new and he couldn't prove his algorithm. I loved to calculate things—no calculator or computer, thank you—but I wasn't going to do the 5,040 distinct determinants possible. There are 9! ways to arrange the nine digits; removing those equivalent by transpose, row exchange, or column exchange, we have $9!/(2 \cdot 3! \cdot 3!) = 5040$. Instead, I considered all arrangements of the form

$$S = a \cdot b \cdot c + d \cdot e \cdot f + g \cdot h \cdot i$$

for the digits 1 through 9 and made a table of the 280 possible values for S. Of course, subtracting two of these values gives a possible determinant value, provided that the two expressions can co-exist in the same square array. Thus, I had every value that could occur as well as one or more determinants yielding that value. This took me all of spring break but it became the first paper [1] I wrote with the master.

After that, he asked me to call him Vern and began sending mathematical correspondence to me, always signed VEH. He kept me busy with proofreading and rewriting papers from foreign authors who had good ideas but no command of English grammar. I helped to package up and mail reprints to authors, refereed articles submitted to the *Quarterly*, and wrote rejection letters, which always thanked the person for his submission and suggested how to rewrite the paper. Hoping I would follow in his footsteps, Vern named me Co-Editor of the journal for the three years 1973–1975, but as a high school teacher, I had few contacts in Academia. When I married Frank Johnson in 1976, Vern said he felt as though his right hand had been cut off, since I changed direction in my life and spent less time "Fibonaturally."

Since the years 1963–1980 are described in detail [3, 4, 2, 5], I will give one digression and then skip to events following Vern's sudden death.

Vern preferred to work at home at the executive desk in his book-lined study. He wrote several letters everyday in his big scrawling hand and without making copies. He kept everything in his head: addresses, telephone numbers, and ongoing correspondence with other mathematicians. At one point, he was working on fifteen research papers at once while supervising several graduate mathematics projects and master theses in progress. He carried on such a prolific correspondence on Fibonacci matters that he often wrote fifty letters in one week and typically slept only four hours a night.

I, often with my two children in tow, went to see Vern once or twice a week for fifteen years. I wrote 47 (or L_8, as he would have counted them) articles with him as co-author, mostly on properties of Pascal's triangle, convolution arrays, and representations, all related to Fibonacci numbers in some way. I typed all of Vern's papers and a book manuscript: he never learned to type. As an aside, for all this work, I used an Olympia standard manual typewriter with π on one key, very cool for the time; does anyone even remember manual typewriters?

While I lived only two miles away, each week I received two or three letters, all eight to ten pages long, all handwritten, all on mathematics and signed VEH, because he wanted to put his thoughts on paper. He would call me for feedback, often before I had received the letters.

One letter began, "Dear Marjorie, Of course you remember Hilbert's Tenth Problem." Of course I did not—who the heck was Hilbert? I couldn't look it up on Google in those days.

It turned out that the famous German mathematician David Hilbert posed a list of twenty-three unsolved mathematics problems to the Paris Conference of the International Congress of Mathematicians in 1900. In 1970, Yuri Matiyasevich utilized a method involving Fibonacci numbers to show that solutions of Diophantine equations may grow exponentially and used that with earlier work by Julia Robinson, Martin Davis, and Hilary Putnam to prove Hilbert's tenth problem unsolvable. Note the key words: Fibonacci numbers.

In 1978, Frank and I found the statue of Fibonacci in Pisa, Italy, per Vern's request. The statue is larger than life-size; Frank climbed onto a rickety 13-foot scaffolding to get a portrait shot. We had it framed and it hung in Vern's office for many years. In fact, it's still there, as well as his built-in walnut bookcases with his collection of math books, because his widow Herta has left it that way. That same portrait graces the Fibonacci Association's webpage. In July 1980, Frank and I took thirty high school students to Europe for a month. I knew something was amiss when I returned to find only two letters in my mailbox. The next morning, I called Vern and learned that he had died the day before. I was shattered; Vern was my mathematical mentor who liked to reflect his ideas off me. It was the saddest time of my life. I felt as though I had lost my father. For months, I could not utter the word Fibonacci without choking up. I kept dreaming that I received a letter from Vern, with the usual cryptic PS scrawled on the envelope, but the envelope was empty.

When Vern died suddenly in 1980, his files were in disarray, and that created a fragile time for the *Fibonacci Quarterly*. Soon after Vern's death, Gerald E. Bergum, then Co-Editor of the *Quarterly*, came to Santa Clara from South Dakota and stayed with my husband and me long enough to bond with my family.

Jerry and I made several visits to Vern's home and cleaned out his desk and his four-drawer filing cabinet. Jerry shipped several cartons of manuscripts home to South Dakota. He wrote to each author because there was no way to tell if the paper submitted was accepted for publication, returned for revision, rejected, or merely ignored. Had Jerry not stepped in and reorganized everything with a strong hand, the *Quarterly* would not be publishing today. Thus it was that, in the fall of 1980, the *Fibonacci Quarterly* moved to South Dakota State University with Editor Gerald E. Bergum for a term of 18 years. His daughters Jennifer and Patty served as secretaries and typists for the *Quarterly*, and his wife Shirley helped with registration at conferences.

2. Just the Facts, Ma'am: A Brief History of the Fibonacci Association

The Fibonacci Association is a nonprofit 501(c)(3) corporation, incorporated in 1962 by Verner E. Hoggatt, Jr., I. Dale Ruggles, and Brother U. Alfred. Dale Ruggles was Vern's officemate at San Jose State; he set up the incorporation paperwork and served on the Editorial Board for seven years. Brother U. Alfred, who changed his name to Brother Alfred Brousseau, served on the Editorial Board, edited and wrote many articles to interest the beginner, and managed subscriptions and association money until his retirement in 1975.

Verner E. Hoggatt, Jr. was Editor of The *Fibonacci Quarterly* 1963–1980; Gerald E. Bergum, 1980-1998; and Curtis Cooper, 1998 to date. Brother Alfred Brousseau was treasurer 1963–1975; Leonard Klosinski, 1976–1979; Marjorie Bicknell-Johnson, 1979–1998; and Peter G. Anderson, 1998 to date. Marjorie Bicknell-Johnson was secretary 1963–2010; current secretary, Art Benjamin, 2010 to date.

The first official meeting of the Board of Directors of the Fibonacci Association was held at San Jose State College on January 20, 1968, with Brother Alfred Brousseau presiding. Other board members present: Verner E. Hoggatt, Jr.; G. L. Alexanderson, Mathematics Department

Chairman, Santa Clara University; George Ledin, Jr., now Professor of Computer Science at Sonoma State University; and Marjorie Bicknell, Adrian C. Wilcox High School.

Vern and Brother Alfred organized informal half-day Fibonacci meetings once or twice a year from 1962 as well as presenting lectures at all local and state conferences for mathematics teachers. A Fibonacci Open House was held at University of San Francisco on January 18, 1969, with the morning devoted to high school students and the afternoon for Fibonacci aficionados; I arranged for mailing invitations to all high schools within 100 miles of San Francisco. Vern corresponded with mathematicians all around the world and dreamed about having an international Fibonacci conference. Unfortunately, the first conference came too late for him to enjoy it.

In 1984, Andreas N. Philippou organized the Fibonacci Association's first international conference in Patras, Greece. The International Conferences on Fibonacci Numbers and Their Applications have been held biennially for 30 years.

- 1984 Patras, Greece
- 1986 San Jose State, San Jose, California
- 1988 Pisa, Italy
- 1990 Wake Forest, North Carolina
- 1992 St. Andrews University, Scotland
- 1994 Washington State, Pullman, Washington
- 1996 Technische Universität, Graz, Austria
- 1998 Rochester Institute of Technology, Rochester, New York
- 2000 Luxembourg City, Luxembourg
- 2002 North Arizona University, Flagstaff, Arizona
- 2004 Technische Universität, Braunschweig, Germany
- 2006 San Francisco State, San Francisco, California
- 2008 Back to University of Patras, Greece
- 2010 Universidad Nacional Autónoma de México, Morelia, México
- 2012 Eszterházy Károly College, Eger, Hungary
- 2014 Back to Rochester Institute of Technology

References

[1] M. Bicknell and V. E. Hoggatt, Jr., *An Investigation of Nine-Digit Determinants,* Mathematics Magazine, May-June 1963, 147–152.

[2] M. Bicknell-Johnson, *In Memoriam: Verner E. Hoggatt, Jr.,* Fibonacci Quarterly, 18.4 (1980), 289.

[3] M. Bicknell-Johnson, *A Short History of The Fibonacci Quarterly*, Fibonacci Quarterly, 25:1 (1987), 2–5.

[4] M. Bicknell-Johnson, *In Memoriam: Brother Alfred Brousseau,* Fibonacci Quarterly, 26.3 (1988), 194.

[5] M. Bicknell-Johnson, *The Fibonacci Quarterly: Fifty Years,* Fibonacci Quarterly, 50:4 (2012), 290–293.

[6] V. E. Hoggatt, Jr., K. Alladi, and P. Erdös, *On Additive Partitions of Integers,* Discrete Math., 22:3 (1978), 201–211.

[7] F. S. Stancliff, *Nine Digit Determinants,* Scripta Mathematica, 19 (1953), 278.

E-mail address: marjohnson@mac.com

Santa Clara, CA

PROBLEM PROPOSALS

COMPILED BY CLARK KIMBERLING

These fourteen problems were posed by participants of the Sixteenth International Conference on Fibonacci Numbers and Their Applications, Rochester Institute of Technology, Rochester, New York, July 24, 2014. A few solutions and partial solutions, received during August-December, are included.

Problem 1, posed by Marjorie Johnson

Prove or disprove that the only Pythagorean triples containing exactly two Fibonacci numbers are $3, 4, 5$ and $5, 12, 13$.

Problem 2, posed by Heiko Harborth and Jens-P. Bode

Two players A and B choose alternatingly an integer. Does there exist a strategy for A to choose integers n, $n+2$, $n+3$, and $n+5$ for some n, or, equivalently, does there exist a strategy for B to prevent A from this objective?

Problem 3, posed by Clark Kimberling

Observe that

$$\begin{aligned}&1/6 + 1/7 + 1/8\\ &< 1/9 + \cdots + 1/13\\ &< 1/14 + \cdots + 1/21\\ &< 1/22 + \cdots + 1/34\end{aligned}$$

Let $H(n) = 1/1 + 1/2 + \cdots + 1/n$, so that the observation can be written using Fibonacci numbers as

$$\begin{aligned}&H(8) - H(5)\\ &< H(13) - H(8)\\ &< H(21) - H(13)\\ &< H(34) - H(21)\end{aligned}$$

More generally, if $x \leq y$, let

$$\begin{aligned}a(1) &= \text{least } k \text{ such that } H(y) - H(x) < H(k) - H(y);\\ a(2) &= \text{least } k \text{ such that } H(a(1)) - H(y) < H(k) - H(a(1));\\ a(n) &= \text{least } k \text{ such that } H(a(n-1)) - H(a(n-2))\\ &\quad < H(k) - H(a(n-1)),\end{aligned}$$

for $n \geq 3$. Prove that if $(x, y) = (5, 8)$, then $a(n) = F(n+6)$, and determine all (x, y) for which $(a(n))$ is linearly recurrent.

Problem 4, posed by Peter Anderson

Let $u_{n+1} = u_n + u_{n-k}$, for $n \geq k$, where $u_i = 0$ for $i = 0, 1, \ldots, k-1$ and $u_k = 1$. Let α be the largest real root of the companion polynomial. For $k = 3$, show how to obtain the Bergman representation of every positive integer. For $k > 4$, show that there is no finite Bergman representation of 2.

Problem 5.1, posed by Dale Gerdemann (problem 5, version 1, as proposed in Rochester)

In "Bergman-Fibonacci" representation,

$$\begin{aligned} 1 &= 1.0 = 1 + 0 \\ 2 &= 1.01 = 1 + 1 \\ 3 &= 10.01 = 2 + 1 \\ 4 &= 101.01 = 3 + 1 \\ 5 &= 1000.1001 = 3 + 2. \end{aligned}$$

What is the ratio of the values of the positive digits to the value of the negative digits? Does it approach a limit?

Empirical solution by Margaret P. Kimberling, Lynda J. Martin, and Peter J. C. Moses. "Bergman-Fibonacci" representations use base $\varphi = (1 + \sqrt{5})/2$ with φ^n replaced by F_{n+1}, so that the five examples are interpreted as

$$\begin{aligned} 1 &= F_1 + F_0 = 1 + 0 \\ 2 &= F_2 + F_{-1} = 1 + 1 \\ 3 &= F_3 + F_{-1} = 2 + 1 \\ 4 &= F_3 + F_1 + F_{-1} = 3 + 1 \\ 5 &= F_4 + (F_0 + F_{-3}) = 3 + 2. \end{aligned}$$

Thus each n has a representation of the form $x.y = u + v$, where u and v are the positive part and negative part, respectively. We claim that the ratios are given by

$$u/v = u(n)/v(n) = (1 + k)/(n - k - 1),$$

where $k = \lfloor (n-1)/(3-\varphi) \rfloor$, and that $\lim_{n\to\infty} u(n)/v(n) = \varphi + 1$.

The claim is based on the following Mathematica code, which finds the Bergman-Fibonacci representation of n, using the first 1000 base 10 digits of φ:

```
phiBase[n_] := Last[#] - Flatten[Position[First[#], 1]] &
[RealDigits[n, GoldenRatio, 1000]];
```

To see the representation for an example, say $n = 12$, add this line of code:

```
test = 12; SplitBy[phiBase[test] + 1, # > 0 &]
```

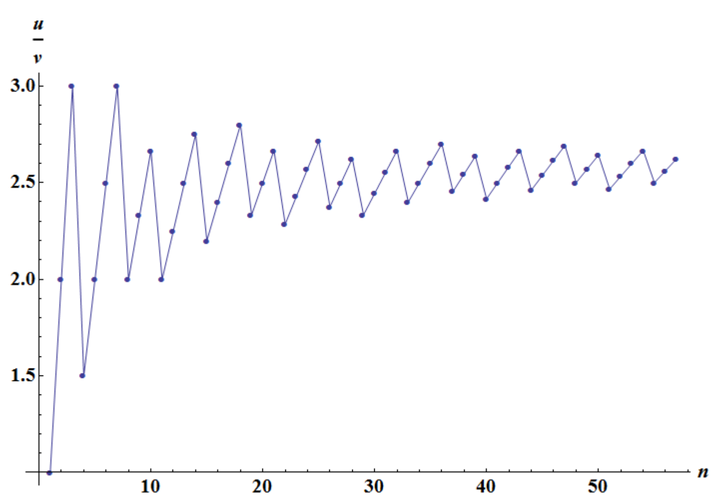
which shows $\{\{6\}, \{0, -2, -1\}\}$, i.e.,

$$F_6 + (F_0 + F_{-2} + F_{-3}) = 8 + [0 + (-1) + 5] = 12.$$

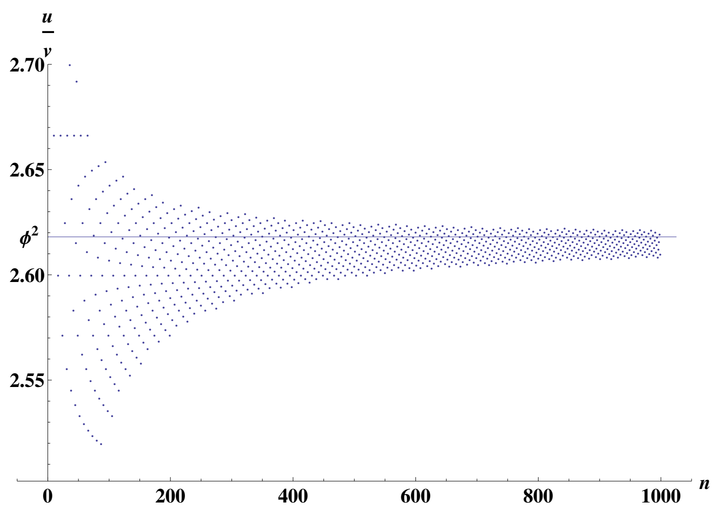
In this example, $u = 8$ and $v = 0 + (-1) + 5 = 4$. The code can be extended to obtain

2	10.01	1/1
3	100.01	2/1
4	101.01	3/1
5	1000.1001	3/2
6	1010.0001	4/2
7	10000.0001	5/2
8	10001.0001	6/2
9	10010.0101	6/3
10	10100.0101	7/3
11	10101.0101	8/3
12	100000.101001	8/4
13	100010.001001	9/4
14	100100.001001	10/4
15	100101.001001	11/4
16	101000.100001	11/5

The denominators in column 3 form the sequence $(1, 1, 1, 2, 2, 2, 2, 3, 3, 3, 4, \ldots)$, of which the difference sequence is $(0, 0, 1, 0, 0, 0, 1, 0, 0, 1, 0, \ldots)$, which appears to be the sequence indexed in OEIS [2] as A221150, authored by Neil Sloane in 2013. Information given at A221150 enables use to find $v(n) = n + \lfloor (n-1)/(\varphi - 3) \rfloor$. The difference sequence for the numerators appears to be the binary complement of A221150, $(1, 1, 0, 1, 1, 1, 0, 1, 1, 0, 1, 1, 1, 0, 1, 1, 1, 0, \ldots)$, leading to $u(n) = 1 + \lfloor (n-1)/(3 - 3\varphi) \rfloor$. The fractions u/v for $2 \le n \le 60$ and $2 \le n \le 1000$ are depicted here:



u/v for $2 \le n \le 60$



u/v for $2 \leq n \leq 1000$

As a somewhat randomly selected example take $n = 46792386$. Then **SplitBy[phiBase[test] + 1, # > 0 &] gives**

$$\{\{37, 35, 28, 26, 23, 19, 17, 13, 11, 9, 6, 4\},$$
$$\{0, -5, -7, -9, -11, -15, -17, -22, -27, -33, -35\}\},$$

which represents

$$\begin{aligned} u &= F_{37} + F_{35} + F_{28} + F_{26} + F_{23} + F_{19} + F_{17} + F_{13} + F_{11} \\ &\quad + F_9 + F_6 + F_4 \\ &= 33859288; \\ v &= F_0 + F_{-5} + F_{-7} + F_{-9} + F_{-11} + F_{-15} + F_{-17} + F_{-22} + F_{-27} \\ &\quad + F_{-33} + F_{-35} \\ &= 12933098. \end{aligned}$$

It can now be checked that u/v in this case agrees with the asserted formula. Finally, it is easy to check that if $u(n)$ and $v(n)$ are as asserted, then $\lim_{n\to\infty} u(n)/v(n) = \varphi + 1$.

References.

[1] Bergman, G. *A number system with an irrational base,* Mathematics Magazine 31 (1957-58) 98-110.

[2] Online Encyclopedia of Integer Sequences, **https://oeis.org/**

Partial solution by Dale Gerdemann. This is a partial proof of the following statement: In Bergman-Fibonacci representation (golden ratio base with each φ^n replaced by f_n, where $f_0 = 1$, $f_1 = 1$, $f_n = f_{n-1} + f_{n-2}$, $f_n = f_{n+2} - f_{n-1}$), the ratio of the positively indexed Fibonacci numbers to the negatively indexed Fibonacci numbers converges to $\varphi + 1$. I limit myself here to proving the weaker statement that if this sequence converges, then it converges to $\varphi + 1$. My strategy is to find a convergent subsequence consisting of the simplest Bergman-Fibonacci representations and then to employ this basic fact about limits: *Every subsequence of a convergent sequence converges, and its limit is the limit of the original sequence.*

The simplest Bergman-Fibonacci representations are for the odd-indexed Lucas numbers, and the second simplest are for the even indexed Lucas numbers, where the Lucas numbers are indexed here starting with $L_0 = -1$ and $L_1 = 2$:

$$\begin{aligned} L_{2n-1} &= f_{2n-2} + f_{-2n+2} \\ L_{2n} &= f_{2n-2} + f_{2n-4} + \cdots + f_0 + \cdots + f_{-2n+4} + f_{-2n+2}. \end{aligned}$$

For an inductive proof, note that these two statements are true for the following base case: $L_4 = f_2 + f_0 + f_{-2}$. For the inductive steps, note that

$$\begin{aligned} L_{2n+1} &= L_{2n-1} + L_{2n} \\ &= 2f_{2n-2} + f_{2n-4} + \cdots + f_0 + \cdots + f_{-2n+4} + 2f_{-2n+2} \\ &= f_{2n} + f_{-2n} \\ L_{2n+2} &= L_{2n+1} + L_{2n} \\ &= f_{2n} + f_{2n-2} + \cdots + f_0 + \cdots + f_{-2n+2} + f_{-2n}. \end{aligned}$$

Now consider the positive-indexed to negative-indexed ratio for the subsequence of odd-indexed Lucas numbers:

$$\cdots \ \frac{f_{2n-2}}{f_{-2n+2}},\ \frac{f_{2n}}{f_{-2n}},\ \frac{f_{2n+2}}{f_{-2n-2}},\ \ldots\ .$$

Since the denominators are even-indexed, the negative indexing can be eliminated:

$$\cdots \ \frac{f_{2n-2}}{f_{2n-4}},\ \frac{f_{2n}}{f_{2n-2}},\ \frac{f_{2n+2}}{f_{2n}},\ \ldots\ .$$

Now, as is well known, the ratio of adjacent Fibonacci numbers converges to φ, so that the ratio of these two-apart Fibonacci numbers converges to $\varphi^2 = \varphi + 1$.

Problem 5.2, posed by Dale Gerdemann (problem 5, version 2)

Golden ratio base differs from more familiar integer bases in that it uses both positive and negative powers of the base to represent an integer. For example, the number $m = 100$ is represented as the sum

$$\varphi^9 + \varphi^6 + \varphi^3 + \varphi + \varphi^{-4} + \varphi^{-7} + \varphi^{-10},$$

where $\varphi = (1+\sqrt{5})/2$. Here the contribution of the positive powers is much greater than the contribution of the negative powers. Note what happens, however, when the powers of φ are replaced by corresponding Fibonacci numbers (using the combinatorial definition: $f_0 = 1$, $f_1 = 1$, $f_n = f_{n-1} + f_{n-2}$, $f_n = f_{n+2} - f_{n-1}$):

$$\begin{aligned} &f_9 + f_6 + f_3 + f_1 + f_{-4} + f_{-7} + f_{-10} \\ &= 55 + 13 + 3 + 1 + 2 - 8 + 34 \\ &= 100 \end{aligned}$$

This replacement does not change the sum, which remains 100. However, the negatively indexed Fibonacci numbers play a larger role than the corresponding negative powers in golden ratio base. Here the positively indexed Fibonacci numbers sum to 72, the negative ones sum to 28, and the ratio $72/28 = 2.571...$. Prove that as m increases, this ratio approaches $\varphi + 1$.

Problem 6, posed by Curtis Cooper

The following two statements are true: If $g^5 = 2$, then

$$\frac{\sqrt[3]{5g^2+1}+\sqrt[3]{35g^2+g-43}}{\sqrt[3]{5g^2+1}-\sqrt[3]{35g^2+g-43}} = \frac{2+g-g^2}{-g+g^2},$$

and if $g^7 = 2$, then

$$\frac{\sqrt[5]{15g^3+11g^2+15g+12}+\sqrt[5]{-270g^4-259g^3+346g^2+315g+14}}{\sqrt[5]{15g^3+11g^2+15g+12}-\sqrt[5]{-270g^4-259g^3+346g^2+315g+14}}$$
$$= \frac{2+g-g^2}{-g+g^2}.$$

Find similar true statements for $g^k = 2$ where $k \geq 9$ is an odd integer.

Solutions by Sam Northshield for $k = 7, 9, 11,$ **and** 13. For $k = 7$, we present a solution distinct from the one stated just above (g^4 does not appear in our new solution).

$$\frac{\sqrt[5]{A}+\sqrt[5]{B}}{\sqrt[5]{A}-\sqrt[5]{B}} = \frac{2+g-g^2}{-g+g^2} \text{ if } g^7 = 2,$$

$$\begin{aligned} A &= -15g^3 - 6239g^2 + 255g - 6438, \\ B &= 112561g^3 + 20246g^2 - 160155g - 6836. \end{aligned}$$

$$\frac{\sqrt[7]{A}+\sqrt[7]{B}}{\sqrt[7]{A}-\sqrt[7]{B}} = \frac{2+g-g^2}{-g+g^2} \text{ if } g^9 = 2,$$

$$\begin{aligned} A &= 8980553g^4 + 7941290g^3 + 15149890g^2 + 6386905g + 11823140, \\ B &= -45991056g^4 - 420491442g^3 - 440508591g^2 + 579500187g + 511466434. \end{aligned}$$

$$\frac{\sqrt[9]{A}+\sqrt[9]{B}}{\sqrt[9]{A}-\sqrt[9]{B}} = \frac{2+g-g^2}{-g+g^2} \text{ if } g^{11} = 2,$$

$$\begin{aligned} A &= a_5g^5 + a_4g^4 + a_3g^3 + a_2g^2 + a_1g + a_0, \\ B &= b_5g^5 + b_4g^4 + b_3g^3 + b_2g^2 + b_1g + b_0, \\ b_5 &= -2832370277, b_4 = 2254685169, b_3 = 4298350067, \\ b_2 &= -4610451384, b_1 = -2556248098, b_0 = 3738894258, \\ a_5 &= 1050574, a_4 = -915414, a_3 = 9829317, \\ a_2 &= -12489450, a_1 = 8175912, a_0 = -8267688. \end{aligned}$$

$$\frac{\sqrt[11]{A}+\sqrt[11]{B}}{\sqrt[11]{A}-\sqrt[11]{B}} = \frac{2+g-g^2}{-g+g^2} \text{ if } g^{13} = 2,$$

$$
\begin{aligned}
&A = a_6g^6 + a_5g^5 + a_4g^4 + a_3g^3 + a_2g^2 + a_1g + a_0,\\
&B = b_6g^6 + b_5g^5 + b_4g^4 + b_3g^3 + b_2g^2 + b_1g + b_0,\\
&b_6 = 18362391990345, b_5 = 10955091993365, b_4 = -54313592877440,\\
&b_3 = -15431135576532, b_2 = 68772473586419, b_1 = 5062921298005,\\
&b_0 = -36107722357990, a_6 = -22949055914, a_5 = 10769387302,\\
&a_4 = -30534819159, a_3 = 46896418382, a_2 = -29067883130,\\
&a_1 = 33833389975, a_0 = -6861047820.
\end{aligned}
$$

$$\frac{\sqrt[13]{A} + \sqrt[13]{B}}{\sqrt[13]{A} - \sqrt[13]{B}} = \frac{2 + g - g^2}{-g + g^2} \text{ if } g^{15} = 2,$$

$$
\begin{aligned}
&A = a_7g^7 + a_6g^6 + a_5g^5 + a_4g^4 + a_3g^3 + a_2g^2 + a_1g + a_0,\\
&B = b_7g^7 + b_6g^6 + b_5g^5 + b_4g^4 + b_3g^3 + b_2g^2 + b_1g + b_0,\\
&b_7 = 1242970243369974777790866, b_6 = 43189622456246393414224,\\
&b_5 = -258642712235742814743726, b_4 = -136428165027671534593750,\\
&b_3 = 213681762408969527031250, b_2 = 199976876120553414562602,\\
&b_1 = -70774416610255087951747, b_0 = -129089005248346771092897,\\
&a_7 = 59165301272956037525, a_6 = 40996615699845889982,\\
&a_5 = 152969005301622874489, a_4 = 53665185894144140125,\\
&a_3 = 148453810204496210375, a_2 = 33066388703204638925,\\
&a_1 = 63819073735217372000, a_0 = 2337520804186801675.
\end{aligned}
$$

Method: I used Maple, which handles large integers easily.

1) Find remainder of

$$(1 + g - g^2)^{2n-1}(a_ng^n + ... + a_0)$$

upon division by $g^{2n+1} - 2$ (where g and all the a_i's are indeterminate). The result is a polynomial, in g, of degree $2n$ with each coefficient b_j a linear combination of the a_i's.

2) Solving $b_{2n} = ... = b_{n+1} = 0$ and $b_n = k$ gives, for the right choice of k, relatively prime integers $a_0, ..., a_n$.

3) Letting $A(g) = \sum a_ig^i$, and letting $B(g) = \sum b_ig^i$ be the remainder of $(1+g-g^2)^{2n-1}A(g)$ upon division by $g^{2n+1} - 2$, implies

$$(1 + g - g^2)^{2n-1}A(g) = (g^{2n+1} - 2)P(g) + B(g)$$

for some polynomial $P(g)$. If $g^{2n+1} = 2$, then

$$B(g)/A(g) = (1 + g - g^2)^{2n-1}$$

or, equivalently,

$$\frac{\sqrt[2n-1]{A(g)} + \sqrt[2n-1]{B(g)}}{\sqrt[2n-1]{A(g)} - \sqrt[2n-1]{B(g)}} = \frac{2 + g - g^2}{-g + g^2}.$$

Problem 7, posed by Sam Northshield

Let

$$f(n+1) = \sum_{k=0}^{n} \sigma^{S_F(k)} \overline{\sigma}^{S_F(n-k)},$$

where $\sigma = (1+i\sqrt{3})/2$, and $S_F(k)$ is the number of terms in the Zeckendorf representation of k. The sequence f begins:

$$1, 1, 2, 3, 2, 4, 3, 3, 6, 4, 6, 6, 4, 8, 6, 7, \ldots$$

and is integer-valued. Define $\sigma(n) = \lfloor n\varphi + 1/\varphi \rfloor$, where $\varphi = (1+\sqrt{5})/2$ and $\tau(n) = \lfloor n\varphi^2 + \varphi \rfloor$, so that these sequences are a complementary pair. Prove or disprove the following chain of inequalities:

$$f(\tau(n)) \geq f(\lfloor n\varphi^2 \rfloor) \geq f(\lfloor n\varphi \rfloor) \geq f(\sigma(n)) \geq f(n) \geq 0.$$

Also, what does the sequence f count?

Problem 8, posed by Larry Ericksen

Let $p_i = \sum_{j=0}^{J_i} c_j 10^j$ be the decimal representation of the ith prime, and let $r_i = \sum_{j=0}^{J_i} 10^{J_i - j} c_j$ be the number obtained by reversing the digits. For what primes p_i is $r_i + p_i$ a square and $r_i - p_i$ a cube? Example: for $p_i = 47$ and $r_i = 74$, we have $r_i + p_i = 11^2$ and $r_i - p_i = 3^3$.

Problem 9, posed by Patrick Dynes

It is known that the sequence of Fibonacci numbers modulo q, where $q \in \mathbb{Z}^+$, repeats with period $\pi(q)$, known as the Pisano period. Given integers $0 \leq r < q$ and n, let $S(q,r,n) = \{F_i : i \leq n \text{ and } F_i \equiv r \pmod{q}\}$. How well can we approximate $|S(q,r,n)|$? Is it possible to develop an asymptotic formula for $|S(q,r,n)|$ that becomes more precise as q and n grow arbitrarily large?

Problem 10, posed by Russell Hendel

Let $\{a_{n,i}\}_{n\geq 0},\ 1 \leq i \leq m$, be a collection of m linear homogeneous recursive nondecreasing sequences with constant coefficients. Define the *merged sequence* as the sequence formed by arranging in nondecreasing order the set-theoretic union of these sequences. Define the weight, w, of a sequence $\{G_n\}_{n\geq 0}$ satisfying $\sum_{i=0}^{p} b_i G_{n-i} = 0$ by $w_G = \sum_{i=0}^{p} |b_i|$.

Problem: Under what conditions does the merged sequence have a lesser order or lesser weight than all contributing sequences?

Example 1. For $i \geq 0$, let $H_i = F_{2i}$ and $J_i = F_{2i+1}$. The merged sequence is the Fibonacci sequence, of order 2 and weight 2, whereas H and J each have order 2 and weight 4, since $H_n = 3H_{n-1} - H_{n-2}$ and $J_n = 3J_{n-1} - J_{n-2}$ This example is generalizable since subsequences whose indices form arithmetic progressions inherit recursivity [1].

Example 2. For $i \geq 0$, let $H_{2i} = F_i$ and $H_{2i+1} = 0$, and similarly, let $J_{2i+1} = F_i$ and $J_{2i} = 0$. The merged sequence, G, satisfies $G_{2i} = G_{2i+1} = F_i$. All three sequences, H, J, and G, satisfy the recursion $K_n = K_{n-2} + K_{n-4}$ of order 4 and hence have the same weight.

Reference. [1] Russell Jay Hendel, "Factorizations of sums of $F(aj-b)$", *The Fibonacci Quarterly*, **45** (2007) 128-133.

Problem 11, posed by Michael Wiener

Given a prime $p > 3$ and $1 < \kappa < p-1$, we call a sequence $(a_n)_n$ in $\mathbb{F}_p$ a Φ_κ-sequence if it is periodic with period $p-1$ and satisfies the linear recurrence $a_n + a_{n+1} = a_{n+\kappa}$ with $a_0 = 1$. Such a sequence is said to be a *complete Φ_κ-sequence* if in addition

$$\{a_0, a_1, \ldots, a_{p-2}\} = \{1, \ldots, p-1\}.$$

For instance, every primitive root b mod p generates a complete Φ_κ-sequence $a_n = b^n$ for some (unique) κ. In 1992 Brison [1] proved that for prime.$p > 3$, every complete Fibonacci sequence ($\kappa = 2$) in $\mathbb{F}_p$ is generated by a Fibonacci primitive root (i.e. a root of $x^2 - x - 1$ that is also a primitive root in $\mathbb{F}_p$). In 2007, Gil, Weiner and Zara [2] studied the Padovan case ($\kappa = 3$) and related cases. In particular, they proved that when $x^3 - x - 1$ has fewer than three distinct roots in $\mathbb{F}_p$, then every complete Padovan sequence is generated by a Padovan primitive root. However, in the case of three distinct roots, they proved this result only for certain primes and conjectured that the statement holds for every p.

1. Given a prime $p > 3$, prove that a Φ_3-sequence is complete if and only if $a_n = b^n$, where b is a primitive root in $\mathbb{F}_3$ that satisfies $b^3 = b + 1$.

2. Prove, more generally, that if prime $p > 3$ and any $1 < \kappa < p-1$, then a Φ_κ-sequence is complete if and only if $a_n = b^n$, where b is a primitive root in $\mathbb{F}_p$ that satisfies $b^\kappa = b + 1$.

References.

[1] Brison, Owen, "Complete Fibonacci sequences in finite fields", *The Fibonacci Quarterly* 30 (1992), no. 4, 295-304.

[2] J. Gil, M. Weiner, and C. Zara, "Complete Padovan sequences in finite fields", *The Fibonacci Quarterly* 45 (2007), no. 1, 64–75.

Problem 12, posed by Clark Kimberling

Let S be the set generated by these rules: $1 \in S$, and if $x \in S$, then $2x \in S$ and $1 - x \in S$; so that S grows in generations:

$$g(1) = \{1\},\ g(2) = \{0, 2\},\ g(3) = \{-1, 4\},\ g(4) = \{-3, -2, 8\},\ \ldots$$

Prove or disprove that each generation contains at least one Fibonacci number or its negative.

Problem 13, posed by Marjorie Johnson

Prove that the Fibonacci representations of squares of even subscripted Fibonacci numbers end with 0001; and that the odd subscripted end with 000101. (Hint, consider sums of Fibonacci numbers having subscripts of the form $4j$ or $4j + 2$.) More difficult and more interesting: find all integers M such that M^2 ends in 0.

Problem 14, posed by Ron Knott, solved by Sam Northshield

As an infinite Mancala game, suppose a line of pots contains pebbles, 1 in the first, 2 in the second, and n in the nth, without end. The pebbles are taken from the leftmost non-empty pot and added, one per pot, to the pots to the right. Prove that the number of pebbles in pot n as it is emptied is $\lfloor n\varphi \rfloor$, where φ is the golden ratio, $(1 + \sqrt{5})/2$. (This is a variation on a comment by Roland Schroeder on the lower Wythoff sequence; see A000201 in the Online Encyclopedia of Integer Sequences.)

Solution by Sam Northshield. Starting with the positive integers, repeat the following procedure:

* Remove the first entry to create a new row.

* If that number was n, then add 1 to each of the first n entries in the new row, obtaining

$$\begin{array}{cccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & \dots \\ & 3 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & \dots \\ & & 4 & 5 & 6 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & \dots \\ & & & 6 & 7 & 7 & 8 & 8 & 9 & 10 & 11 & 12 & \dots \end{array}$$

Lemma.

$$2n - \lfloor n\varphi \rfloor = \lfloor n/\varphi^2 \rfloor = \min\{j : \lfloor j\varphi^2 \rfloor \geq n\}.$$

Proof. The first equality is obvious from $2 - \varphi = 1/\varphi^2$. To prove the second, note that

$$\lfloor n/\varphi^2 \rfloor < n/\varphi^2 \Rightarrow \lfloor n/\varphi^2 \rfloor \varphi^2 < n \Rightarrow \lfloor \lfloor n/\varphi^2 \rfloor \varphi^2 \rfloor < n$$

and

$$n/\varphi^2 < \lceil n/\varphi^2 \rceil \Rightarrow \lceil n/\varphi^2 \rceil \varphi^2 > n \Rightarrow \lfloor \lceil n/\varphi^2 \rceil \varphi^2 \rfloor \geq n.$$

Theorem. *Let d_n denote the first term in the nth row of the array above. Then $d_n = \lfloor n\varphi \rfloor$.*

Proof. We see that the d_{j-1} ones added to the jth row contribute 1 to the value of d_n if $j + d_{j-1} - 1 \geq n$. That is,

$$d_n = n + |\{j \leq n : j + d_{j-1} - 1 \geq n\}|$$

or equivalently,

$$d_n = n + |\{j < n : j + d_{j-1} \geq n\}|$$

Since d_n is strictly increasing, we arrive at the recursive formula

$$d_n = 2n - \min\{j : d_j + j \geq n\},$$

of which the solution is unique (given that $d_1 = 1$) and so it is enough to show that $\lfloor n\varphi \rfloor$ satisfies it; i.e., that

$$\lfloor n\varphi \rfloor = 2n - \min\{j : \lfloor j\varphi \rfloor + j \geq n\}.$$

Since $\lfloor j\varphi \rfloor + j = \lfloor j\varphi^2 \rfloor$, the lemma applies and the proof of the theorem is finished.

E-mail address: ck6@evansville.edu

University of Evansville, Evansville, IN

EXTENDED FIBONACCI ZECKENDORF THEORY

PETER G. ANDERSON

ABSTRACT. We review the well-known and less well-known properties of the two-way infinite Fibonacci Zeckendorf array, namely every positive integer occurs exactly once in the right half (Zeckendorf's theorem), every integer occurs exactly once in a left portion with a ragged boundary (Bunder's theorem), and every pair of positive integers occurs as adjacent entries exactly once (Morrison's theorem). We refine the third statement and show how to locate the given pairs in the array.

1. The Extended Zeckendorf Array

Table 1 shows the upper left-hand corner of the infinite *Fibonacci Zeckendorf array* [4, 5, 6]. The top row consists of the Fibonacci numbers starting $1, 2, 3, \ldots$. Each subsequent row begins with the smallest positive integer that has not yet appeared (the numbers in each row are strictly increasing, so this poses no problem). If a number has Zeckendorf representation [7] $\sum c_i F_i$, then the number to its right in the table is $\sum c_i F_{i+1}$. Each row in this array clearly follows the Fibonacci recurrence rule: $a_{i,j} = a_{i,j-1} + a_{i,j-2}$. It follows immediately that the infinite table contains every positive integer once and only once.

This array is closely connected to the well-known Theorem of Zeckendorf [7].

Theorem 1.1. *Every positive integer n is uniquely a finite sum $\sum_{k \geq 2} c_k F_k$ with $c_k \in \{0, 1\}$ and $c_k + c_{k+1} \leq 1$, for all k.*

Each row of Table 1 can be extended arbitrarily far to the left via *precurrence*: $a_{i,j} = a_{i,j+2} - a_{i,j+1}$. Table 2 shows a fragment of Table 1 precursed several columns. The unshaded right two columns in Table 2 are the initial two columns of Table 1. The unshaded left portion of Table 2 corresponds to those numbers expressed in the table as sums consisting only of Fibonacci numbers with negative subscripts. Bunder [3] showed that every non-zero integer has a unique representation as a sum of Fibonacci numbers with negative subscripts, no two consecutive. He also provided an algorithm to produce such sums.

The present paper deals with *Extended Fibonacci Zeckendorf (EZ)* representations in which we express integers as sums of non-consecutive Fibonacci numbers without restriction on the signs of the subscripts. It is easy to see that without some rules there are an infinite number of ways to express any integer (if k is even $F_k = -F_{-k}$, so there are an infinite number of representations of zero).

The **Main Result** (Theorem 2.11) is that given any pair of positive integers, a and b, there is an Extended Zeckendorf representation $a = \sum c_k F_k$ such that $b = \sum c_k F_{k+1}$. Section 3 discusses methods of determining the $\{c_k\}$. (Theorem 2.11 is slightly stronger than stated here.)

TABLE 1. The Fibonacci Zeckendorf array.

1	2	3	5	8	13	21	34	55	89
4	7	11	18	29	47	76	123	199	322
6	10	16	26	42	68	110	178	288	466
9	15	24	39	63	102	165	267	432	699
12	20	32	52	84	136	220	356	576	932
14	23	37	60	97	157	254	411	665	1076
17	28	45	73	118	191	309	500	809	1309
19	31	50	81	131	212	343	555	898	1453
22	36	58	94	152	246	398	644	1042	1686
25	41	66	107	173	280	453	733	1186	1919
27	44	71	115	186	301	487	788	1275	2063
30	49	79	128	207	335	542	877	1419	2296
33	54	87	141	228	369	597	966	1563	2529
35	57	92	149	241	390	631	1021	1652	2673
38	62	100	162	262	424	686	1110	1796	2906
40	65	105	170	275	445	720	1165	1885	3050
43	70	113	183	296	479	775	1254	2029	3283

TABLE 2. The precursed Fibonacci Zeckendorf array. The unshaded portion on the right repeats the first two columns of the array of Table 1. The unshaded portion on the left contains the numbers represented by sums of only negatively subscripted Fibonacci numbers.

89	-55	34	-21	13	-8	5	-3	2	-1	1	0	1	1	2
123	-76	47	-29	18	-11	7	-4	3	-1	2	1	3	4	7
68	-42	26	-16	10	-6	4	-2	2	0	2	2	4	6	10
102	-63	39	-24	15	-9	6	-3	3	0	3	3	6	9	15
136	-84	52	-32	20	-12	8	-4	4	0	4	4	8	12	20
81	-50	31	-19	12	-7	5	-2	3	1	4	5	9	14	23
115	-71	44	-27	17	-10	7	-3	4	1	5	6	11	17	28
60	-37	23	-14	9	-5	4	-1	3	2	5	7	12	19	31
94	-58	36	-22	14	-8	6	-2	4	2	6	8	14	22	36
128	-79	49	-30	19	-11	8	-3	5	2	7	9	16	25	41
73	-45	28	-17	11	-6	5	-1	4	3	7	10	17	27	44
107	-66	41	-25	16	-9	7	-2	5	3	8	11	19	30	49
141	-87	54	-33	21	-12	9	-3	6	3	9	12	21	33	54
86	-53	33	-20	13	-7	6	-1	5	4	9	13	22	35	57
120	-74	46	-28	18	-10	8	-2	6	4	10	14	24	38	62
65	-40	25	-15	10	-5	5	0	5	5	10	15	25	40	65
99	-61	38	-23	15	-8	7	-1	6	5	11	16	27	43	70

Table 3. Bergman's ϕnary representation of some small integers. The $\bullet$ is the analog of a traditional radix point. Columns are labeled with powers of ϕ and also with Fibonacci numbers.

	5	3	2	1	1	$\bullet$	0	1	-1	2
	ϕ^4	ϕ^3	ϕ^2	ϕ	1	$\bullet$	ϕ^{-1}	ϕ^{-2}	ϕ^{-3}	ϕ^{-4}
1:					1	$\bullet$				
2:				1	0	$\bullet$	0	1		
3:			1	0	0	$\bullet$	0	1		
4:			1	0	1	$\bullet$	0	1		
5:		1	0	0	0	$\bullet$	1	0	0	1
6:		1	0	1	0	$\bullet$	0	0	0	1
7:	1	0	0	0	0	$\bullet$	0	0	0	1
8:	1	0	0	0	1	$\bullet$	0	0	0	1
9:	1	0	0	1	0	$\bullet$	0	1	0	1
10:	1	0	1	0	0	$\bullet$	0	1	0	1
11:	1	0	1	0	1	$\bullet$	0	1	0	1

2. Extended Zeckendorf representations

Bergman [2] introduced the representation of non-negative integers using the irrational base $\phi = \frac{1+\sqrt{5}}{2}$ where

$$n = \sum_{-\infty}^{\infty} c_k \phi^k \tag{2.1}$$

is a finite sum (i.e., Laurent polynomial) with $c_k \in \{0, 1\}$ and $c_k + c_{k+1} \leq 1$, for all k, as in the Zeckendorf and Bunder representations above. The relation

$$\phi^{k+1} = \phi^k + \phi^{k-1}, \text{ for all } k \tag{2.2}$$

yields carrying-and-borrowing rules for this notation. This is also known as the ϕnary number system.

The extension of Bergman's results to ϕnary representations of positive numbers $b\phi + a$, a and b integers, will yield our main result.

The process of determining ϕnary representations is based on two observations of Bergman's. (In the following observations, n is a positive integer.)

Observation 2.1. *If there is a finite sum $n = \sum c_k \phi^k$ with $c_k \in \{0, 1\}$, for all k, then there is a finite sum $n = \sum d_k \phi^k$ with $d_k \in \{0, 1\}$, and $d_k + d_{k+1} \leq 1$, for all k.*

Observation 2.2. *If there is a finite sum $n = \sum c_k \phi^k$ with $c_k \in \{0, 1\}$, for all k, then there is a finite sum $n = \sum d_k \phi^k$ with $d_k \in \{0, 1\}$, for all k, and $d_0 = 0$.*

These are proved by straightforward applications of the carrying-and-borrowing principle.

Bergman's theorem, that every non-negative integer has a representation as in Equation (2.1), follows by induction from these two observations, starting with $\phi^0 = 1$. Table 2 shows the ϕnary representation of integers 1–11 written to mimic binary representation. Each column is labeled with the ϕ^k and also with F_{k+1} per Proposition 2.5.

For Proposition 2.4, we need a third observation corresponding to Bergman's two:

Observation 2.3. *If there is a finite sum* $n = \sum c_k\phi^k$, *with* $c_k \in \{0,1\}$, *for all* k, *then there is a finite sum* $n = \sum d_k\phi^k$ *with* $d_k \in \{0,1\}$, *for all* k, *and* $d_0 = 1$.

Proposition 2.4. *Bergman's* ϕ*nary representation of non-negative integers is unique.*

Proof. Bergman's mechanism of adding 1 to the representation of n to get the representation of $n+1$ can be easily reversed using Observation 2.3, along with the trivial observation that for positive $n = \sum c_k\phi^k$ (the Bergman representation), for at least one $k \geq 0$, $c_k = 1$. □

Proposition 2.5. *When* $n = \sum c_k\phi^k$ *is Bergman's* ϕ*nary representation of* n *then*

$$n = \sum c_k F_{k+1} \tag{2.3}$$

Proof. Bergman's mechanism of adding $\phi^0 = 1$ to the representation of n to get the representation of $n+1$ is unchanged when the powers of ϕ are replaced by the Fibonacci numbers. We replace ϕ^0 with F_1 and, generally, ϕ^k with F_{k+1}. (However, it is important to regard numbers such as $\cdots, F_2, F_1, F_0, F_{-1}, \cdots$ as Fibonacci numbers with generic subscripts, $\cdots, F_{k+2}, F_{k+1}, F_k, F_{k-1}, \cdots$.) □

We call Eq. (2.3) the *Bergman-Zeckendorf (BZ) representation* of n.

Proposition 2.6. *If* $n = \sum c_kF_{k+1}$ *is the BZ representation of* n, *then* $0 = \sum c_kF_k$.

Proof. Replace the notion of repeatedly adding $F_1 = 1$ in the proof of Proposition 2.5 by adding $F_0 = 0$. □

In Proposition 2.6, we have Bergman's representation of ϕ^{-1} corresponding to an EZ representation of zero. We exploit this below.

Proposition 2.7. *The EZ representations of zero given in Theorem 2.6 are the* only *EZ representations of zero.*

Proof. Suppose $0 = \sum c_kF_k$ is an EZ, specifically a finite sum. For sufficiently large m, the number $u_m = \sum c_kF_{k+m}$ will be positive, because the Fibonacci numbers $\{F_{k+m}\}$ must all be positive. Consequently, the sequence $\{u_m\}_{m=0}^{\infty}$, which obeys the Fibonacci recurrence, must satisfy $u_1 = u_2 > 0$, and EZ representation we have for u_1 must be its BZ representation. □

The following propositions extend Bergman's representation to numbers of the form $b\phi + a$. In the following, u_0 and u_1 are integers, and $u_{n+1} = u_n + u_{n-1}$ for all n.

Proposition 2.8. *If* $u_1\phi + u_0 > 0$, *there exists* n_0 *such that for any* $n \geq n_0$ *we have* $u_n > 0$.

Proof. Use the matrix form of the Fibonacci recurrence, $(u_{n+1}, u_n) = (u_n, u_{n-1})\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}$. The positive eigenvalue of the matrix, $(1+\sqrt{5})/2$, corresponds to the eigenvector $(\phi, 1)$. Consequently, if the scalar product $u_1\phi + u_0 = (u_1, u_0)\cdot(\phi, 1) > 0$ and n is sufficiently large, then the components of the vector$(u_{n+1}, u_n) = (u_1, u_0)\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}^n$ will be positive. □

Proposition 2.9. *We have* $(u_1\phi + u_0)\phi^n = u_{n+1}\phi + u_n$.

Proof. This follows immediately from $\phi^2 = \phi + 1$. □

Proposition 2.10. *If* $b\phi + a > 0$, a *and* b *integers, there is a* ϕ*nary representation of* $b\phi + a$.

Proof. Let $B\phi + A = (b\phi + a)\phi^n = b\phi^{n+1} + a\phi^n$ and n be large enough so $A > 0, B > 0$. $B\phi + A$ has ϕnary representation $\sum c_k\phi^k$. Consequently, $b\phi + a = \sum c_{k-n}\phi^k$. □

We may now give the refined statement and proof of our main result.

Theorem 2.11. *If $b\phi + a > 0$, a and b integers, there is an Extended Zeckendorf representation $a = \sum c_k F_k$ such that $b = \sum c_k F_{k+1}$.*

Proof. Because ϕnary representations of ϕ^{-1} correspond to EZ representations of zero, and $F_1 = F_{-1} = 1$, the desired coefficients are in the ϕnary representation $b\phi^{-1} + a\phi^{-2} = \sum c_k \phi^k$. This can be achieved by a minor modification of the above. □

3. Illustrations and Finding the Coefficients

Below are terms of sequences defined using the Fibonacci recurrence with initial values (a, b) of $(6, 5)$ and $(5, 6)$. The initial numbers in each list are expressed as sums of Fibonacci numbers using the usual Zeckendorf expansion. Eventually, at $(27, 43)$ in the left example, $(17, 28)$ in the right, the expansion of the second value in the pair is clearly the Fibonacci shift of the first. From that point on, the list reverses (using precursion) back to the starting pair of values. During this reversal, the Fibonacci numbers in the right-hand summations are also precursed, leading to the desired expansion of 5 as the shift of 6 and vice versa.

6	=	1	+	5			5	=	5				
5	=	5	+				6	=	1	+	5		
11	=	3	+	8			11	=	3	+	8		
16	=	3	+	13			17	=	1	+	3	+	13
27	=	1	+	5	+	21	28	=	2	+	5	+	21
43	=	1	+	8	+	34	17	=	1	+	3	+	13
27	=	1	+	5	+	21	11	=	1	+	2	+	8
16	=	0	+	3	+	13	6	=	0	+	1	+	5
11	=	1	+	2	+	8	5	=	1	+	1	+	3
5	=	-1	+	1	+	5							
6	=	2	+	1	+	3							

Notice that the pairs of values $(6, 5)$ and $(5, 6)$ are in the shaded regions of Table 2.

The above may be thought of as an algorithm—admittedly inefficient—for locating the coefficients of Theorem 2.11.

A second—also inefficient—algorithm is to determine the ϕnary representation

$$b\phi^{-1} + a\phi^{-2} = \sum c_i \phi^i \tag{3.1}$$

The coefficients $\{c_i\}$ of Eq. 3.1 are again those of Theorem 2.11.

A third algorithm, in the spirit of the greedy change-making algorithm to find the usual Zeckendorf coefficients, is as follows.

We are given (a, b) such that $a + b\phi > 0$. Iteratively, locate the largest n such that $[(a, b) - (F_n, F_{n+1})] \cdot (1, \phi) \geq 0$ and replace (a, b) with the difference $(a, b) - (F_n, F_{n+1})$. Terminate the algorithm when $(a, b) = (0, 0)$.

Below, we use this algorithm on our example pairs $(6, 5)$ and $(5, 6)$.

			$(F_n$	$F_{n+1})$		$(a$	$b)$	$a+b\phi$
						(6	5)	14.09017
(6	5)	-	(3	5)	=	(3	0)	3.00000
(6	5)	-	(5	8)	=	(1	-3)	-3.85410
(3	0)	-	(1	1)	=	(2	-1)	0.38197
(3	0)	-	(1	2)	=	(2	-2)	-1.23607
(2	-1)	-	(2	-1)	=	(0	0)	0.00000
(2	-1)	-	(-1	1)	=	(3	-2)	-0.23607

			$(F_n$	$F_{n+1})$		$(a$	$b)$	$a+b\phi$
						(5	6)	14.70820
(5	6)	-	(3	5)	=	(2	1)	3.61803
(5	6)	-	(5	8)	=	(0	-2)	-3.23607
(2	1)	-	(1	1)	=	(1	0)	1.00000
(2	1)	-	(1	2)	=	(1	-1)	-0.61803
(1	0)	-	(1	0)	=	(0	0)	0.00000
(1	0)	-	(0	1)	=	(1	-1)	-0.61803

4. Generalizations to Other Recurrences: Success and Failure

Now consider k-th order "generalized Fibonacci sequences" of the form $u_n = \sum_{i=1}^{k} R_i u_{n-i}$, starting with k initial values $0, \ldots, 0, 1$.

Zeckendorf representations and arrays exist, for these sequences, as above. That is, the initial row of the array is the sequence $a_{i,j} = u_j$ suitable shifted so the first two elements are 1 and an integer larger than 1. Subsequent rows begin with the smallest number that has not yet appeared, with the elements of that row being Zeckendorf shifts of the first element. These arrays contain each positive integer exactly once. (Zeckendorf representations based on this recurrence are, as with the Fibonacci case, determined by the greedy change-making algorithm.)

The k-bonacci numbers for which $\{R_i\} = (1, 1, \ldots, 1)$ were addressed in [1], which proved the analogy of Theorem 2.11: a sequence of k positive numbers $(a_1, \ldots, a_k)$ possesses a k-bonacci extended Zeckendorf representation for each a_i such that the representation of a_{i+i} is the shift of that of a_i, for all $1 \leq i < k$.

However, for the case of $\{R_i\} = (1, 0, \ldots, 0, 1)$ the ϕnary representation theory does not apply, starting with $k = 4$: $u_n = u_{n-1} + u_{n-4}$. This sequence begins

$$0, 0, 0, 1, 1, 1, 1, 2, 3, 4, 5, 7, 10, 14, 19, 26, 36, 50, 69, 95, 131, 181, 250, 345, 476, 657, 907, \ldots$$

The Bergman/Zeckendorf coefficients $\{c_i\}$ for this sequence require

- $c_i \in \{0, 1\}$.
- Every non-zero c_i is preceded by and followed by at least three zeros.

There is no finite Bergman representation for 2 or $1 + \phi$. That is, the sequences $\{2u_i\}$ and $\{u_i + u_{i+1}\}$ are not in the Zeckendorf array.

For the $k = 5$, the periodic sequence, $\mathcal{S}$, with period five,

$$0, 1, 1, 0, -1, -1, \ldots$$

satisfies $u_n = u_{n-1} + u_{n-5}$. The Zeckendorf array for this sequence, as usual, contains every positive integer exactly once. The sum of two sequences that satisfies a given recurrence will also satisfy that recurrence, so the sum of $\mathcal{S}$ with any row, $\mathcal{R}$, of the Zeckendorf array will be eventually positive, yet will contain infinitely many values in common with $\mathcal{R}$.

References

[1] P. G. Anderson and C. Cooper, *Every Positive k-bonacci-like Sequences Eventually Agrees with a Row of the k-Zeckendorf Array* The Fibonacci Quarterly, **49.4** (2011) 303–309.

[2] G. Bergman, *A Number System with an Irrational Base* Mathematics Magazine **31.2** (1957) 98–110.

[3] M. W. Bunder, *Zeckendorf representations using negative Fibonacci numbers*, The Fibonacci Quarterly, **30.2** (1992), 111–115.

[4] C. Cooper, *The k-Zeckendorf array*, Presented at the Fourteenth International Conference on Fibonacci Numbers and Their Applications, Morelia, Mexico, July 2011.

[5] C. Kimberling, *The Zeckendorf array equals the Wythoff array*, Fibonacci Quarterly **33.1** (1995) 3–8.

[6] D. R. Morrison, *A Stolarsky array of Wythoff pairs*, A Collection of Manuscripts Related to the Fibonacci Sequence, 18th Anniversary Volume, V. E. Hoggatt, Jr. and Marjorie Bicknell-Johnson, eds., 1980, 134–136.

[7] E. Zeckendorf, *Représentation des nombres naturels par une somme de nombres de Fibonacci ou de nombres de Lucas*, Bull. Soc. Roy. Liege, **41** (1972), 179–182.

MSC2010: 11B39, 33C05

Computer Science Department, Rochester Institute of Technology, Rochester, NY 14531
E-mail address: `pga@cs.rit.edu`

PULSATED FIBONACCI RECURRENCES

KRASSIMIR T. ATANASSOV, DARYL R. DEFORD, AND ANTHONY G. SHANNON

Abstract. In this note we define a new type of pulsated Fibonacci sequence. Properties are developed with a successor operator. Some examples are given.

1. Introduction

The motivation for this work goes back to some research of Hall [9], Neumann [14], and Stein [19] on finite models of identities. In order to answer the question of whether every member of a variety is a quasi–group given that every finite member is, Stein [18] found it necessary to examine the intersection of Fibonacci sequences.

Subba Rao [20, 21], Horadam [10], and Shannon [17] investigated the intersection of Fibonacci and Lucas sequences and their generalizations with asymptotic proofs, while Péter Kiss adopted a different approach and supplied many relevant historical references [11]. Atanassov developed coupled recursive sequence which had some obvious intersections [1, 5]. Not considered here are various sequences, such as diatomic sequences, which by their very definitions intersect with many other sequences [14].

In this paper, following previous research (see [2, 3, 4]), a new type of pulsated Fibonacci sequence is developed: 'pulsated' because, in a sense, these sequences expand and contract with regular movements.

2. Definitions

Let a, b, and c be three fixed real numbers. Let us construct the following two recurrent sequences, $\{\alpha_n\}$ and $\{\beta_n\}$ with initial conditions:

$$\alpha_0 = \beta_0 = a, \tag{2.1}$$

$$\alpha_1 = 2b, \tag{2.2}$$

$$\beta_1 = 2c, \tag{2.3}$$

satisfying the combined recurrence relations:

$$\alpha_{2k} = \beta_{2k} = \alpha_{2k-2} + \frac{\alpha_{2k-1} + \beta_{2k-1}}{2}, \tag{2.4}$$

$$\alpha_{2k+1} = \alpha_{2k} + \beta_{2k-1}, \tag{2.5}$$

$$\beta_{2k+1} = \beta_{2k} + \alpha_{2k-1}, \tag{2.6}$$

for every natural number $k \geq 1$. We refer to this pair of intertwined sequences as the $(a; 2b; 2c)$–Pulsated Fibonacci sequence. The first values of the sequence are given in the following table:

Key words and phrases. Fibonacci Sequence, Systems of Recurrences, Successor Operator.

TABLE 1. Initial values for the $(a; 2b; 2c)$–Pulsated Fibonacci sequence.

n	α_{2k+1}	$\alpha_{2k} = \beta_{2k}$	β_{2k+1}
0	–	a	–
1	$2b$	–	$2c$
2	–	$a+b+c$	–
3	$a+b+3c$	–	$a+3b+c$
4	–	$2a+3b+3c$	–
5	$3a+6b+4c$	–	$3a+4b+6c$
6	–	$5a+8b+8c$	–
7	$8a+12b+14c$	–	$8a+14b+12c$
8	–	$13a+21b+21c$	–

Theorem 2.1. *For every natural number $k \geq 1$, with the elements of the Fibonacci sequence denoted $\{F_n\}$,*

$$\alpha_{2k} = \beta_{2k} = F_{2k-1}a + F_{2k}b + F_{2k}c, \tag{2.7}$$

$$\alpha_{4k-1} = F_{4k-2}a + (F_{4k-1} - 1)b + (F_{4k-1} + 1)c, \tag{2.8}$$

$$\beta_{4k-1} = F_{4k-2}a + (F_{4k-1} + 1)b + (F_{4k-1} - 1)c, \tag{2.9}$$

$$\alpha_{4k+1} = F_{4k}a + (F_{4k+1} + 1)b + (F_{4k+1} - 1)c, \tag{2.10}$$

$$\beta_{4k+1} = F_{4k}a + (F_{4k+1} - 1)b + (F_{4k+1} + 1)c. \tag{2.11}$$

Proof. We proceed by mathematical induction. Obviously, for $k = 1$ the assertion is valid. Let us assume that for some natural number $k \geq 1$, (2.7)–(2.11) hold. For the natural number $k+1$, first, we check that

$$\begin{aligned}
& \alpha_{4k+2} && (2.12)\\
= \; & \beta_{4k+2} && (2.13)\\
= \; & \alpha_{4k} + \tfrac{\alpha_{4k+1}+\beta_{4k+1}}{2} && (2.14)\\
= \; & F_{4k-1}a + F_{4k}b + F_{4k}c + \tfrac{F_{4k}a+(F_{4k+1}+1)b+(F_{4k+1}-1)c+F_{4k}a+(F_{4k+1}-1)b+(F_{4k+1}+1)c}{2} && (2.15)\\
= \; & F_{4k-1}a + F_{4k}b + F_{4k}c + F_{4k}a + F_{4k+1}b + F_{4k+1}c. && (2.16)
\end{aligned}$$

Secondly, we check that

$$\begin{aligned}
& \alpha_{4k+1} && (2.17)\\
= \; & \alpha_{4k+2} + \beta_{4k+1} && (2.18)\\
= \; & F_{4k+1}a + F_{4k+2}b + F_{4k+2}c + F_{4k}a + (F_{4k+1} - 1)b + (F_{4k+1} + 1)c && (2.19)\\
= \; & F_{4k+2}a + (F_{4k+3} - 1)b + (F_{4k+3} + 1)c. && (2.20)
\end{aligned}$$

All of the other equalities are checked analogously. □

For example, when $c = -b$, the Pulsated Fibonacci sequence has the form shown in Table 2, while when $c = b$ we obtain Table 3.

TABLE 2. Initial values for the $(a; 2b; -2b)$–Pulsated Fibonacci sequence.

n	α_{2k+1}	$\alpha_{2k} = \beta_{2k}$	β_{2k+1}
0	–	a	–
1	$2b$	–	$-2b$
2	–	a	–
3	$a - 2b$	–	$a + 2b$
4	–	$2a$	–
5	$3a + 2b$	–	$3a - 2b$
6	–	$5a$	–
7	$8a - 2b$	–	$8a + 2b$
8	–	$13a$	–

TABLE 3. Initial values for the $(a; 2b; 2b)$–Pulsated Fibonacci sequence.

n	α_{2k+1}	$\alpha_{2k} = \beta_{2k}$	β_{2k+1}
0	–	a	–
1	$2b$	–	$2b$
2	–	$a + 2b$	–
3	$a + 4b$	–	$a + 4b$
4	–	$2a + 6b$	–
5	$3a + 10b$	–	$3a + 10b$
6	–	$5a + 16b$	–
7	$8a + 26b$	–	$8a + 26b$
8	–	$13a + 42b$	–

Where the coefficients can be easily derived from the result of Theorem 1 by substitution.

3. DISCUSSION

We note that the recursive definitions of α and β may be rewritten in the following form:

$$\alpha_k = \begin{cases} \alpha_{k-2} + \frac{\alpha_{k-1}+\beta_{k-1}}{2} & k \equiv 0 \pmod 2 \\ \alpha_{k-1} + \beta_{k-2} & k \equiv 1 \pmod 2 \end{cases} \tag{3.1}$$

and

$$\beta_k = \begin{cases} \alpha_{k-2} + \frac{\alpha_{k-1}+\beta_{k-1}}{2} & k \equiv 0 \pmod 2 \\ \beta_{k-1} + \alpha_{k-2} & k \equiv 1 \pmod 2 \end{cases} \tag{3.2}$$

This interpretation permits the statement of this problem in terms of the successor operator method introduced by DeTemple and Webb in [7]. Thus, we may define helper sequences

$$w_n = \alpha_{2n}, \tag{3.3}$$

$$x_n = \alpha_{2n+1}, \tag{3.4}$$

$$y_n = \beta_{2n}, \tag{3.5}$$

$$z_n = \beta_{2n+1}. \tag{3.6}$$

This allows us to rewrite (3.1) and (3.2) as

$$\begin{aligned} w_n = y_n = &\quad w_{n-1} + \tfrac{1}{2}x_{n-1} + \tfrac{1}{2}z_{n-1}, && (3.7)\\ x_n = &\quad w_n + z_{n-1}, && (3.8)\\ z_n = &\quad y_n + x_{n-1}. && (3.9) \end{aligned}$$

Which in terms of the successor operator E gives the following linear system of sequences:

$$\begin{bmatrix} E-1 & -\frac{1}{2} & 0 & -\frac{1}{2} \\ -E & E & 0 & -1 \\ -1 & -\frac{1}{2} & E & -\frac{1}{2} \\ 0 & -1 & -E & E \end{bmatrix} \begin{bmatrix} w_n \\ x_n \\ y_n \\ z_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{3.10}$$

Thus, the determinant of this system gives the characteristic polynomial of a recurrence relation that annihilates all of the sequences. The determinant is equal to $E(E^3-2E^2-2E+1)$ and hence the sequences $\{w_n\}$, $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ all satisfy the third order homogeneous, linear recurrence relation

$$t_n = 2t_{n-1} + 2t_{n-2} - t_{n-3}. \tag{3.11}$$

This recurrence (3.11) has eigenvalues $\{-1, \frac{3\pm\sqrt{5}}{2}\}$, and, with initial values of unity yields the 'coupled' sequence $\{1, 1, 1, 3, 7, 19, 49, 129, 337, \ldots\}$ [6]. This sequence appears in the OEIS as A061646, with a variety of combinatorial interpretations [16]. Additionally, the polynomial factors further as $E(E+1)(E^2-3E+1)$. From this factorization the sequence $\{w_n\}$ and $\{y_n\}$ (the even α and β terms) satisfy the second order relation

$$t_n = 3t_{n-1} - t_{n-2}, \tag{3.12}$$

which is also satisfied by alternate terms of the Fibonacci sequence (A001519 and A001906 [16]).

Finally, putting the sequences back together we would expect to need a sixth order recurrence. Instead, we find that both of the original α_n and β_n sequences satisfy the fourth order recurrence

$$t_n = t_{n-1} + t_{n-3} + t_{n-4}. \tag{3.13}$$

This recurrence (3.13) has roots $\{\pm i, \frac{1\pm\sqrt{5}}{2}\}$ and with unit initial values yields the sequence $\{1, 1, 1, 1, 3, 5, 7, 11, 19, 31, 49, 79, 129, \ldots\}$, contained in the OEIS as A126116 [16], of which the couple sequence above is a subsequence. The connections among all these sequence are not surprising since, as is well known, $i^2 = -1$ and $\left(\frac{1+\sqrt{5}}{2}\right)^2 = \frac{3+\sqrt{5}}{2}$, and so on.

4. Concluding Comments

In summary then, we have that the given recursive sequences satisfy the following recurrences:

Sequence	Recurrence Relation
α_n and β_n	$t_n = t_{n-1} + t_{n-3} + t_{n-4}$
$w_n = \alpha_{2n} = \beta_{2n} = y_n$	$t_n = 3t_{n-1} - t_{n-2}$
$x_n = \alpha_{2n+1}$ and $z_n = \beta_{2n+1}$	$t_n = 2t_{n-1} + 2t_{n-2} - t_{n-3}$

The two sequences discussed in [2, 3] we called 2–Pulsated Fibonacci sequences (from (a;b) and (a;b;c)–types). In [4] they were extended to what were called s–Pulsated Fibonacci sequences, where $s \geq 3$. In future research, it is planned to extend the present

2–Pulsated Fibonacci sequences from $(a; 2b; 2c)$–type, to s–Pulsated Fibonacci sequences from $(a; 2b_1; \ldots, 2b_s)$–type. Other related possibilities for research concern

- conjectures on the number of distinct prime divisors of these sequences [13, 22],
- connections with geometry [6, 8, 12].

Acknowledgments

An earlier draft of this paper was presented at the Sixteenth International Conference on Fibonacci Numbers and their Application at Rochester Institute of Technology, July 20–26, 2014, and gratitude is expressed to a number of participants who subsequently suggested relevant references for this work.

References

[1] K. T. Atanassov, *On a Second New Generalization of the Fibonacci Sequence*, The Fibonacci Quarterly, **24(4)**, (1986), 362–265.
[2] K. T. Atanassov, *Pulsating Fibonacci Sequences*, Notes on Number Theory and Discrete Mathematics, **19(3)** (2013), 12–14.
[3] K. T. Atanassov, *Pulsating Fibonacci Sequences*, Notes on Number Theory and Discrete Mathematics, **19(4)** (2013), 33–36.
[4] K. T. Atanassov, *n–pulsated Fibonacci Sequences. Part 2.*, Notes on Number Theory and Discrete Mathematics, **20(1)**, (2013), 32–35.
[5] K. T. Atanassov, L. Atanassova, D. Sasselov, *Recurrent Formulas of the Generalized Fibonacci and Tribonacci Sequences*, The Fibonacci Quarterly, **23(1)**, (1985), 21–28.
[6] K. T. Atanassov, V. Atanassova, A. Shannon, J. Turner, *New Visual Perspectives on Fibonacci Numbers*, World Scientific, New Jersey (2002).
[7] D. DeTemple, W. Webb, *Combinatorial Reasoning: An Introduction to the Art of Counting*, Wiley, New Jersey (2014).
[8] P. Frankl, R. M. Wilson, *Intersection Theory with Geometric Consequences*, Combinatorica, **1(4)**, (1981), 357–368.
[9] M. Hall, Jr., *The Theory of Groups*, MacMillan, New York, (1959).
[10] A. F. Horadam, *Generalization of Two Theorems of K. Subba Rao*, Bulletin of the Calcutta Mathematical Society, **58(1)**, (1966), 23–29.
[11] P. Kiss, *On Common Terms of Linear Recurrences*, Acta Mathematica Academiae Scientiarum Hungarica, **40(1–2)**, (1982), 119–123.
[12] H. V. Krishna, *A Note on Number Quartets*, Mathematics Student, **40(1)**, (1973), 1.
[13] R. R. Laxton, *On a Problem of M. Ward*, The Fibonacci Quarterly, **12(1)**, (1974), 41–44.
[14] H. Neumann, *Varieties of Groups. (Ergebnisse der Mathematik und ihrer Grenzgebiete, (Bd. 37).*, Springer, Berlin, (1967).
[15] S. Northshield, *Two Analogues of Stern's Diatomic Sequence*, Sixteenth International Conference on Fibonacci Numbers and their Applications, RIT, New York 2014.
[16] OEIS Foundation Inc. (2014), The On-Line Encyclopedia of Integer Sequences, `http://oeis.org.`
[17] A. G. Shannon, *Intersections of Second Order Linear Recursive Sequences*, The Fibonacci Quarterly, **21(1)** (1983), 6–12.
[18] S. K. Stein, *The Intersection of Fibonacci Sequences*, Michigan Mathematical Journal, **9**, (1962), 399–402.
[19] S. K. Stein, *Finite Models of Identities*, Proceedings of the American Mathematical Society, **14**, (1963), 216–222.
[20] K. S. Rao, *Some Properties of the Fibonacci Numbers –I*, Bulletin of the Calcutta Mathematical Society, **46**, (1954), 253–257.
[21] K. S. Rao, *Some Properties of the Fibonacci Numbers –II*, Mathematics Student, **27**, (1959), 19–23.
[22] M. Ward, *The Laws of Apparition and Repetition of Primes in a Cubic Recurrence*, Transactions of the American Mathematical Society, **79** (1955), 72–90.

MSC2010: 11B39

Department of Bioinformatics and Mathematical Modeling, Institute of Biophysics and Biomedical Sciences, Bulgarian Academy of Sciences, Sofia–1113, Bulgaria,
E-mail address: `krat@bas.bg`

Department of Mathematics, Dartmouth College, Hanover, NH 03755,
E-mail address: `ddeford@math.dartmouth.edu`

Faculty of Engineering and Information Technology, University of Technology, Sydney NSW 2007, Australia
E-mail address: `t.shannon@warrane.unsw.edu.au, Anthony.Shannon@uts.edu.au`

COMBINATORIAL PROOFS OF FIBONOMIAL IDENTITIES

ARTHUR T. BENJAMIN AND ELIZABETH REILAND

ABSTRACT. Fibonomial coefficients are defined like binomial coefficients, with integers replaced by their respective Fibonacci numbers. For example, $\binom{10}{3}_F = \frac{F_{10}F_9F_8}{F_3F_2F_1}$. Remarkably, $\binom{n}{k}_F$ is always an integer. In 2010, Bruce Sagan and Carla Savage derived two very nice combinatorial interpretations of Fibonomial coefficients in terms of tilings created by lattice paths. We believe that these interpretations should lead to combinatorial proofs of Fibonomial identities. We provide a list of simple looking identities that are still in need of combinatorial proof.

1. INTRODUCTION

What do you get when you cross Fibonacci numbers with binomial coefficients? Fibonomial coefficients, of course! Fibonomial coefficients are defined like binomial coefficients, with integers replaced by their respective Fibonacci numbers. Specifically, for $n \geq k \geq 1$,

$$\binom{n}{k}_F = \frac{F_n F_{n-1} \cdots F_{n-k+1}}{F_1 F_2 \cdots F_k}$$

For example, $\binom{10}{3}_F = \frac{F_{10}F_9F_8}{F_3F_2F_1} = \frac{55 \cdot 34 \cdot 21}{1 \cdot 1 \cdot 2} = 19{,}635$. Fibonomial coefficients resemble binomial coefficients in many ways. Analogous to the Pascal Triangle boundary conditions $\binom{n}{1} = n$ and $\binom{n}{n} = 1$, we have $\binom{n}{1}_F = F_n$ and $\binom{n}{n}_F = 1$. We also define $\binom{n}{0}_F = 1$.

Since $F_n = F_{k+(n-k)} = F_{k+1}F_{n-k} + F_kF_{n-k-1}$, Pascal's recurrence $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$ has the following analog.

Identity 1.1. *For $n \geq 2$,*

$$\binom{n}{k}_F = F_{k+1}\binom{n-1}{k}_F + F_{n-k-1}\binom{n-1}{k-1}_F.$$

As an immediate corollary, it follows that for all $n \geq k \geq 1$, $\binom{n}{k}_F$ is an integer. Interesting integer quantities usually have combinatorial interpretations. For example, the binomial coefficient $\binom{a+b}{a}$ counts lattice paths from $(0,0)$ to (a,b) (since such a path takes $a+b$ steps, a of which are horizontal steps and the remaining b steps are vertical). As described in [1] and elsewhere, the Fibonacci number F_{n+1} counts the ways to tile a strip of length n with squares (of length 1) and dominos (of length 2). As we'll soon discuss, Fibonomial coefficients count, appropriately enough, tilings of lattice paths!

2. COMBINATORIAL INTERPRETATIONS

In 2010 [9], Bruce Sagan and Carla Savage provided two elegant counting problems that are enumerated by Fibonomial coefficients. The first problem counts restricted linear tilings and the second problem counts unrestricted bracelet tilings as described in the next two theorems.

Theorem 2.1. *For $a, b \geq 1$, $\binom{a+b}{a}_F$ counts the ways to draw a lattice path from $(0,0)$ to (a,b), then tile each row above the lattice path with squares and dominos, then tile each column below the lattice path with squares and dominos, with the restriction that the column tilings are not allowed to start with a square.*

Let's use the above theorem to see what $\binom{6}{3}_F = \frac{F_6F_5F_4}{F_1F_2F_3} = \frac{8\cdot5\cdot3}{1\cdot1\cdot2} = 60$ is counting. There are $\binom{6}{3} = 20$ lattice paths from $(0,0)$ to $(3,3)$ and each lattice path creates an integer partition (m_1, m_2, m_3) where $3 \geq m_1 \geq m_2 \geq m_3 \geq 0$, where m_i is the length of row i. Below the path the columns form a complementary partition (n_1, n_2, n_3) where $0 \leq n_1 \leq n_2 \leq n_3 \leq 3$. For example, the lattice path below has horizontal partition $(3,1,1)$ and vertical partition $(0,2,2)$. The first row can be tiled $F_4 = 3$ ways (namely *sss* or *sd* or *ds* where *s* denotes a square and *d* denotes a domino). The next rows each have one tiling. The columns, of length 0, 2 and 2 can only be tiled in 1 way with the empty tiling, followed by tilings *d* and *d* since the vertical tilings are not allowed to begin with a square. For another example, the lattice path associated with partition $(3,2,2)$ (with complementary vertical partition $(0,0,2)$) can be tiled 12 ways. These lattice paths are shown below.

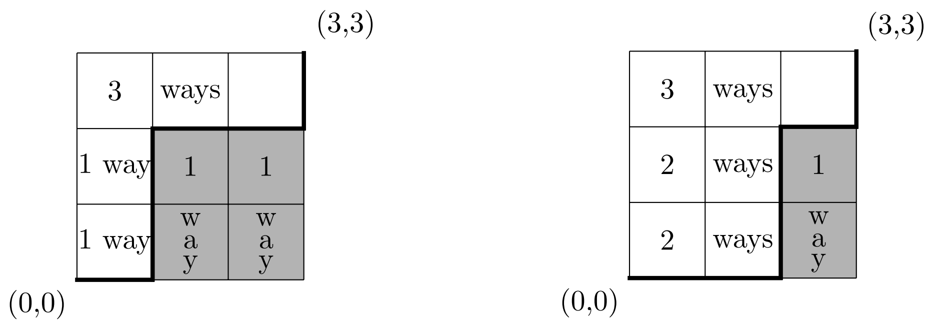


FIGURE 1. The rows of the lattice path $(3,1,1)$ can be tiled 3 ways. The columns below the lattice path, with vertical partition $(0,2,2)$ can be tiled 1 way since those tilings may not start with squares. This lattice path contributes 3 tilings to $\binom{6}{3}_F$. The lattice path $(3,2,2)$ contributes 12 tilings to $\binom{6}{3}_F$.

The lattice path associated with $(3,1,0)$ has no legal tilings since its vertical partition is $(1,2,2)$ and there are no legal tilings of the first column since it has length 1. There are 10 lattice paths that yield at least one valid tiling. Specifically, the paths associated with horizontal partitions $(3,3,3)$, $(3,2,2)$, $(3,1,1)$, $(3,0,0)$, $(2,2,2)$, $(2,1,1)$, $(2,0,0)$, $(1,1,1)$, $(1,0,0)$, $(0,0,0)$ contribute, respectively, $27+12+3+3+8+2+2+1+1+1 = 60$ tilings to $\binom{6}{3}_F$.

More generally, for the Fibonomial coefficient $\binom{a+b}{a}_F$, we sum over the $\binom{a+b}{a}$ lattice paths from $(0,0)$ to (a,b) which corresponds to an integer partition $(m_1, m_2, \ldots, m_b)$ where $a \geq m_1 \geq m_2 \cdots \geq m_b \geq 0$, and has a corresponding vertical partition $(n_1, n_2, \ldots, n_a)$ where $0 \leq n_1 \leq n_2 \cdots \leq n_a \leq b$. Recalling that $F_0 = 0$ and $F_{-1} = 1$, this lattice path contributes

$$F_{m_1+1}F_{m_2+1}\cdots F_{m_b+1}\ F_{n_1-1}F_{n_2-1}\cdots F_{n_a-1}$$

tilings to $\binom{a+b}{a}_F$.

The second combinatorial interpretation of Fibonomial coefficients utilizes circular tilings, or *bracelets*. A bracelet tiling is just like a linear tiling using squares and dominos, but bracelets also allow a domino to cover the first and last cell of the tiling. As shown in [1], for $n \geq 1$, the Lucas number L_n counts bracelet tilings of length n. For example, there are $L_3 = 4$ tilings of length 3, namely sss, sd, ds and $d's$ where d' denotes a domino that covers the first and last cell. Note that $L_2 = 3$ counts ss, d and d' where the d' tiling is a single domino that starts at cell 2 and ends on cell 1. For combinatorial convenience, we say there are $L_0 = 2$ empty tilings. The next combinatorial interpretation of Sagan and Savage has the advantage that there is no restriction on the vertical tilings.

Theorem 2.2. *For $a, b \geq 1$, $2^{a+b}\binom{a+b}{a}_F$ counts the ways to draw a lattice path from $(0,0)$ to (a,b), then assign a bracelet to each row above the lattice path and to each column below the lattice path.*

Specifically, the lattice path from $(0,0)$ to (a,b) that generates the partition $(m_1, m_2, \ldots, m_b)$ above the path and the partition $(n_1, n_2, \ldots, n_a)$ below the path contributes

$$L_{m_1} L_{m_2} \cdots L_{m_b} \; L_{n_1} L_{n_2} \cdots L_{n_a}$$

bracelet tilings to $2^{a+b}\binom{a+b}{a}_F$. Note that each empty bracelet contributes a factor of 2 to this product. For example, the lattice path from $(0,0)$ to $(3,3)$ with partition $(3,1,1)$ above the path and $(0,2,2)$ below the path contributes $L_3 L_1 L_1 L_0 L_2 L_2 = 72$ bracelet tilings enumerated by $2^6\binom{6}{3}_F = 64 \times 60 = 3840$.

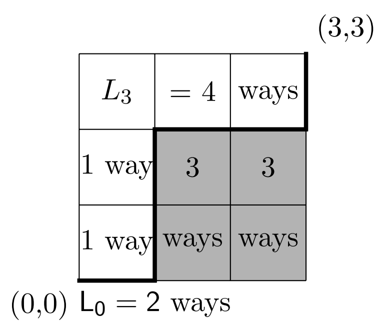


FIGURE 2. The rows above the lattice path can be tiled with bracelets in 4 ways and the columns below the path can be tiled with bracelets in $L_0 L_2 L_2 = 2 \times 3 \times 3 = 18$ ways. This contributes 72 bracelet tilings to $2^6\binom{6}{3}_F = 3840$.

In their paper, Sagan and Savage extend their interpretation to handle Lucas sequences, defined by $U_0 = 0$, $U_1 = 1$ and for $n \geq 2$, $U_n = aU_{n-1} + bU_{n-2}$. Here U_{n+1} enumerates the total weight of all tilings of length n where the weight of a tiling with i squares and j dominos is $a^i b^j$. (Alternatively, if a and b are positive integers, U_{n+1} counts colored tilings of length n where there are a colors for squares and b colors for dominos.) Likewise the number of weighted bracelets of length n is given by $V_n = aV_{n-1} + bV_{n-2}$ with initial conditions $V_0 = 2$ and $V_1 = a$ (so the empty bracelet has a weight of 2). This leads to a combinatorial interpretation of *Lucasnomial coefficients* $\binom{n}{k}_U$, defined like the Fibonomial coefficients. For example,

$$\binom{10}{3}_U = \frac{U_{10} U_9 U_8}{U_1 U_2 U_3}.$$

Both of the previous combinatorial interpretations work exactly as before, using weighted (or colored) tilings of lattice paths.

3. Combinatorial Proofs

Now that we know what they are counting, we should be able to provide combinatorial proofs of Fibonomial coefficient identities. For example, Identity 1.1 can be rewritten as follows.

Identity 3.1. *For $m, n \geq 1$,*

$$\binom{m+n}{m}_F = F_{m+1}\binom{m+n-1}{m}_F + F_{n-1}\binom{m+n-1}{m-1}_F.$$

Combinatorial Proof: The left side counts tilings of lattice paths from $(0,0)$ to (m,n). How many of these tiled lattice paths end with a vertical step? As shown below, in all of these lattice paths, the first row has length m and can be tiled F_{m+1} ways. The rest depends on the lattice path from $(0,0)$ to $(m,n-1)$. Summing over all possible lattice paths from $(0,0)$ to $(m,n-1)$ there are $\binom{m+n-1}{m}_F$ tiled lattice paths for the rest of the lattice. Hence the number of tiled lattice paths ending in a vertical step is $F_{m+1}\binom{m+n-1}{m}_F$.

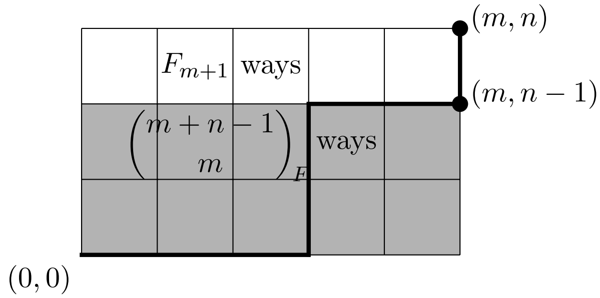


Figure 3. There are $F_{m+1}\binom{m+n-1}{m}_F$ tiled lattice paths that end with a vertical step.

How many tiled lattice paths end with a horizontal step? In all such paths, the last column has length n and can be tiled F_{n-1} ways (beginning with a domino). Summing over all lattice paths from $(0,0)$ to $(m-1,n)$ there are $\binom{m+n-1}{m-1}_F$ tiled lattice paths for the rest of the lattice. Hence the number of tiled lattice paths ending in a horizontal step, as illustrated below, is $F_{n-1}\binom{m+n-1}{m-1}_F$.

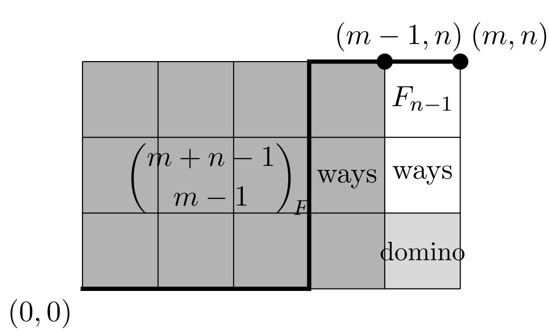


Figure 4. There are $F_{n-1}\binom{m+n-1}{m-1}_F$ tiled lattice paths that end with a horizontal step.

Combining the two previous cases, the total number of tiled lattice paths from $(0,0)$ to (m,n) is $F_{m+1}\binom{m+n-1}{m}_F + F_{n-1}\binom{m+n-1}{m-1}_F$. □

Replacing linear tilings with bracelets and removing the initial domino restriction for vertical tilings, we can apply the same logic as before to get

$$2^{m+n}\binom{m+n}{m}_F = 2^{m+n-1}L_m\binom{m+n-1}{m}_F + 2^{m+n-1}L_n\binom{m+n-1}{m-1}_F.$$

Dividing both sides by 2^{m+n-1} gives us

Identity 3.2. *For $m,n \geq 1$,*

$$2\binom{m+n}{m}_F = L_m\binom{m+n-1}{m}_F + L_n\binom{m+n-1}{m-1}_F.$$

In full disclosure, Identities 3.1 and 3.2 are used by Sagan and Savage to prove their combinatorial interpretations, so it is no surprise that these identities would have easy combinatorial proofs. The same is true for the weighted (or colorized) version of these identities for Lucasnomial coefficients.

Identity 3.3. *For $m,n \geq 1$,*

$$\binom{m+n}{m}_U = U_{m+1}\binom{m+n-1}{m}_U + U_{n-1}\binom{m+n-1}{m-1}_U.$$

Identity 3.4. *For $m,n \geq 1$,*

$$2\binom{m+n}{m}_U = V_m\binom{m+n-1}{m}_U + V_n\binom{m+n-1}{m-1}_U.$$

By considering the number of vertical steps that a lattice path ends with, Reiland [8] proved

Identity 3.5. *For $m,n \geq 1$,*

$$\binom{m+n}{m}_F = \sum_{j=0}^{n} F_{m+1}^{j}F_{n-j-1}\binom{m-1+n-j}{m-1}_F$$

Combinatorial Proof: We count the tiled lattice paths from $(0,0)$ to (m,n) by considering the number j of vertical steps at the end of the path, where $0 \leq j \leq n$. Such a tiling begins with j full rows, which can be tiled F_{m+1}^{j} ways. Since the lattice path must have a horizontal step from $(m-1,n-j)$ to $(m,n-j)$, the last column will have height $n-j$ and can be tiled (without starting with a square) in F_{n-j-1} ways. The rest of the tiling consists of a tiled lattice path from $(0,0)$ to $(m-1,n-j)$ which can be created in $\binom{m-1+n-j}{m-1}_F$ ways. (Note that when $j=n-1$, the summand is 0, since $F_0=0$, as is appropriate since the last column can't have height 1 without starting with a square; also, when $j=n$, $F_{-1}=1$, so the summand simplifies to F_{m+1}^{n}, as required.) All together, the number of tilings is $\sum_{j=0}^{n} F_{m+1}^{j}F_{n-j-1}\binom{m-1+n-j}{m-1}_F$, as desired.

□

By the exact same logic, using bracelet tilings, we get

Identity 3.6. *For $m,n \geq 1$,*

$$2^{m+n}\binom{m+n}{m}_F = \sum_{j=0}^{n} L_m^{j}L_{n-j}2^{m+n-1-j}\binom{m-1+n-j}{m-1}_F$$

Replacing F with U and replacing L with V, the last two identities are appropriately colorized as well.

4. Open Problems

What follows is a list of Fibonomial identities that are still in need of combinatorial proof. Some of these identities have extremely simple algebraic proofs (and some hold for more general sequences than Fibonomial sequences) so one would expect them to have elementary combinatorial proofs as well.

Many simple identities appear in Fibonacci Quarterly articles by Gould [4, 5].

$$\binom{n}{k}_F \binom{k}{j}_F = \binom{n}{j}_F \binom{n-j}{k-j}_F$$

$$\binom{n}{k}_F = \sum_{j=k}^{n} \frac{F_j - F_{j-k}}{F_k} \binom{j-1}{k-1}_F$$

$$F_k \binom{n}{k}_F = F_n \binom{n-1}{k-1}_F = F_{n-k+1} \binom{n}{k-1}_F$$

Here is another basic identity for generalized binomial coefficients, first noted by Fontené [3] and further developed by Trojovský [10]

$$\binom{n}{k}_F - \binom{n-1}{k}_F = \binom{n-1}{k-1}_F \frac{F_n - F_k}{F_{n-k}}.$$

Here are some alternating sum identities, provided by Lind [7] and Cooper and Kennedy [2], respectively, that might be amenable to sign-reversing involutions:

$$\sum_{j=0}^{k+1} (-1)^{j(j+1)/2} \binom{k+1}{j}_F \binom{n-1}{k}_F = 0.$$

$$\sum_{j=0}^{k} (-1)^{j(j+1)/2} \binom{k}{j}_F F_{n-j}^{k-1} = 0.$$

Here are some special cases of very intriguing formulas that appear in a recent paper by Kilic, Akkus and Ohtsuka [6].

$$\sum_{k=0}^{2n+1} \binom{2n+1}{k}_F = \prod_{k=0}^{n} L_{2k}$$

$$\sum_{k=0}^{2n} (-1)^k \binom{4n}{2k}_F = (-1)^n \prod_{k=1}^{2n} L_{2k-1}$$

We have just scratched the surface here. There are countless others!

References

[1] A. T. Benjamin and J. J. Quinn. *Proofs That Really Count: The Art of Combinatorial Proof*, The Dolciani Mathematical Expositions, 27, Mathematical Association of America, Washington DC, 2003.

[2] C. Cooper and R. E. Kennedy. *Proof of a Result by Jarden by Generalizing a Proof by Carlitz*, Fib. Quart **33.4** (1995) 304–310.

[3] G. Fontené. *Généralisation d'une Formule Connue*, Nouvelle. Ann. Math., **15** (1915), 112.

[4] H. W. Gould. *The bracket function and the Fontené-Ward Generalized Binomial Coefficients with Application to Fibonomial Coefficients*, Fib. Quart. **7** (1969) 23–40, 55.

[5] H. W. Gould. *Generalization of Hermite's Divisibility Theorems and the Mann-Shanks Primality Criterion for s-Fibonomial Arrays*, Fib. Quart. **12** (1974) 157–166.

[6] E. Kilic, A. Akkus, and H. Ohtsuka. *Some Generalized Fibonomial Sums related with the Gaussian q-Binomial sums*, Bull. Math. Soc. Sci. Math. Roumanie, **55** (2012), 51–61.

[7] D. A Lind. *A Determinant Involving Generalized Binomial Coefficients*, Fib. Quart **9.2** (1971) 113–119, 162.

[8] E. Reiland. *Combinatorial Interpretations of Fibonomial Identities*, Senior Thesis, Harvey Mudd College, Claremont, CA 2011.

[9] B. Sagan and C. Savage. *Combinatorial Interpretations of Binomial Coefficient Analogues Related to Lucas Sequences*, Integers **10** (2010), 697–703.

[10] P. Trojovský. *On some Identities for the Fibonomial Coefficients via Generating Function*, Discrete Appl. Math.**155** (2007) no. 15, 2017–2024.

AMS Classification Numbers: 05A19, 11B39

Department of Mathematics, Harvey Mudd College, Claremont, CA 91711
E-mail address: benjamin@hmc.edu

Department of Applied Mathematics and Statistics, Johns Hopkins University, Baltimore, MD 21218
E-mail address: ereiland@jhu.edu

BENFORD BEHAVIOR OF ZECKENDORF DECOMPOSITIONS

ANDREW BEST, PATRICK DYNES, XIXI EDELSBRUNNER, BRIAN MCDONALD, STEVEN J. MILLER, KIMSY TOR, CAROLINE TURNAGE-BUTTERBAUGH, AND MADELEINE WEINSTEIN

ABSTRACT. A beautiful theorem of Zeckendorf states that every integer can be written uniquely as the sum of non-consecutive Fibonacci numbers $\{F_i\}_{i=1}^{\infty}$. A set $S \subset \mathbb{Z}$ is said to satisfy Benford's law if the density of the elements in S with leading digit d is $\log_{10}(1 + \frac{1}{d})$. We prove that, as $n \to \infty$, for a randomly selected integer m in $[0, F_{n+1})$ the distribution of the leading digits of the Fibonacci summands in its Zeckendorf decomposition converge to Benford's law almost surely. Our results hold more generally; instead of looking at the distribution of leading digits of summands in Zeckendorf decompositions, one obtains similar theorems concerning how often values in sets with positive density inside the Fibonacci numbers are attained in these decompositions.

CONTENTS

1. INTRODUCTION

1.1. History.

The Fibonacci numbers have fascinated professional mathematicians and amateurs for centuries. The purpose of this article is to review the connection between two interesting results, namely Zeckendorf's theorem and Benford's law of digit bias, and to discuss density results that arise in special subsets of the Fibonacci numbers.

A beautiful theorem due to Zeckendorf [28] states that every positive integer may be written uniquely as a sum of non-adjacent Fibonacci numbers. The standard proof is by straightforward induction and the greedy algorithm (though see [18] for a combinatorial approach). For this theorem to hold we must normalize the Fibonacci numbers by taking $F_1 = 1$ and $F_2 = 2$

This research was conducted as part of the 2014 SMALL REU program at Williams College and was supported by NSF grants DMS 1347804 and DMS 1265673, Williams College, and the Clare Boothe Luce Program of the Henry Luce Foundation. It is a pleasure to thank them for their support, and the participants there and at the 16^{th} International Conference on Fibonacci Numbers and their Applications for helpful discussions. We also thank the referee for several comments which improved the exposition.

(and of course $F_{n+1} = F_n + F_{n-1}$), for if our series began with two 1's or with a 0 the decompositions of many numbers into non-adjacent summands would not be unique.

In 1937 the physicist Frank Benford [2], then working for General Electric, observed that the distributions of the leading digits of numbers in many real and mathematical data sets were not uniform. In fact, the leading digits of numbers from various sources such as atomic weights, baseball statistics, numbers in periodicals and values of mathematical functions or sequences seemed biased towards lower values; for instance, a leading digit of 1 occurred about 30% of the time, while a leading digit of 9 occurred less than 5% of the time. We now say a data set satisfies Benford's law (base B) if the probability of a first digit base B of d is $\log_B(1 + 1/d)$, or more generally the probability that the significand[1] is at most s is $\log_B(s)$. Benford's law has applications in disciplines ranging from accounting (where it is used to detect fraud) to zoology and population growth, and many areas between. While this bias is often initially surprising, it is actually very natural as Benford's law is equivalent to the logarithms of the set being equidistributed modulo 1. For more on Benford's law see [15, 16, 21, 24], as well as [20] for a compilation of articles on its theory and applications.

Obviously, we would not be discussing Benford's law if it had no connection to the Fibonacci numbers. A fascinating result, originally published in [5] (see also [21, 27]), states that the Fibonacci numbers follow Benford's law of digit bias.[2] There are many questions that may be asked concerning the connection between the Fibonacci numbers and Benford's law. This research was motivated by the study of the distribution of leading digits of Fibonacci summands in Zeckendorf decompositions. Briefly, our main result is that the distribution of leading digits of summands in Zeckendorf decompositions converges to Benford's law. Our result is more universal, and in fact holds for special sequences with density. We first set some notation, and then precisely state our results.

1.2. **Preliminaries.**

Let $S \subset \{F_i\}_{i=1}^{\infty}$, and let $q(S, n)$ be the density of S over the Fibonacci numbers in the interval $[0, F_n]$. That is,

$$q(S,n) \;=\; \frac{\#\{F_i \in S \;:\; 1 \le i \le n\}}{n}. \tag{1.1}$$

When $\lim_{n\to\infty} q(S, n)$ exists, we define the *asymptotic density* $q(S)$ as

$$q(S) \;:=\; \lim_{n\to\infty} q(S,n). \tag{1.2}$$

For the sake of completeness, we define a mapping between the positive integers and their Zeckendorf decompositions. We first note that a *legal* Zeckendorf decomposition is the unique decomposition of a number into non-adjacent Fibonacci numbers.

Definition 1.1. *Let $m \in \mathbb{N}$. The function ZD injectively maps each $m \in \mathbb{N}$ to the set of its Zeckendorf summands. Conversely, ZD^{-1} injectively maps each legal set of Zeckendorf summands to the positive integer that set represents.*

For example, $\mathrm{ZD}(10) = \{2, 8\}$ and $\mathrm{ZD}^{-1}(\{8, 34\}) = 42$; however, $\mathrm{ZD}^{-1}(\{8, 13\})$ is undefined, as $21 = 8 + 13$ is not a legal Zeckendorf decomposition.

[1] If $x > 0$ we may write $x = S_B(x)10^{k(x)}$, where $S_B(x) \in [1, B)$ is the significand and $k(x) \in \mathbb{Z}$ is the exponent.

[2] The main idea of the proof is to note that $\log_{10}\left(\frac{1+\sqrt{5}}{2}\right)$ is irrational, and then use Weyl's criterion and Binet's formula to show the logarithms of the Fibonacci numbers converge to being equidistributed modulo 1.

Let $m \in \mathbb{N}$ be chosen uniformly at random from the interval $[0, F_{n+1})$. We define two useful random variables:

$$X_n(m) \ := \ \#\mathrm{ZD}(m), \qquad Y_n(m) \ := \ \#\mathrm{ZD}(m) \cap S. \tag{1.3}$$

In our main result, we show that the density of S in a typical Zeckendorf decomposition is asymptotic to the density of S in the set of Fibonacci numbers.

Theorem 1.2 (Density Theorem for Zeckendorf Decompositions)**.** *Let $S \subset \{F_i\}_{i=1}^{\infty}$ with asymptotic density $q(S)$ in the Fibonacci numbers. For $m \in \mathbb{N}$ chosen uniformly at random from the interval $[0, F_{n+1})$, let $X_n(m)$ and $Y_n(m)$ be defined as above. Then for any $\varepsilon > 0$, we have with probability $1 + o(1)$ that*

$$\left| \frac{Y_n(m)}{X_n(m)} \ - \ q(S) \right| \ < \ \varepsilon. \tag{1.4}$$

We now define a method of constructing a random Zeckendorf decomposition, which plays a central role in our proofs. Essentially, we want to select a random subset of the Fibonacci numbers which satisfy the criterion of being a legal Zeckendorf decomposition. We fix a probability $p \in (0, 1)$ and let $A_n(p)$ be a random subset of Fibonacci numbers at most F_n. Let $A_0(p) = \emptyset$, and define $A_n(p)$ recursively for $n > 0$ as follows. We set

$$A_n(p) \ = \ \begin{cases} A_{n-1}(p) & \text{if } F_{n-1} \ \in \ A_{n-1}(p) \\ A_{n-1}(p) \cup F_n & \text{with probability } p \text{ if } F_{n-1} \ \notin \ A_{n-1}(p) \\ A_{n-1}(p) & \text{otherwise,} \end{cases} \tag{1.5}$$

and define

$$A(p) \ := \ \bigcup_n A_n(p). \tag{1.6}$$

This random process leads to the following result.

Theorem 1.3 (Density Theorem for Random Decompositions)**.** *Let $S \subset \{F_i\}_{i=1}^{\infty}$ have asymptotic density $q(S)$ over the Fibonacci numbers. Then, with probability 1, $S \cap A(p)$ has asymptotic density $q(S)$ in $A(p)$.*

We use Theorem 1.3 with the clever choice of probability of $p = 1/\varphi^2$ to prove Theorem 1.2. The reason for this choice is that this random Zeckendorf decomposition is similar to the Zeckendorf decomposition of an integer chosen uniformly at random.

We now describe some situations where Theorem 1.3 applies. There are many interesting situations where $S \subset \{F_i\}_{i=1}^{\infty}$ has a limiting density over the Fibonacci numbers. As the Fibonacci numbers follow Benford's law, the set S_d of Fibonacci number with a fixed leading digit $1 \leq d \leq 9$ has asymptotic density $q(S_d) = \log{(1 + 1/d)}$ in the Fibonacci numbers. By an extension of Benford's law, the Fibonacci numbers in which a finite amount of leading digits are fixed also have asymptotic density over the Fibonacci numbers. Conversely, we could fix a finite set of digits at the right and obtain similar results. For example, if we look at the Fibonacci numbers modulo 2 we get $1, 0, 1, 1, 0, 1, 1, 0, \dots$; thus in the limit one-third of the Fibonacci numbers are even, and the asymptotic density exists. These arguments immediately imply Benford behavior of the Zeckendorf decompositions.

Corollary 1.4 (Benford Behavior in Zeckendorf Decompositions)**.** *Fix positive integers D and B, and let*

$$\mathcal{D}_D \ := \ \{(d_1, \dots, d_D) : d_1 \geq 1, d_i \in \{0, 1, \dots, B-1\}\}; \tag{1.7}$$

to each $(d_1, \dots, d_D) \in \mathcal{D}_D$ *we associate the set* $S_{d_1,\dots,d_D}$ *of Fibonacci numbers whose significand starts* $d_1.d_2d_3 \cdots d_D$. *With probability* 1, *for each* $(d_1, \dots, d_D)$ *we have the asymptotic density of* $S_{d_1,\dots,d_D} \cap A(p)$ *equals* $\log_B(d_1.d_2d_3 \cdots d_D)$, *and thus with probability* 1 *Benford's law holds.*

Proof. As D is fixed and finite, there are only finitely many starting blocks for significands in $\mathcal{D}_D$. By Theorem 1.3 for each of these the asymptotic density of $S_{d_1,\dots,d_D} \cap S(p)$ equals the corresponding Benford probability; as the intersection of finitely many events that each happen with probability 1 happens with probability 1, we see that with probability 1, all the significands of length D happen with the correct probability. Sending $D \to \infty$ yields the desired Benford behavior. $\square$

As a check of our Benfordness results, we performed two simple experiments. The first was an exhaustive search of all $m \in [F_{25}, F_{26}) = [121393, 196418)$. We performed a chi-square goodness of fit test on the distribution of first digits of summands for each m and Benford's law. There are eight degrees of freedom, and 99.74% of the time our chi-square values were below the 95% confidence threshold of 15.51, and 99.99% of the time they were below the 99% confidence threshold of 20.09. We then randomly chose a number in $[10^{60000}, 10^{60001})$, and found a chi-square value of 8.749. See Figure 1 for a comparison between the observed digit frequencies and Benford's law.

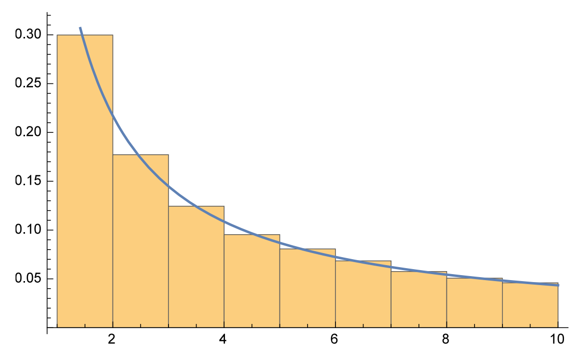


FIGURE 1. Comparison of the frequencies of leading digits in Zeckendorf decompositions of a large random integer, approximately $7.94 \cdot 10^{60000}$, and Benford's law (the solid curve is $1/(x \log 10)$, the Benford density).

To prove our main results we first state and prove some lemmas about random legal decompositions. The key observation is that for an appropriate choice of p, the set $A(p)$ derived from the random process defined in (1.5) acts similarly to the Zeckendorf decomposition of a randomly chosen integer $m \in [0, F_{n+1})$. Theorem 1.2 thus becomes a consequence Theorem 1.3, which we prove through Chebyshev's inequality.

2. Proof of Theorem 1.2

In this section, we assume the validity of Theorem 1.3 in order to prove Theorem 1.2. The proof of Theorem 1.3 is given in §3. We begin with a useful lemma on the probability that $ZD^{-1}(A(p))$ equals m. We find that $m \in [0, F_{n+1})$ are almost uniformly chosen.

Lemma 2.1. *With* $A_n(p)$ *defined as in* (1.5), $ZD^{-1}(A_n(p)) \in [0, F_{n+1})$ *is a random variable. For a fixed integer* $m \in [0, F_{n+1})$ *with the Zeckendorf decomposition* $m = F_{a_1} + F_{a_2} + \cdots + F_{a_k}$, *where* $k \in \mathbb{N}$, $1 \le a_1$, $a_1 + 1 < a_2$, $\ldots$, $a_{k-1} + 1 < a_k$, *we have*

$$\text{Prob}\left(ZD^{-1}\left(A_n(p)\right) = m\right) \;=\; \begin{cases} p^k(1-p)^{n-2k} & \text{if } m \in [0, F_n) \\ p^k(1-p)^{n-2k+1} & \text{if } m \in [F_n, F_{n+1}). \end{cases} \tag{2.1}$$

Proof. With probability $(1-p)^{a_1-1}p$, F_{a_1} is the smallest element of $A_n(p)$. For $j \in \mathbb{Z}$, suppose that $F_{a_1}, F_{a_2}, \ldots, F_{a_{j-1}}$ be the $j-1$ smallest elements of $A_n(p)$. With probability $(1-p)^{a_j - a_{j-1}-2}p$, F_{a_j} is the next smallest element of $A_n(p)$; the reason we have a -2 in the exponent is that once we select $F_{a_{j-1}}$ we cannot have $F_{a_{j-1}+1}$, and thus there are $a_j - a_{j-1} - 2$ Fibonacci numbers between $F_{a_{j-1}+1}$ and F_{a_j-1} which we could have selected (but did not). Continuing, we find $\text{ZD}^{-1}\left(A_n(p)\right) = m$ if and only if the k smallest elements of $A_n(p)$ are $F_{a_1}, F_{a_2}, \ldots, F_{a_k}$ and $F_j \notin A_n(p)$ for $j > a_k$; note if $a_k = n$ then we are done determining if we have or do not have summands, while if $a_k < n$ we must elect not to have $F_{a_k+1}, \ldots, F_n$ and thus need another $n - a_k - 1$ factors of $1-p$. Then, by these calculations, $\text{ZD}^{-1}\left(A_n(p)\right) = m$ with probability

$$\text{Prob}\left(\text{ZD}^{-1}\left(A_n(p)\right) \;=\; m\right) \;=\; (1-p)^{a_1-1}p\left(\prod_{j=2}^{k}(1-p)^{a_j-a_{j-1}-2}p\right)(1-p)^{n-a_k-\delta_k}, \tag{2.2}$$

where $\delta_k = 1$ if $a_k < n$ and 1 if $a_k = n$. The first case happens when $m \in [0, F_n)$ and the second when $m \in [F_n, F_{n+1})$; (2.1) now follows from simple algebra. □

The key idea in proving Theorem 1.2 is to consider the special case of $p = 1/\varphi^2$ in Lemma 2.1, where $\varphi := \frac{1+\sqrt{5}}{2}$ is the golden mean.[3] The reason this is an exceptionally useful choice is that initially the probability of choosing m in our random process $A(p)$ depends on the *number* of summands of m; however, for $p = 1/\varphi^2$ we have $p^k(1-p)^{-2k} = 1$. Thus in this case, for m an integer in $[0, F_{n+1})$ we see that (2.2) reduces to

$$\text{Prob}\left(\text{ZD}^{-1}\left(A_n(\varphi^{-2})\right) = m\right) \;=\; \begin{cases} \varphi^{-n} & \text{if } m \in [0, F_n) \\ \varphi^{-(n+1)} & \text{if } m \in [F_n, F_{n+1}). \end{cases} \tag{2.3}$$

Note this is nearly independent of m; all that matters is whether or not it is larger than F_n. The desired result follows from straightforward algebra.[4]

We now are ready to prove Theorem 1.2.

Proof of Theorem 1.2. For a fixed $\varepsilon > 0$, let

$$E(n,\varepsilon) \;:=\; \left\{m \in \mathbb{Z} \,\cap\, [0, F_{n+1}) \;:\; \left|\frac{Y_n(m)}{X_n(m)} \;-\; q(S)\right| \;\ge\; \varepsilon\right\}. \tag{2.4}$$

[3]For us, the importance of φ is that it is the largest root of the characteristic polynomial for the Fibonacci recurrence, and by Binet's formula it governs the growth of the sequence.

[4]As a quick check, note $F_n\varphi^{-n} + (F_{n+1} - F_n)\varphi^{-(n+1)} = 1$, as required for a probability.

By Theorem 1.3, for m chosen uniformly at random from the integers in $[0, F_{n+1})$, we have

$$\begin{aligned}
\operatorname{Prob}(m \in E(n,\varepsilon)) &= \sum_{x \in E(n,\varepsilon)} \frac{1}{F_{n+1}} \\
&= O\left(\sum_{x \in E(n,\varepsilon)} \operatorname{Prob}\left(\mathrm{ZD}^{-1}\left(A_n(\varphi^{-2})\right) = x\right)\right) \\
&= O\left(\operatorname{Prob}\left(\mathrm{ZD}^{-1}\left(A_n(\varphi^{-2})\right) \in E(n,\varepsilon)\right)\right) = o(1). \qquad (2.5)
\end{aligned}$$

We conclude that $\left|\frac{Y_n(m)}{X_n(m)} - q(S)\right| < \varepsilon$ with probability $1 + o(1)$. □

3. Proof of Theorem 1.3

In this section, we prove Theorem 1.3. We first prove some useful lemmas.

Lemma 3.1. *Let $A(p) \subset \{F_n\}_{n=1}^{\infty}$ be constructed as in (1.5) with probability parameter $p \in (0,1)$. Then*

$$\operatorname{Prob}(F_k \in A(p)) = \frac{p}{p+1} + O(p^k). \qquad (3.1)$$

Proof. By conditioning on whether $F_{k-2} \in A(p)$, we obtain a recurrence relation:[5]

$$\begin{aligned}
\operatorname{Prob}(F_k \in A(p)) &= \operatorname{Prob}(F_k \in A(p) \mid F_{k-2} \in A(p)) \cdot \operatorname{Prob}(F_{k-2} \in A(p)) \\
&\quad + \operatorname{Prob}(F_k \in A(p) \mid F_{k-2} \notin A(p)) \cdot \operatorname{Prob}(F_{k-2} \notin A(p)) \\
&= p \cdot \operatorname{Prob}(F_{k-2} \in A(p)) + p(1-p) \cdot \operatorname{Prob}(F_{k-2} \notin A(p)) \\
&= p^2 \cdot \operatorname{Prob}(F_{k-2} \in A(p)) + p - p^2. \qquad (3.2)
\end{aligned}$$

As $\operatorname{Prob}(F_1 \in A(p)) = p$ and $\operatorname{Prob}(F_2 \in A(p)) = (1-p)p = p - p^2$, we have

$$\operatorname{Prob}(F_k \in A(p)) = (\operatorname{Prob}(F_1 \in A(p)))^2 \cdot \operatorname{Prob}(F_{k-2} \in A(p)) + \operatorname{Prob}(F_2 \in A(p)). \qquad (3.3)$$

From induction and the geometric series formula we immediately obtain for all k that

$$\operatorname{Prob}(F_k \in A(p)) = \sum_{j=1}^{k} (-1)^{j+1} p^j = \frac{p}{1+p} + O(p^k), \qquad (3.4)$$

completing the proof. □

Lemma 3.2. *Let W_n be the random variable defined by $W_n := \#A_n(p)$. Then*

$$\mathbb{E}[W_n] = \frac{np}{1+p} + O(1) \quad \text{and} \quad \operatorname{Var}(W_n) = O(n). \qquad (3.5)$$

Proof. Define the indicator function $\chi(F_k)$ for $k \in \mathbb{N}$ by

$$\chi(F_k) := \begin{cases} 1 & \text{if } F_k \in A(p) \\ 0 & \text{if } F_k \notin A(p). \end{cases} \qquad (3.6)$$

[5] We can also give a simple heuristic suggesting the main term of the answer. For k large, the probability F_k occurs should roughly be the same as the probability that F_{k-1} is used; call this x. Then $x \approx (1-x)p$ (to have F_k we must first not have taken F_{k-1}, and then once this happens we choose F_k with probability p), which implies $x \approx p/(1+p)$ as claimed.

We note that $W_n = \sum_{k=1}^{n} \chi(F_k)$ and by linearity of expectation have

$$\begin{aligned}\mathbb{E}[W_n] &= \sum_{k=1}^{n} \mathbb{E}[\chi(F_k)] \\ &= \sum_{k=1}^{n} \operatorname{Prob}(F_k \in A(p)) \\ &= \sum_{k=1}^{n} \left(\frac{p}{1+p} + O(p^k)\right) \\ &= \frac{np}{1+p} + O(1). \end{aligned} \tag{3.7}$$

To find the variance we use that it equals $\mathbb{E}[W_n^2] - \mathbb{E}[W_n]^2$. Without loss of generality, when we expand below we may assume $i \leq j$ and double the contribution of certain terms. As we cannot have F_i and F_{i+1}, there are dependencies. While we could determine the variance exactly with a bit more work, for our applications we only need to bound its order of magnitude.

$$\begin{aligned}\mathbb{E}[W_n^2] &= \mathbb{E}\left[\left(\sum_{k=1}^{n} \chi(F_k)\right)^2\right] \\ &= \mathbb{E}\left[\sum_{i,j \leq n} \chi(F_i) \cdot \chi(F_j)\right] \\ &= \sum_{i,j \leq n} \mathbb{E}[\chi(F_i) \cdot \chi(F_j)] \\ &= \sum_{i,j \leq n} \operatorname{Prob}(F_i \in A(p)) \operatorname{Prob}(F_j \in A(p) | F_i \in A(p)) \\ &= \sum_{i \leq n} \operatorname{Prob}(F_i \in A(p)) + 2 \sum_{i+2 \leq j \leq n} \operatorname{Prob}(F_i \in A(p)) \operatorname{Prob}(F_j \in A(p) | F_i \in A(p)) \\ &= O(n) + 2 \sum_{i+2 \leq j \leq n} \operatorname{Prob}(F_i \in A(p)) \operatorname{Prob}(F_{j-i-1} \in A(p)) \\ &= O(n) + 2 \sum_{i+2 \leq j \leq n} \left(\frac{p}{1+p}\right)^2 \left(1 + O\left(p^{\min(i,j-i)}\right)\right) \\ &\leq O(n) + \left(\frac{np}{1+p}\right)^2 + O\left(\sum_{i+2 \leq j \leq n} p^{\min(i,j-i)}\right). \end{aligned} \tag{3.8}$$

For a fixed $k = 1, 2, \ldots, n-1$, there are less than n pairs (i, j) with $k = i < j - i$ and $i + 2 \leq j \leq n$. Similarly, there are less than n pairs (i, j) with $k = i - j \leq i$, $i + 2 \leq j \leq n$. Therefore, there are less than $2n$ pairs (i, j) for which $\min(i, j - i) = k$. Thus

$$\sum_{i+2 \leq j \leq n} p^{\min(i,j-i)} \ < \ 2n \sum_{k=1}^{n-1} p^k \ = \ O(n), \tag{3.9}$$

and therefore

$$\mathbb{E}[W_n^2] \;=\; \left(\frac{np}{1+p}\right)^2 + O(n) \;=\; \mathbb{E}[W_n]^2 + O(n). \tag{3.10}$$

We conclude that

$$\operatorname{Var}(W_n) \;=\; O(n), \tag{3.11}$$

completing the proof. □

Corollary 3.3. *Let W_n be the random variable defined by $W_n := \#A_n(p)$. With probability $1 + o(1)$,*

$$\left|W_n - \frac{np}{1+p}\right| \;<\; n^{2/3}. \tag{3.12}$$

Proof. From (3.7) we know $\mathbb{E}[W_n] = \frac{np}{1+p} + O(1)$. For n large, if $\left|W_n - \frac{np}{1+p}\right| \geq n^{2/3}$ then $|W_n - \mathbb{E}[W_n]| \geq n^{2/3}/2015$. By Chebyshev's inequality we have

$$\operatorname{Prob}\left(|W_n - \mathbb{E}[W_n]| \geq \frac{n^{2/3}}{2015}\right) \;\leq\; \frac{2015^2 \operatorname{Var}(W_n)}{n^{4/3}} \;=\; o(1) \tag{3.13}$$

as by (3.11) the variance of W_n is of order n. □

Lemma 3.4. *Let $S \subset \{F_n\}_{n=1}^{\infty}$ with asymptotic density $q(S)$ in the Fibonacci numbers. Let Z_n be the random variable defined by $Z_n := \#A_n(p) \cap S$. Then*

$$\begin{aligned}
\mathbb{E}[Z_n] \;&=\; \frac{npq(S)}{1 \;+\; p} \;+\; o(n) \\
\operatorname{Var}(Z_n) \;&=\; o(n^2).
\end{aligned} \tag{3.14}$$

Proof. Define the indicator function $\psi(F_k)$ for $k \in \mathbb{N}$ by

$$\psi(F_k) \;=\; \begin{cases} 1 & \text{if } F_k \in S \\ 0 & \text{if } F_k \notin S. \end{cases} \tag{3.15}$$

Then we have

$$\begin{aligned}
\mathbb{E}[Z_n] \;&=\; \sum_{k=1}^{n} \psi(F_k) \operatorname{Prob}\left(F_k \in A(p)\right) \\
&=\; \sum_{k=1}^{n} \psi(F_k) \left(\frac{p}{1+p} + O(p^k)\right) \\
&=\; O(1) + \frac{p}{1+p} \sum_{k=1}^{n} \psi(F_k) \\
&=\; \frac{npq(S)}{1+p} + o(n)
\end{aligned} \tag{3.16}$$

since $\lim_{n\to\infty} q(S, n) = q(S)$.

Similarly to the calculation in Lemma 3.2, we compute

$$\begin{aligned}
\mathbb{E}[Z_n^2] &= \sum_{i,j\le n}\psi(F_i)\psi(F_j)\mathrm{Prob}\left(F_i\in A(p)\right)\mathrm{Prob}\left(F_j\in A(p)|F_i\in A(p)\right)\\
&= O(n)+2\sum_{i+2\le j\le n}\psi(F_i)\psi(F_j)\mathrm{Prob}\left(F_i\in A(p)\right)\mathrm{Prob}\left(F_{j-i-1}\in A(p)\right)\\
&= O(n)+2\sum_{i+2\le j\le n}\psi(F_i)\psi(F_j)\left(\frac{p}{1+p}\right)^2\left(1+O\left(p^{\min(i,j-i)}\right)\right)\\
&= O(n)+2\left(\frac{p}{1+p}\right)^2\sum_{i+2\le j\le n}\psi(F_i)\psi(F_j)\\
&= o(n^2)+\left(\frac{npq(S)}{1+p}\right)^2. \qquad (3.17)
\end{aligned}$$

In the calculation above, the only difficulty is in the second to last line, where we argue that the main term of the i and j double sum is $n^2q(S)^2/2$. To see this, note by symmetry that up to contributions of size $O(n)$ we can remove the restrictions on i and j (and thus have each range from 1 to n) if we then take half of the resulting sum. Thus, the restricted double sum becomes $\frac{1}{2}\left(\sum_{i\le n}\psi(F_i)\right)\left(\sum_{j\le n}\psi(F_j)\right)$, which as $n\to\infty$ converges to $\frac{1}{2}q(S)n\cdot q(S)n$ (up to an error of size $o(n^2)$, of course). Therefore, we have

$$\mathrm{Var}(Z_n) \;=\; \mathbb{E}[Z_n^2]-\mathbb{E}[Z_n]^2 \;=\; o(n^2), \qquad (3.18)$$

which completes the proof. □

Corollary 3.5. *Let Z_n be the random variable defined by $Z_n := \#A_n(p)\cap S$, and let $g(n) = n^{1/2}\mathrm{Var}(Z_n)^{-1/4}$. Then*

$$\mathrm{Prob}\left(|Z_n-\mathbb{E}[Z_n]|>\frac{\mathbb{E}[Z_n]}{g(n)}\right) \;\le\; \frac{\mathrm{Var}(Z_n)g(n)^2}{\mathbb{E}[Z_n]^2} \;=\; o(1). \qquad (3.19)$$

Proof. The proof follows immediately by Chebyshev's inequality and the order of magnitude of the various quantities. □

Armed with the above results, we can now prove our main theorem.

Proof of Theorem 1.3. Let

$$\begin{aligned}
e_1(n) &= n^{-1/3},\\
e_2(n) &= \frac{1}{n}\left(\frac{\mathbb{E}[Z_n]}{g(n)}+\left|E[Z_n]-\frac{npq(S)}{1+p}\right|\right). \qquad (3.20)
\end{aligned}$$

Note that both are of order $o(1)$. We combine Corollaries 3.3 and 3.5 to see that with probability $1+o(1)$ we have

$$\begin{aligned}
Z_n &\le \frac{npq(S)}{1+p}(1+e_2(n)),\\
W_n &\ge \frac{np}{1+p}(1-e_1(n)). \qquad (3.21)
\end{aligned}$$

Therefore, for any $\varepsilon > 0$ we have with probability 1 that

$$\lim_{n\to\infty} \frac{Z_n}{W_n} \ \leq \ \lim_{n\to\infty} \frac{q(S)(1+e_2(n))}{1-e_1(n)} \ = \ q(S). \tag{3.22}$$

A similar argument gives $q(S)$ as a lower bound for $\lim_{n\to\infty} Z_n/W_n$, and thus with probability 1

$$\lim_{n\to\infty} \frac{Z_n}{W_n} \ = \ q(S), \tag{3.23}$$

as desired. □

4. Conclusion and Future Work

We were able to handle the behavior of almost all Zeckendorf decompositions by finding a correspondence between these and a special random process, replacing the deterministic behavior for each $m \in [0, F_n)$ with random behavior which is easier to analyze. The key observation was that this correspondence held when choosing $p = 1/\varphi^2$. This allowed us to prove not just Benford behavior for the leading digits of summands in almost all Zeckendorf decompositions, but also similar results for other sequences with density.

In [4] we revisit these problems for more general recurrences, where there is an extensive literature (see among others [1, 8, 9, 10, 11, 12, 13, 14, 17, 19, 22, 23, 25, 26]). Similar to other papers in the field (for example, [18] versus [22], or [6] versus [7]), the arguments are often easier for the Fibonacci numbers, as we have simpler and more explicit formulas at our disposal. In the more general case we introduce the notion of a super-legal decomposition, which aids in the arguments.

Instead of choosing our integers uniformly in $[0, F_{n+1})$ one can consider other models, such as choosing elements in $[0, M)$ with $M \to \infty$ or various sub-intervals of $[0, F_{n+1})$. For most of these choices we expect to see similar behavior; we analyze many of these cases in [3].

Acknowledgements

This research was conducted as part of the 2014 SMALL REU program at Williams College and was supported by NSF grants DMS 1347804 and DMS 1265673, Williams College, and the Clare Boothe Luce Program of the Henry Luce Foundation. It is a pleasure to thank them for their support, and the participants there and at the 16$^{\text{th}}$ International Conference on Fibonacci Numbers and their Applications for helpful discussions. We also thank the referee for several comments which improved the exposition.

References

[1] H. Alpert, *Differences of multiple Fibonacci numbers*, Integers: Electronic Journal of Combinatorial Number Theory **9** (2009), 745–749.

[2] F. Benford, *The Law of Anomalous Numbers*, Proceedings in the American Philosophical Society (1937) 551-572. www.jstor.org/stable/984802.

[3] A. Best, P. Dynes, X. Edelsbrunner, B. McDonald, S. J. Miller, K. Tor, C. Turnage-Butterbaugh, M. Weinstein, *Gaussian Behavior of the Number of Summands in Zeckendorf Decompositions in Small Intervals*, Fibonacci Quarterly **52**, no. 5 (2014), 47–53.

[4] A. Best, P. Dynes, X. Edelsbrunner, B. McDonald, S. J. Miller, K. Tor, C. Turnage-Butterbaugh, M. Weinstein, *Benford Behavior of Generalized Zeckendorf Decompositions*, http://arxiv.org/pdf/1412.6839v1.

[5] J. Brown and R. Duncan, *Modulo One Uniform Distribution of the Sequence of Logarithms of Certain Recursive Sequences*, Fibonacci Quarterly **8** (1970) 482-486. http://www.mathstat.dal.ca/FQ/Scanned/8-5/brown.pdf.

[6] M. Catral, P. Ford, P. Harris, S. J. Miller and D. Nelson, *Generalizing Zeckendorf's Theorem: The Kentucky Sequence*, Fibonacci Quarterly **52**, no. 5 (2014), 68–90.
[7] M. Catral, P. Ford, P. Harris, S. J. Miller and D. Nelson, *The Generacci Recurrences and Zeckendorf's Theorem*, preprint.
[8] D. E. Daykin, *Representation of Natural Numbers as Sums of Generalized Fibonacci Numbers*, J. London Mathematical Society **35** (1960), 143–160.
[9] P. Demontigny, T. Do, A. Kulkarni, S. J. Miller, D. Moon and U. Varma, *Generalizing Zeckendorf's Theorem to f-decompositions*, Journal of Number Theory **141** (2014), 136–158.
[10] P. Demontigny, T. Do, A. Kulkarni, S. J. Miller and U. Varma, *A Generalization of Fibonacci Far-Difference Representations and Gaussian Behavior*, to appear in the Fibonacci Quarterly. http://arxiv.org/pdf/1309.5600v2.
[11] M. Drmota and J. Gajdosik, *The distribution of the sum-of-digits function*, J. Théor. Nombres Bordeaux **10** (1998), no. 1, 17–32.
[12] P. Filipponi, P. J. Grabner, I. Nemes, A. Pethö, and R. F. Tichy, *Corrigendum to: "Generalized Zeckendorf expansions"*, Appl. Math. Lett., **7** (1994), no. 6, 25–26.
[13] P. J. Grabner and R. F. Tichy, *Contributions to digit expansions with respect to linear recurrences*, J. Number Theory **36** (1990), no. 2, 160–169.
[14] P. J. Grabner, R. F. Tichy, I. Nemes, and A. Pethö, *Generalized Zeckendorf expansions*, Appl. Math. Lett. **7** (1994), no. 2, 25–28.
[15] T. Hill, *The first-digit phenomenon*, American Scientist **86** (1996), 358–363.
[16] T. Hill, *A statistical derivation of the significant-digit law*, Statistical Science **10** (1996), 354–363.
[17] T. J. Keller, *Generalizations of Zeckendorf's theorem*, Fibonacci Quarterly **10** (1972), no. 1 (special issue on representations), pages 95–102.
[18] M. Kologlu, G. Kopp, S. J. Miller and Y. Wang, *On the Number of Summands in Zeckendorf Decompositions*, Fibonacci Quarterly **49** (2011), no. 2, 116-130.
[19] T. Lengyel, *A Counting Based Proof of the Generalized Zeckendorf's Theorem*, Fibonacci Quarterly **44** (2006), no. 4, 324–325.
[20] S. J. Miller (editor), *The Theory and Applications of Benford's Law*, Princeton University Press, 2015.
[21] S. J. Miller and R. Takloo-Bighash, *An Invitation to Modern Number Theory*, Princeton University Press, Princeton, NJ, 2006.
[22] S. J. Miller and Y. Wang, *From Fibonacci numbers to Central Limit Type Theorems*, Journal of Combinatorial Theory, Series A **119** (2012), no. 7, 1398–1413.
[23] S. J. Miller and Y. Wang, *Gaussian Behavior in Generalized Zeckendorf Decompositions*, Combinatorial and Additive Number Theory, CANT 2011 and 2012 (Melvyn B. Nathanson, editor), Springer Proceedings in Mathematics & Statistics (2014), 159–173.
[24] R. A. Raimi, *The first digit problem*, Amer. Math. Monthly **83** (1976), no. 7, 521–538.
[25] W. Steiner, *Parry expansions of polynomial sequences*, Integers **2** (2002), Paper A14.
[26] W. Steiner, *The Joint Distribution of Greedy and Lazy Fibonacci Expansions*, Fibonacci Quarterly **43** (2005), 60–69.
[27] L. C. Washington, *Benford's Law for Fibonacci and Lucas Numbers*, The Fibonacci Quarterly, **19.2** (1981), 175–177.
[28] E. Zeckendorf, *Représentation des nombres naturels par une somme des nombres de Fibonacci ou de nombres de Lucas*, Bulletin de la Société Royale des Sciences de Liège **41** (1972), pages 179–182.

E-mail address: `ajb5@williams.edu`

Department of Mathematics and Statistics, Williams College, Williamstown, MA 01267

E-mail address: `pdynes@clemson.edu`

Department of Mathematical Sciences, Clemson University, Clemson, SC 29634

E-mail address: `xe1@williams.edu`

Department of Mathematics and Statistics, Williams College, Williamstown, MA 01267

E-mail address: `bmcdon11@u.rochester.edu`

Department of Mathematics, University of Rochester, Rochester, NY 14627

E-mail address: sjm1@williams.edu, Steven.Miller.MC.96@aya.yale.edu

Department of Mathematics and Statistics, Williams College, Williamstown, MA 01267

E-mail address: ktor.student@manhattan.edu

Department of Mathematics, Manhattan College, Riverdale, NY 10471

E-mail address: cturnagebutterbaugh@gmail.com

Department of Mathematics, North Dakota State University, Fargo, ND 58102

E-mail address: mweinstein@g.hmc.edu

Department of Mathematics, Harvey Mudd College, Claremont, CA 91711

GAUSSIAN BEHAVIOR OF THE NUMBER OF SUMMANDS IN ZECKENDORF DECOMPOSITIONS IN SMALL INTERVALS

ANDREW BEST, PATRICK DYNES, XIXI EDELSBRUNNER, BRIAN MCDONALD, STEVEN J. MILLER, KIMSY TOR, CAROLINE TURNAGE-BUTTERBAUGH, AND MADELEINE WEINSTEIN

Abstract. Zeckendorf's theorem states that every positive integer can be written uniquely as a sum of non-consecutive Fibonacci numbers F_n, with initial terms $F_1 = 1, F_2 = 2$. We consider the distribution of the number of summands involved in such decompositions. Previous work proved that as $n \to \infty$ the distribution of the number of summands in the Zeckendorf decompositions of $m \in [F_n, F_{n+1})$, appropriately normalized, converges to the standard normal. The proofs crucially used the fact that all integers in $[F_n, F_{n+1})$ share the same potential summands.

We generalize these results to subintervals of $[F_n, F_{n+1})$ as $n \to \infty$; the analysis is significantly more involved here as different integers have different sets of potential summands. Explicitly, fix an integer sequence $\alpha(n) \to \infty$. As $n \to \infty$, for almost all $m \in [F_n, F_{n+1})$ the distribution of the number of summands in the Zeckendorf decompositions of integers in the subintervals $[m, m+F_{\alpha(n)})$, appropriately normalized, converges to the standard normal. The proof follows by showing that, with probability tending to 1, m has at least one appropriately located large gap between indices in its decomposition. We then use a correspondence between this interval and $[0, F_{\alpha(n)})$ to obtain the result, since the summands are known to have Gaussian behavior in the latter interval. We also prove the same result for more general linear recurrences.

Contents

1. Introduction

1.1. **History.** Let $\{F_n\}$ denote the Fibonacci numbers, normalized so that $F_1 = 1$, $F_2 = 2$[1], and $F_{n+1} = F_n + F_{n-1}$. An interesting equivalent definition of the Fibonacci numbers is that they are the unique sequence of positive integers such that every positive integer has a unique

This research was conducted as part of the 2014 SMALL REU program at Williams College and was supported by NSF grant DMS1347804 and DMS1265673, Williams College, and the Clare Boothe Luce Program of the Henry Luce Foundation. It is a pleasure to thank the participants of the SMALL REU and the 16th International Conference on Fibonacci Numbers and their Applications for helpful discussions.

[1] We define the sequence this way to retain uniqueness in our decompositions

legal decomposition as a sum of non-adjacent terms. This equivalence is known as Zeckendorf's theorem [25] and has been extended by many authors to a variety of other sequences.

For the Fibonacci numbers, Lekkerkerker [19] proved that the average number of summands needed in the Zeckendorf decomposition of an integer $m \in [F_n, F_{n+1})$ is $\frac{n}{\varphi^2+1} + O(1)$, where $\varphi = \frac{1+\sqrt{5}}{2}$, the golden mean, is the largest root of the Fibonacci recurrence. This has been extended to other positive linear recurrence sequences, and much more is known. Namely, the distribution of the number of summands converges to a Gaussian as $n \to \infty$. There are several different methods of proof, from continued fractions to combinatorial perspectives to Markov processes. See [9, 11, 12, 13, 14, 15, 16, 20, 17, 18, 21, 22, 23, 24] for a sampling of results and methods along these lines, [1, 6, 7, 8, 10, 11] for generalizations to other types of representations, and [2, 5] for related questions on the distribution of gaps between summands.

The analysis in much of the previous work was carried out for $m \in [F_n, F_{n+1})$. The advantage of such a localization[2] is that each m has the same candidate set of summands and is of roughly the same size. The purpose of this work is to explore some of the above questions on a significantly smaller scale and determine when and how often we obtain Gaussian behavior. Note that we cannot expect such behavior to hold for all sub-intervals of $[F_n, F_{n+1})$, even if we require the size to grow with n. To see this, consider the interval

$$[F_{2n} + F_n + F_{n-2} + \cdots + F_{\lfloor n^{1/4} \rfloor}, F_{2n} + F_{n+1} + F_{\lfloor n^{1/4} \rfloor}). \tag{1.1}$$

The integers in the above interval that are less than $F_{2n} + F_{n+1}$ have on the order of $n/2$ summands, while those that are larger have at most on the order of $n^{1/4}$ summands. Thus the behavior cannot be Gaussian.[3]

1.2. Main Result.

We first introduce some notation before stating our main result. Fix any increasing positive integer valued function $\alpha(n)$ with

$$\lim_{n\to\infty} \alpha(n) \ = \ \lim_{n\to\infty} (n - \alpha(n)) \ = \ \infty. \tag{1.2}$$

Fix a non-decreasing positive function $q(n) < n - \alpha(n)$ taking on even integer values with the restrictions that $q(n) \to \infty$ and $q(n) = o(\sqrt{n})$. For $m \in [F_n, F_{n+1})$ with decomposition

$$m \ = \ \sum_{j=1}^{n} a_j F_j, \tag{1.3}$$

define

$$\begin{aligned} C_1(m) &:= (a_1, a_2, ..., a_{\alpha(n)}), \\ C_2(m) &:= (a_{\alpha(n)+1}, ..., a_{\alpha(n)+q(n)}), \text{ and} \\ C_3(m) &:= (a_{\alpha(n)+q(n)+1}, ..., a_n). \end{aligned} \tag{1.4}$$

Note that each $a_i \in \{0, 1\}$ for all $1 \le i \le n$. Let $s(m)$ be the number of summands in the decomposition of m. That is, let

$$s(m) \ := \ \sum_{j=1}^{n} a_j. \tag{1.5}$$

[2] As the sequence $\{F_n\}$ is exponentially growing, it is easy to pass from m in this interval to $m \in [0, F_n)$.

[3] Though in this situation it would be interesting to investigate separately the behavior on both sides.

Similarly, let $s_1(m)$, $s_2(m)$, and $s_3(m)$ be the number of summands contributed by $C_1(m)$, $C_2(m)$, and $C_3(m)$ respectively. Note that no two consecutive a_j's equal 1.

Our main result, given in the following theorem, extends the Gaussian behavior of the number of summands in Zeckendorf decompositions to smaller intervals. Note that requiring m to be in $[F_n, F_{n+1})$ is not a significant restriction because given any m, there is always an n such that this holds.

Theorem 1.1 (Gaussianity on small intervals)**.** *For $\alpha(n)$ satisfying* (1.2)*, the distribution of the number of summands in the decompositions of integers in the interval $[m, m + F_{\alpha(n)})$ converges to a Gaussian distribution when appropriately normalized for almost all $m \in [F_n, F_{n+1})$. Specifically, this means that using the notation from equations* (1.3) *and* (1.4)*, the Gaussian behavior holds for all m where there is a gap of length at least 2 in the $C_2(m)$ (and $q(n) = o(\sqrt{n})$ is an increasing even function that diverges to infinity).*

2. Preliminaries

In order to prove Theorem 1.1, we establish a correspondence between the decompositions of integers in the interval $[m, m + F_{\alpha(n)})$ and those in $[0, F_{\alpha(n)})$.

Lemma 2.1. *Let $x \in [m, m + F_{\alpha(n)})$. If there are at least two consecutive 0's in $C_2(m)$, then $C_3(x)$ is constant, and hence $s_3(x)$ is constant as well.*

Proof. Assume there are at least two consecutive 0's in $C_2(m)$. Then for some $k \in [\alpha(n) + 2, \alpha(n) + q(n))$, we have $a_{k-1} = a_k = 0$. Let m' denote the integer obtained by truncating the decomposition of m at $a_{k-2}F_{k-2}$. (Note that if $a_{k-2} = 1$, we include F_{k-2} in the truncated decomposition, and if $a_{k-2} = 0$ we do not.) Then $m' < F_{k-1}$. Since $F_{\alpha(n)} \leq F_{k-2}$, it follows that for any $h < F_{\alpha(n)}$ we have

$$m' + h \ < \ F_{k-1} + F_{k-2} \ = \ F_k, \tag{2.1}$$

and thus the decomposition of $m' + h$ has largest summand no greater than F_{k-1}. Therefore, the Zeckendorf decomposition of $m+h$ is obtained simply by concatenating the decompositions for $m - m'$ and $m' + h$. Hence $C_3(m + h) = C_3(m - m') = C_3(m)$. □

With this lemma, we see that the distribution of the number of summands involved in the decomposition of $x \in [m, m + F_{\alpha(n)})$ depends (up to a shift) only on what happens in $C_1(x)$ and $C_2(x)$, ***provided*** that there is a gap between summands of length at least two somewhere in $C_2(m)$. In light of this stipulation, we will show the following items in order to prove our main theorem.

- With high probability, m is of the desired form (i.e., there is a gap between summands of length at least two in $C_2(m)$).
- When m is of the desired form, the distribution of the number of summands involved in $C_1(x)$ for $x \in [m, m + F_{\alpha(n)})$ converges to Gaussian when appropriately normalized.
- The summands involved in $C_2(x)$ produce a negligible error term (i.e., there are significantly fewer summands from $C_2(x)$ than there are from $C_1(m)$).

We address the first point with the following lemma.

Lemma 2.2. *With probability $1 + o(1)$, there are at least 2 consecutive 0's in $C_2(m)$ if m is chosen uniformly at random from the integers in $[F_n, F_{n+1})$.*

Proof. Suppose m is not of the desired form. Recalling that $q(n)$ takes on even integer values, it follows that either $C_2(m) = (1,0,1,0,\ldots,1,0)$ or $C_2(m) = (0,1,0,1,\ldots,0,1)$. For each of these two cases, we now count the total number of ways to choose the coefficients for $C_3(m)$ and $C_1(m)$.

In the former case, we have $a_{\alpha(n)+q(n)} = 0$. Thus the number of ways to choose the coefficients for $C_3(m)$ is equal to the number of ways to legally construct

$$\sum_{j=\alpha(n)+q(n)+1}^{n} a_j F_j \tag{2.2}$$

with no nonzero consecutive coefficients and $a_n = 1$ (since $m \in [F_n, F_{n+1})$ we must select F_n). There are $F_{n-\alpha(n)-q(n)-1}$ ways to make such a construction, so we conclude that the number of ways to choose the coefficients for $C_3(m)$ is equal to $F_{n-\alpha(n)-q(n)-1}$. To see this, we argue as in [2, 5]. By shifting indices, the number of legal constructions here is the same as the number of legal ways to choose the coefficients in

$$\sum_{j=1}^{n-\alpha(n)-q(n)} \widetilde{a}_j F_j \tag{2.3}$$

where we must choose the final summand. By Zeckendorf's theorem, this is equivalent to counting the number of elements in $[F_{n-\alpha(n)-q(n)}, F_{n-\alpha(n)-q(n)+1})$, which by the Fibonacci recurrence is just $F_{n-\alpha(n)-q(n)-1}$. Thus the number of ways to choose the coefficients for $C_3(m)$ is equal to $F_{n-\alpha(n)-q(n)-1}$. Similarly, since $a_{\alpha(n)} = 0$ the number of ways to choose the coefficients for $C_1(m)$ is equal to $F_{\alpha(n)}$. Thus, if $C_2(m) = (1,0,1,0,\ldots,1,0)$, there are $F_{n-3-\alpha(n)-q(n)} F_{\alpha(n)}$ ways to choose the coefficients for $C_3(m)$ and $C_1(m)$.

A similar counting argument shows that if $C_2(m) = (0,1,0,1,\ldots,0,1)$, then the coefficients for $C_3(m)$ and $C_1(m)$ can be chosen in $F_{n-\alpha(n)-q(n)-2} F_{\alpha(n)+1}$ different ways. Therefore, since $q(n) \to \infty$ as $n \to \infty$, the probability of m not being of the desired form is

$$\frac{F_{n-\alpha(n)-q(n)-1} F_{\alpha(n)} + F_{n-\alpha(n)-q(n)-2} F_{\alpha(n)+1}}{F_{n-1}} \ \sim\ \frac{2}{\sqrt{5}} \phi^{-q(n)} \ =\ o(1). \tag{2.4}$$

□

Assuming m is of the desired form, we now consider the distribution of $s(x)$ for $x \in [m, m + F_{\alpha(n)})$.

Lemma 2.3. *If m has at least 2 consecutive 0's in $C_2(m)$, then for all $x \in [m, m + F_{\alpha(n)})$, we have*

$$0 \ \le\ s(x) - s_3(m) - s(t(x)) \ <\ q(n), \tag{2.5}$$

where $t(x)$ denotes some bijection

$$t : \mathbb{Z} \cap [m, m + F_{\alpha(n)}) \to \mathbb{Z} \cap [0, F_{\alpha(n)}). \tag{2.6}$$

Proof. First, note that the number of summands in the decomposition of x with indices $i \in [\alpha(n),\ \alpha(n) + q(n))$ must be less than $q(n)$. Next, let m_0 be the sum of the terms in the decomposition of x truncated at $a_{\alpha(n)-1} F_{\alpha(n)-1}$. Define the bijection t by

$$t(m+h) \ :=\ \begin{cases} m_0 + h, & \text{if } m_0 + h < F_{\alpha(n)} \\ m_0 + h - F_{\alpha(n)} & \text{if } m_0 + h \ge F_{\alpha(n)}. \end{cases} \tag{2.7}$$

For any $x \in [m, m + F_{\alpha(n)})$, the decompositions of $t(x)$ and x agree for the terms with index less than $\alpha(n)$. Furthermore, the decompositions of x and m agree for terms with index greater than $\alpha(n) + q(n)$. Therefore, the number of summands in the decomposition of x with indices $i \in [\alpha(n), \alpha(n) + q(n))$ is equal to $s(x) - s_3(m) - s(t(x))$. Combining this with our initial observation, the lemma now follows. □

As a result of this lemma, the distribution of $s(x)$ over the integers in $[m, m + F_\alpha)$ is a shift of its distribution over $[0, F_{\alpha(n)})$, up to an error bounded by $q(n)$. With this fact, we are now ready to prove the main theorem.

3. Proof of Theorem 1.1

We now prove our main result. The key idea is that with probability approaching 1, we have a gap of length at least 2 in the middle summands of our decompositions, and this allows us to use our bijection to reduce questions on the distribution of the number of summands in $[m, m+F_{\alpha(n)})$ to similar statements on $[0, F_{\alpha(n)})$. In doing so, the fluctuations in the difference between the two quantities is bounded by $q(n)$, which is a free parameter in our splitting of the decomposition, and can therefore be taken to be sufficiently small.

Proof. For a fixed $m \in [F_n, F_{n+1})$ with two consecutive 0's somewhere in $C_2(m)$, we define random variables X_n and Y_n by

$$X_n \ := \ s(X), \quad Y_n \ := \ s(Y), \tag{3.1}$$

where X is chosen uniformly at random from $\mathbb{Z} \cap [m, m + F_{\alpha(n)})$ and Y is chosen uniformly at random from $\mathbb{Z} \cap [0, F_{\alpha(n)})$. Let

$$X_n' \ := \ \frac{1}{\sigma_x(n)}(X_n - \mathbb{E}[X_n]), \tag{3.2}$$

and

$$Y_n' \ := \ \frac{1}{\sigma_y(n)}(Y_n - \mathbb{E}[Y_n]), \tag{3.3}$$

where $\sigma_x(n)$ and $\sigma_y(n)$ are the standard deviations of X_n and Y_n, respectively, so that X_n and Y_n are normalized with mean 0 and variance 1. It is known that the densities of Y_n' converge to the density of the standard normal[4], and we claim that X_n' converges to the standard normal as well. Though we only need the order of magnitude of $\sigma_y(n)$, for completeness we remark that the mean of Y_n is $\frac{n}{\varphi+2} + O(1)$ and the variance $\sigma_y(n)^2$ is $\frac{\varphi n}{5(\varphi+2)} + O(1)$, where $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden mean.

Let f_n and g_n be the cumulative density functions for X_n' and Y_n', respectively. By Lemma 2.3, we have

$$g_n\left(x - \frac{q(n)}{\sigma_y(n)}\right) \ \leq \ f_n(x) \ \leq \ g_n\left(x + \frac{q(n)}{\sigma_y(n)}\right). \tag{3.4}$$

Since $\sigma_y(n) \to \infty$, we may add the restriction to $q(n)$ that

$$q(n) \ = \ o\left(\sigma_y(n)\right) \ = \ o\left(\sqrt{n}\right). \tag{3.5}$$

[4] Many of the references give proofs both for the case of the Fibonacci numbers as well as for more general recurrences; see [20] for a simple proof using just Stirling's formula, which yields that the mean grows on the order of $\alpha(n)$ and the standard deviation grows on the order of $\sqrt{\alpha(n)}$.

Since $\{g_n\}_n$ converges pointwise to the cumulative distribution function for the standard normal, say $g(x)$, and since

$$\lim_{n\to\infty} g_n\left(x - \frac{q(n)}{\sigma_y(n)}\right) \;=\; \lim_{n\to\infty} g_n\left(x + \frac{q(n)}{\sigma_y(n)}\right) \;=\; g(x), \tag{3.6}$$

it follows that $\{f_n\}_n$ also converges pointwise to $g(x)$. $\square$

4. Conclusion and Future Work

We were able to handle the behavior of the number of Zeckendorf summands of numbers drawn from small intervals by finding a correspondence between Zeckendorf decompositions in the interval $[m, m + F_{\alpha(n)})$ and in the interval $[0, F_{\alpha(n)})$ when a certain gentle condition is placed on the integers m we consider. The key step was to show that almost surely an integer m chosen uniformly at random from $[F_n, F_{n+1})$ will permit the construction of a bijection onto the interval $[0, F_{\alpha(n)})$. Our results follow from previous results on the Gaussian behavior of the number of Zeckendorf summands in this interval.

Our arguments hold for more general recurrence relations (see [4]), though the arguments become more technical. There are two approaches to proving an analogue of the key step, specifically showing that for almost all m we have a sufficiently large gap in the middle section. One approach is to appeal to some high powered machinery that shows the distribution of the longest gap between summands for $m \in [F, F_{n+1})$ is strongly concentrated about $C \log\log n$, where C is some constant depending on the recurrence. Results along these lines are known for many recurrences; see [3, 5]. Of course, these results contain far more than we need; we do not need to know there is a gap as large as $C \log\log n$, but rather just that there is a gap a little longer than the length of the recurrence.

References

[1] H. Alpert, *Differences of multiple Fibonacci numbers*, Integers: Electronic Journal of Combinatorial Number Theory **9** (2009), 745–749.

[2] O. Beckwith, A. Bower, L. Gaudet, R. Insoft, S. Li, S. J. Miller and P. Tosteson, *The Average Gap Distribution for Generalized Zeckendorf Decompositions*, Fibonacci Quarterly **51** (2013), 13–27.

[3] I. Ben-Ari and S. J. Miller, *A Probabilistic Approach to Generalized Zeckendorf Decompositions*, preprint. `http://arxiv.org/pdf/1405.2379`.

[4] A. Best, P. Dynes, X. Edelsbrunner, B. McDonald, S. J. Miller, K. Tor, C. Turnage-Butterbaugh and M. Weinstein, *Gaussian Distribution of the Number of Summands in Generalized Zeckendorf Decompositions in Small Intervals*, preprint, `http://arxiv.org/pdf/1501.06833`.

[5] A. Bower, R. Insoft, S. Li, S. J. Miller and P. Tosteson, *The Distribution of Gaps between Summands in Generalized Zeckendorf Decompositions*, to appear in the Journal of Combinatorial Theory, Series A. `http://arxiv.org/abs/1402.3912`.

[6] M. Catral, P. Harris, S. J. Miller and D. Nelson, *The Fibonacci Quilt and Zeckendorf Decompositions*, preprint.

[7] M. Catral, P. Harris, S. J. Miller and D. Nelson, *Generalizing Zeckendorf's Theorem: The Kentucky Sequence*, Fibonacci Quarterly **52**, no. 5 (2014), 68–90.

[8] M. Catral, P. Harris, S. J. Miller and D. Nelson, *The Generacci Recurrences and Zeckendorf's Theorem*, preprint.

[9] D. E. Daykin, *Representation of Natural Numbers as Sums of Generalized Fibonacci Numbers*, J. London Mathematical Society **35** (1960), 143–160.

[10] P. Demontigny, T. Do, A. Kulkarni, S. J. Miller, D. Moon and U. Varma, *Generalizing Zeckendorf's Theorem to f-decompositions*, Journal of Number Theory **141** (2014), 136–158.

[11] P. Demontigny, T. Do, A. Kulkarni, S. J. Miller and U. Varma, *A Generalization of Fibonacci Far-Difference Representations and Gaussian Behavior*, to appear in the Fibonacci Quarterly.`http://arxiv.org/pdf/1309.5600v2`.

[12] M. Drmota and J. Gajdosik, *The distribution of the sum-of-digits function*, J. Théor. Nombrés Bordeaux **10** (1998), no. 1, 17–32.

[13] P. Filipponi, P. J. Grabner, I. Nemes, A. Pethö, and R. F. Tichy, *Corrigendum to: "Generalized Zeckendorf expansions"*, Appl. Math. Lett., **7** (1994), no. 6, 25–26.

[14] P. J. Grabner and R. F. Tichy, *Contributions to digit expansions with respect to linear recurrences*, J. Number Theory **36** (1990), no. 2, 160–169.

[15] P. J. Grabner, R. F. Tichy, I. Nemes, and A. Pethö, *Generalized Zeckendorf expansions*, Appl. Math. Lett. **7** (1994), no. 2, 25–28.

[16] T. J. Keller, *Generalizations of Zeckendorf's theorem*, Fibonacci Quarterly **10** (1972), no. 1 (special issue on representations), pages 95–102.

[17] M. Lamberger and J. M. Thuswaldner, *Distribution properties of digital expansions arising from linear recurrences*, Math. Slovaca **53** (2003), no. 1, 1–20.

[18] T. Lengyel, *A Counting Based Proof of the Generalized Zeckendorf's Theorem*, Fibonacci Quarterly **44** (2006), no. 4, 324–325.

[19] C. G. Lekkerkerker, *Voorstelling van natuurlyke getallen door een som van getallen van Fibonacci*, Simon Stevin **29** (1951-1952), 190–195.

[20] M. Koloğlu, G. Kopp, S. J. Miller and Y. Wang, *On the number of summands in Zeckendorf decompositions*, Fibonacci Quarterly **49** (2011), no. 2, 116–130.

[21] S. J. Miller and Y. Wang, *From Fibonacci numbers to Central Limit Type Theorems*, Journal of Combinatorial Theory, Series A **119** (2012), no. 7, 1398–1413.

[22] S. J. Miller and Y. Wang, *Gaussian Behavior in Generalized Zeckendorf Decompositions*, Combinatorial and Additive Number Theory, CANT 2011 and 2012 (Melvyn B. Nathanson, editor), Springer Proceedings in Mathematics & Statistics (2014), 159–173.

[23] W. Steiner, *Parry expansions of polynomial sequences*, Integers **2** (2002), Paper A14.

[24] W. Steiner, *The Joint Distribution of Greedy and Lazy Fibonacci Expansions*, Fibonacci Quarterly **43** (2005), 60–69.

[25] E. Zeckendorf, *Représentation des nombres naturels par une somme des nombres de Fibonacci ou de nombres de Lucas*, Bulletin de la Société Royale des Sciences de Liége **41** (1972), pages 179–182.

E-mail address: ajb5@williams.edu

Department of Mathematics and Statistics, Williams College, Williamstown, MA 01267

E-mail address: pdynes@clemson.edu

Department of Mathematical Sciences, Clemson University, Clemson, SC 29634

E-mail address: xe1@williams.edu

Department of Mathematics and Statistics, Williams College, Williamstown, MA 01267

E-mail address: bmcdon11@u.rochester.edu

Department of Mathematics, University of Rochester, Rochester, NY 14627

E-mail address: sjm1@williams.edu, Steven.Miller.MC.96@aya.yale.edu

Department of Mathematics and Statistics, Williams College, Williamstown, MA 01267

E-mail address: ktor.student@manhattan.edu

Department of Mathematics, Manhattan College, Riverdale, NY 10471

E-mail address: cturnagebutterbaugh@gmail.com

Department of Mathematics, North Dakota State University, Fargo, ND 58102

E-mail address: mweinstein@g.hmc.edu

Department of Mathematics, Harvey Mudd College, Claremont, CA 91711

CONVOLUTIONS OF TRIBONACCI, FUSS–CATALAN, AND MOTZKIN SEQUENCES

DANIEL BIRMAJER, JUAN B. GIL, AND MICHAEL D. WEINER

Abstract. We introduce a class of sequences, defined by means of partial Bell polynomials, that contains a basis for the space of linear recurrence sequences with constant coefficients as well as other well-known sequences like Catalan and Motzkin. For the family of 'Bell sequences' considered in this paper, we give a general multifold convolution formula and illustrate our result with a few explicit examples.

1. Introduction

Given numbers a and b, not both equal to zero, and given a sequence $c_1, c_2, \dots$, we consider the sequence (y_n) given by

$$y_0 = 1, \quad y_n = \sum_{k=1}^{n} \binom{an+bk}{k-1} \frac{(k-1)!}{n!} B_{n,k}(1!c_1, 2!c_2, \dots) \text{ for } n \geq 1, \tag{1.1}$$

where $B_{n,k}$ denotes the (n,k)-th partial Bell polynomial defined as

$$B_{n,k}(x_1, \dots, x_{n-k+1}) = \sum_{\alpha \in \pi(n,k)} \frac{n!}{\alpha_1! \alpha_2! \cdots \alpha_{n-k+1}!} \left(\tfrac{x_1}{1!}\right)^{\alpha_1} \cdots \left(\tfrac{x_{n-k+1}}{(n-k+1)!}\right)^{\alpha_{n-k+1}}$$

with $\pi(n,k)$ denoting the set of multi-indices $\alpha \in \mathbb{N}_0^{n-k+1}$ such that $\alpha_1 + \alpha_2 + \cdots = k$ and $\alpha_1 + 2\alpha_2 + 3\alpha_3 + \cdots = n$. For more about Bell polynomials, see e.g. [4, Chapter 3]. In general, there is no need to impose any restriction on the entries $x_1, x_2, \dots$ other than being contained in a commutative ring. Here we are mainly interested in $\mathbb{Z}$ and $\mathbb{Z}[x]$.

The class of sequences (1.1) turns out to offer a unified structure to a wide collection of known sequences. For instance, with $a = 0$ and $b = 1$, any linear recurrence sequence with constant coefficients $c_1, c_2, \dots, c_d$, can be written as a linear combination of sequences of the form (1.1). In fact, if (a_n) is a recurrence sequence satisfying $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_d a_{n-d}$ for $n \geq d$, then there are constants $\lambda_0, \lambda_1, \dots, \lambda_{d-1}$ (depending on the initial values of the sequence) such that $a_n = \lambda_0 y_n + \lambda_1 y_{n-1} + \cdots + \lambda_{d-1} y_{n-d+1}$ with

$$y_0 = 1, \quad y_n = \sum_{k=1}^{n} \frac{k!}{n!} B_{n,k}(1!c_1, 2!c_2, \dots) \text{ for } n \geq 1.$$

For more details about this way of representing linear recurrence sequences, cf. [3].

On the other hand, if $a = 1$ and $b = 0$, we obtain sequences like Catalan and Motzkin by making appropriate choices of c_1 and c_2, and by setting $c_j = 0$ for $j \geq 3$. These and other concrete examples will be discussed in sections 3 and 4.

In this paper, we focus on convolutions and will use known properties of the partial Bell polynomials to prove a multifold convolution formula for (1.1).

2. Convolution Formula

Our main result is the following formula.

Theorem 2.1. *Let $y_0 = 1$ and for $n \geq 1$,*

$$y_n = \sum_{k=1}^{n} \binom{an+bk}{k-1} \frac{(k-1)!}{n!} B_{n,k}(1!c_1, 2!c_2, \dots).$$

For $r \geq 1$, we have

$$\sum_{m_1+\cdots+m_r=n} y_{m_1} \cdots y_{m_r} = r \sum_{k=1}^{n} \binom{an+bk+r-1}{k-1} \frac{(k-1)!}{n!} B_{n,k}(1!c_1, 2!c_2, \dots). \tag{2.1}$$

In order to prove this theorem, we recall a convolution formula for partial Bell polynomials that was given by the authors in [2, Section 3, Corollary 11].

Lemma 2.2. *Let $\alpha(\ell, m)$ be a linear polynomial in ℓ and m. For any $\tau \neq 0$, we have*

$$\sum_{\ell=0}^{k} \sum_{m=\ell}^{n} \frac{\binom{\alpha(\ell,m)}{k-\ell}\binom{\tau-\alpha(\ell,m)}{\ell}\binom{n}{m}}{\alpha(\ell,m)(\tau-\alpha(\ell,m))\binom{k}{\ell}} B_{m,\ell} B_{n-m,k-\ell} = \frac{\tau-\alpha(0,0)+\alpha(k,n)}{\tau\alpha(k,n)(\tau-\alpha(0,0))} \binom{\tau}{k} B_{n,k}.$$

This formula is key for proving Theorem 2.1. For illustration purposes, we start by proving the special case of a simple convolution (i.e. $r = 2$).

Lemma 2.3. *The sequence (y_n) defined by (1.1) satisfies*

$$\sum_{m=0}^{n} y_m\, y_{n-m} = 2 \sum_{k=1}^{n} \binom{an+bk+1}{k-1} \frac{(k-1)!}{n!} B_{n,k}(1!c_1, 2!c_2, \dots).$$

Proof. We begin by assuming $a, b \geq 0$. For $n \geq 0$ we can rewrite y_n as

$$y_n = \sum_{k=0}^{n} \frac{1}{an+bk+1} \binom{an+bk+1}{k} \frac{k!}{n!} B_{n,k}(1!c_1, 2!c_2, \dots). \tag{2.2}$$

By definition,

$$\begin{aligned}
&\sum_{m=0}^{n} y_m\, y_{n-m} \\
&\quad= \sum_{m=0}^{n} \left[\sum_{\ell=0}^{m} \tfrac{1}{am+b\ell+1}\tbinom{am+b\ell+1}{\ell}\tfrac{\ell!}{m!} B_{m,\ell}\right]\left[\sum_{j=0}^{n-m} \tfrac{1}{a(n-m)+bj+1}\tbinom{a(n-m)+bj+1}{j}\tfrac{j!}{(n-m)!} B_{n-m,j}\right] \\
&\quad= \sum_{m=0}^{n}\sum_{k=0}^{n}\sum_{\ell=0}^{k} \frac{\binom{am+b\ell+1}{\ell}\binom{a(n-m)+b(k-\ell)+1}{k-\ell}}{(am+b\ell+1)(a(n-m)+b(k-\ell)+1)} \frac{\ell!}{m!}\frac{(k-\ell)!}{(n-m)!} B_{m,\ell}B_{n-m,k-\ell} \\
&\quad= \sum_{k=0}^{n} \frac{k!}{n!} \sum_{\ell=0}^{k}\sum_{m=\ell}^{n} \frac{\binom{a(n-m)+b(k-\ell)+1}{k-\ell}\binom{am+b\ell+1}{\ell}\binom{n}{m}}{(am+b\ell+1)(a(n-m)+b(k-\ell)+1)\binom{k}{\ell}} B_{m,\ell}B_{n-m,k-\ell} \\
&\quad= \sum_{k=0}^{n} \frac{k!}{n!} \left[\sum_{\ell=0}^{k}\sum_{m=\ell}^{n} \frac{\binom{\alpha(\ell,m)}{k-\ell}\binom{\tau-\alpha(\ell,m)}{\ell}\binom{n}{m}}{(\tau-\alpha(\ell,m))\alpha(\ell,m)\binom{k}{\ell}} B_{m,\ell}B_{n-m,k-\ell}\right]
\end{aligned}$$

with $\alpha(\ell, m) = a(n-m) + b(k-\ell) + 1$ and $\tau = an + bk + 2$.

Thus, by Lemma 2.2,

$$\begin{aligned}
\sum_{m=0}^{n} y_m\, y_{n-m} &= \sum_{k=0}^{n} \frac{k!}{n!} \left[\frac{\tau - \alpha(0,0) + \alpha(k,n)}{\tau\alpha(k,n)(\tau - \alpha(0,0))} \binom{\tau}{k} B_{n,k}(1!c_1, 2!c_2, \dots) \right] \\
&= \sum_{k=0}^{n} \frac{k!}{n!} \left[\frac{2}{(an+bk+2)} \binom{an+bk+2}{k} B_{n,k}(1!c_1, 2!c_2, \dots) \right] \\
&= 2\sum_{k=0}^{n} \binom{an+bk+1}{k-1} \frac{(k-1)!}{n!} B_{n,k}(1!c_1, 2!c_2, \dots).
\end{aligned}$$

For any fixed n, both sides of the claimed equation are polynomials in a and b. Since they coincide on an open subset of $\mathbb{R}^2$, they must coincide for all real numbers a and b. $\square$

Proof of Theorem 2.1. We proceed by induction in r. The case $r = 2$ was discussed in the previous lemma. Assume the formula (2.1) holds for products of length less than $r > 2$.

As before, we temporarily assume that both a and b are positive. For $n \geq 0$ we rewrite

$$\sum_{m_1+\cdots+m_{r-1}=n} y_{m_1} \cdots y_{m_{r-1}} = \sum_{k=0}^{n} \frac{r-1}{an+bk+r-1} \binom{an+bk+r-1}{k} \frac{k!}{n!} B_{n,k}(1!c_1, 2!c_2, \dots).$$

Thus

$$\begin{aligned}
\sum_{m_1+\cdots+m_r=n} y_{m_1} \cdots y_{m_r} &= \sum_{m=0}^{n} y_m \sum_{m_1+\cdots+m_{r-1}=n-m} y_{m_1} \cdots y_{m_{r-1}} \\
&= \sum_{m=0}^{n} y_m \sum_{j=0}^{n-m} \tfrac{r-1}{a(n-m)+bj+r-1} \tbinom{a(n-m)+bj+r-1}{j} \tfrac{j!}{(n-m)!} B_{n-m,j}.
\end{aligned}$$

Writing y_m as in (2.2), we then get

$$\begin{aligned}
&\frac{1}{r-1} \sum_{m_1+\cdots+m_r=n} y_{m_1} \cdots y_{m_r} \\
&\quad = \sum_{m=0}^{n} \left[\sum_{\ell=0}^{m} \frac{\binom{am+b\ell+1}{\ell}\ell!}{(am+b\ell+1)m!} B_{m,\ell} \right] \left[\sum_{j=0}^{n-m} \frac{\binom{a(n-m)+bj+r-1}{j} j!}{(a(n-m)+bj+r-1)(n-m)!} B_{n-m,j} \right] \\
&\quad = \sum_{m=0}^{n} \sum_{k=0}^{n} \sum_{\ell=0}^{k} \frac{\binom{a(n-m)+b(k-\ell)+r-1}{k-\ell}\binom{am+b\ell+1}{\ell}}{(am+b\ell+1)(a(n-m)+b(k-\ell)+r-1)} \frac{\ell!}{m!} \frac{(k-\ell)!}{(n-m)!} B_{m,\ell} B_{n-m,k-\ell} \\
&\quad = \sum_{k=0}^{n} \frac{k!}{n!} \left[\sum_{\ell=0}^{k} \sum_{m=\ell}^{n} \frac{\binom{\alpha(\ell,m)}{k-\ell}\binom{\tau-\alpha(\ell,m)}{\ell}\binom{n}{m}}{(\tau - \alpha(\ell,m))\alpha(\ell,m)\binom{k}{\ell}} B_{m,\ell} B_{n-m,k-\ell} \right]
\end{aligned}$$

with $\alpha(\ell, m) = a(n-m) + b(k-\ell) + r - 1$ and $\tau = an + bk + r$.

Finally, by Lemma 2.2,

$$\begin{aligned}\sum_{m_1+\cdots+m_r=n} y_{m_1}\cdots y_{m_r} &= (r-1)\sum_{k=0}^{n}\frac{k!}{n!}\left[\frac{\tau-\alpha(0,0)+\alpha(k,n)}{\tau\alpha(k,n)(\tau-\alpha(0,0))}\binom{\tau}{k}B_{n,k}(1!c_1,2!c_2,\dots)\right]\\ &= (r-1)\sum_{k=0}^{n}\frac{k!}{n!}\left[\frac{r\binom{an+bk+r}{k}}{(an+bk+r)(r-1)}B_{n,k}(1!c_1,2!c_2,\dots)\right]\\ &= r\sum_{k=0}^{n}\binom{an+bk+r-1}{k-1}\frac{(k-1)!}{n!}B_{n,k}(1!c_1,2!c_2,\dots).\end{aligned}$$

As in the previous lemma, this equation actually holds for all $a,b\in\mathbb{R}$ as claimed. □

3. Examples: Fibonacci, Tribonacci, Jacobsthal

As mentioned in the introduction, sequences of the form (1.1) with $a=0$ and $b=1$ can be used to describe linear recurrence sequences with constant coefficients. In this case, (1.1) takes the form

$$y_n=\sum_{k=0}^{n}\frac{k!}{n!}B_{n,k}(1!c_1,2!c_2,\dots)\text{ for } n\ge 0, \tag{3.1}$$

and the convolution formula (2.1) turns into

$$\begin{aligned}\sum_{m_1+\cdots+m_r=n} y_{m_1}\cdots y_{m_r} &= r\sum_{k=1}^{n}\binom{k+r-1}{k-1}\frac{(k-1)!}{n!}B_{n,k}(1!c_1,2!c_2,\dots)\\ &= \sum_{k=1}^{n}\binom{k+r-1}{k}\frac{k!}{n!}B_{n,k}(1!c_1,2!c_2,\dots).\end{aligned}$$

One can obtain (with a similar proof) the more general formula

$$\sum_{m_1+\cdots+m_r=n} y_{m_1-\delta}\cdots y_{m_r-\delta}=\sum_{k=0}^{n-\delta r}\binom{k+r-1}{k}\frac{k!}{(n-\delta r)!}B_{n-\delta r,k}(1!c_1,2!c_2,\dots)$$

for any integer $\delta\ge 0$, assuming $y_{-1}=y_{-2}=\cdots=y_{-\delta}=0$.

Example 3.1 (Fibonacci)**.** *Consider the sequence defined by*

$$f_0=0,\ \ f_1=1,\ \ \textit{and}\ \ f_n=f_{n-1}+f_{n-2}\ \ \textit{for } n\ge 2.$$

Choosing $c_1=c_2=1$ and $c_j=0$ for $j\ge 3$ in (3.1), for $n\ge 1$ we have

$$f_n=y_{n-1}=\sum_{k=0}^{n-1}\frac{k!}{(n-1)!}B_{n-1,k}(1,2,0,\dots)=\sum_{k=0}^{n-1}\binom{k}{n-1-k},$$

and

$$\sum_{m_1+\cdots+m_r=n} f_{m_1}\cdots f_{m_r}=\sum_{k=0}^{n-r}\binom{k+r-1}{k}\binom{k}{n-r-k}.$$

Example 3.2 (Tribonacci)**.** *Let (t_n) be the sequence defined by*

$$t_0=t_1=0,\ \ t_2=1,\ \ \textit{and}\ \ t_n=t_{n-1}+t_{n-2}+t_{n-3}\ \textit{for } n\ge 3.$$

Choosing $c_1 = c_2 = c_3 = 1$ and $c_j = 0$ for $j \geq 4$ in (3.1), for $n \geq 2$ we have

$$t_n = y_{n-2} = \sum_{k=0}^{n-2} \frac{k!}{(n-2)!} B_{n-2,k}(1!, 2!, 3!, 0, \dots),$$

and since $B_{n,k}(1!, 2!, 3!, 0, \dots) = \frac{n!}{k!} \sum_{\ell=0}^{k} \binom{k}{k-\ell}\binom{k-\ell}{n+\ell-2k} = \frac{n!}{k!} \sum_{\ell=0}^{k} \binom{k}{\ell}\binom{\ell}{n-k-\ell}$, we get

$$t_n = \sum_{k=0}^{n-2} \sum_{\ell=0}^{k} \binom{k}{\ell}\binom{\ell}{n-2-k-\ell},$$

and

$$\sum_{m_1+\cdots+m_r=n} t_{m_1} \cdots t_{m_r} = \sum_{k=0}^{n-2r} \sum_{\ell=0}^{k} \binom{k+r-1}{k}\binom{k}{\ell}\binom{\ell}{n-2r-k-\ell}.$$

Example 3.3 (Jacobsthal)**.** *The Jacobsthal polynomials are obtained by the recurrence*

$$J_0 = 0, \quad J_1 = 1, \ \text{and}$$
$$J_n = J_{n-1} + 2xJ_{n-2} \ \text{ for } n \geq 2.$$

Choosing $c_1 = 1$, $c_2 = 2x$, and $c_j = 0$ for $j \geq 3$ in (3.1), for $n \geq 1$ we get

$$J_n = y_{n-1} = \sum_{k=0}^{n-1} \frac{k!}{(n-1)!} B_{n-1,k}(1, 2(2x), 0, \dots) = \sum_{k=0}^{n-1} \binom{k}{n-1-k} (2x)^{n-1-k},$$

and

$$\begin{aligned} \sum_{m_1+\cdots+m_r=n} J_{m_1} \cdots J_{m_r} &= \sum_{k=0}^{n-r} \frac{k!}{(n-r)!} \binom{k+r-1}{k} B_{n-r,k}(1, 4x, 0, \dots) \\ &= \sum_{k=0}^{n-r} \binom{k+r-1}{k}\binom{k}{n-r-k} (2x)^{n-r-k}. \end{aligned}$$

4. Examples: Fuss–Catalan, Motzkin

All of the previous examples are related to the family (3.1). However, there are many other cases of interest. For example, let us consider the case when $a = 1$, $b = 0$, and $c_j = 0$ for $j \geq 3$. Since $B_{n,k}(c_1, 2c_2, 0, \dots) = \frac{n!}{k!}\binom{k}{n-k} c_1^{2k-n} c_2^{n-k}$, the family (1.1) can be written as

$$y_0 = 1, \quad y_n = \sum_{k=1}^{n} \frac{1}{k} \binom{n}{k-1}\binom{k}{n-k} c_1^{2k-n} c_2^{n-k} \text{ for } n \geq 1, \tag{4.1}$$

and the convolution formula (2.1) becomes

$$\sum_{m_1+\cdots+m_r=n} y_{m_1} \cdots y_{m_r} = \sum_{k=1}^{n} \frac{r}{k} \binom{n+r-1}{k-1}\binom{k}{n-k} c_1^{2k-n} c_2^{n-k}. \tag{4.2}$$

Example 4.1 (Catalan)**.** *If we let $c_1 = 2$ and $c_2 = 1$ in* (4.1)*, for $n \geq 1$ we get*

$$\begin{aligned} y_n &= \sum_{k=1}^{n} \frac{1}{k}\binom{n}{k-1}\binom{k}{n-k}2^{2k-n} \\ &= \frac{1}{n+1}\sum_{k=1}^{n}\binom{n+1}{k}\binom{k}{n-k}2^{2k-n} \\ &= \frac{1}{n+1}\binom{2(n+1)}{n} = \frac{1}{n+2}\binom{2(n+1)}{n+1} = C_{n+1}. \end{aligned}$$

Here we used the identity

$$\sum_{k=\lceil \frac{n}{2}\rceil}^{n} \binom{x}{k}\binom{k}{n-k}2^{2k} = 2^n\binom{2x}{n} \tag{4.3}$$

from Gould's collection [5, Identity (3.22)]*. As for convolutions,* (4.2) *leads to*

$$\begin{aligned} \sum_{m_1+\cdots+m_r=n} C_{m_1+1}\cdots C_{m_r+1} &= \sum_{k=1}^{n}\frac{r}{k}\binom{n+r-1}{k-1}\binom{k}{n-k}2^{2k-n} \\ &= \frac{r}{n+r}\sum_{k=1}^{n}\binom{n+r}{k}\binom{k}{n-k}2^{2k-n}. \end{aligned}$$

Using again (4.3)*, we arrive at the identity*

$$\sum_{m_1+\cdots+m_r=n} C_{m_1+1}\cdots C_{m_r+1} = \frac{r}{n+r}\binom{2(n+r)}{n}.$$

Example 4.2 (Motzkin)**.** *Let us now consider* (4.1) *with $c_1 = 1$ and $c_2 = 1$. For $n \geq 1$,*

$$y_n = \sum_{k=1}^{n}\frac{1}{k}\binom{n}{k-1}\binom{k}{n-k} = \frac{1}{n+1}\sum_{k=1}^{n}\binom{n+1}{k}\binom{k}{n-k}.$$

These are the Motzkin numbers M_n. Moreover,

$$\sum_{m_1+\cdots+m_r=n} M_{m_1}\cdots M_{m_r} = \frac{r}{n+r}\sum_{k=0}^{n}\binom{n+r}{k}\binom{k}{n-k}.$$

We finish this section by considering the sequence (with $b \neq 0$):

$$y_0 = 1, \quad y_n = \sum_{k=1}^{n}\binom{bk}{k-1}\frac{(k-1)!}{n!}B_{n,k}(1!c_1, 2!c_2, \dots) \text{ for } n \geq 1.$$

Example 4.3 (Fuss–Catalan)**.** *If $c_1 = 1$ and $c_j = 0$ for $j \geq 2$, then the above sequence becomes*

$$y_0 = 1, \quad y_n = \binom{bn}{n-1}\frac{(n-1)!}{n!} = \frac{1}{(b-1)n+1}\binom{bn}{n}.$$

Denoting $C_n^{(b)} = y_n$, and since $r\binom{bn+r-1}{n-1}\frac{(n-1)!}{n!} = \frac{r}{bn+r}\binom{bn+r}{n}$, we get the identity

$$\sum_{m_1+\cdots+m_r=n} C_{m_1}^{(b)}\cdots C_{m_r}^{(b)} = \frac{r}{bn+r}\binom{bn+r}{n}.$$

References

[1] E. T. Bell, *Exponential polynomials*, Ann. of Math. **35** (1934), 258–277.
[2] D. Birmajer, J. Gil, and M. Weiner, *Some convolution identities and an inverse relation involving partial Bell polynomials*, Electron. J. Combin. **19** (2012), no. 4, Paper 34, 14 pp.
[3] D. Birmajer, J. Gil, and M. Weiner, *Linear recurrence sequences and their convolutions via Bell polynomials*, preprint, http://arxiv.org/abs/1405.7727.
[4] L. Comtet, *Advanced Combinatorics: The Art of Finite and Infinite Expansions*, D. Reidel Publishing Co., Dordrecht, 1974.
[5] H. W. Gould, *Combinatorial Identities*, Morgantown Printing and Binding Co., 1972.

MSC2010: 11B37, 05A19

Department of Mathematics, Nazareth College, 4245 East Ave., Rochester, NY 14618, U.S.A.
E-mail address: abirmaj6@naz.edu

Penn State Altoona, 3000 Ivyside Park, Altoona, PA 16601, U.S.A.
E-mail address: jgil@psu.edu

Penn State Altoona, 3000 Ivyside Park, Altoona, PA 16601, U.S.A.
E-mail address: mdw8@psu.edu

STEINHAUS TRIANGLES WITH GENERALIZED PASCAL ADDITION

JENS-P. BODE AND HEIKO HARBORTH

Abstract. For a given row of 0s and 1s the following row is determined by the sums mod 2 of s consecutive entries each. If this operation is repeated as long as possible then a generalized Steinhaus triangle is obtained which is called balanced if there are as many 0s as 1s. Necessary conditions for the existence of balanced Steinhaus triangles are determined. Constructions are given in most of the cases for odd s and in some cases for even s.

1. Introduction

Consider a sequence $a = (a_1, a_2, \dots, a_n)$ of length n with $a_i \in \{0, 1\}$. For $n \geq s \geq 2$ the derivative $a' = (a'_1, a'_2, \dots, a'_{n-s+1})$ is defined by $a'_i = (a_i + a_{i+1} + \dots + a_{i+s-1}) \bmod 2$ being the addition as in a generalized Pascal triangle [2]. A given sequence a determines a finite sequence $\nabla = \nabla_s(a) = (a, a', a'', \dots, a^{(t-1)})$ of length $t = \lceil n/(s-1) \rceil$ where $a^{(t-1)}$ is of length less than s and thus has no further derivative. The sequence ∇ can be represented in triangular arrangements, for example, as in Figure 1 for $s = 3$ and $a = 1101010$. These triangles will be

$$
\begin{array}{ccccccc}
1&1&0&1&0&1&0\\
&&0&0&1&0&1\\
&&&&1&1&0\\
&&&&&&0
\end{array}
\qquad
\begin{array}{ccccccc}
1&1&0&1&0&1&0\\
&0&0&1&0&1&\\
&&1&1&0&&\\
&&&0&&&
\end{array}
\qquad
\begin{array}{ccccccc}
1&1&0&1&0&1&0\\
0&0&1&0&1&&\\
1&1&0&&&&\\
0&&&&&&
\end{array}
$$

Figure 1. Triangular arrangements of $\nabla_3(1101010)$.

called (generalized) Steinhaus triangles since they have been introduced for $s = 2$ in [9], that is, using the classical Pascal addition. A Steinhaus triangle is called balanced if there occur as many 0s as 1s in the whole triangle as for example in Figure 1. As Steinhaus did for $s = 2$ we will ask for the existence of balanced Steinhaus triangles for general s and all lengths n.

A first solution for $s = 2$ has been presented in [8] already. A further generalization where $a_i \in \{0, 1, \dots, m-1\}$ and $a'_i = (a_i + a_{i+1}) \bmod m$ and corresponding references can be found in [3, 4, 5]. For $a'_i = |a_i - a_{i+1}|$ see [1]. Classical Steinhaus triangles ($s = 2$) have been interpreted as incidence matrices of so-called Steinhaus graphs (see for example [6, 7]).

2. Necessary conditions

At first the number $P(n, s)$ of entries in a Steinhaus triangle $\nabla_s(a_1, \dots, a_n)$ will be determined.

Theorem 2.1. *For $n = (t-1)(s-1) + j$ with $1 \leq j \leq s-1$ and $t \geq 1$ it holds*

$$P(n, s) = \frac{t(n+j)}{2}.$$

Proof. In row r of the t rows there are $(t-r)(s-1)+j$ entries for $1 \le r \le t$. The sum of the t elements of this arithmetic progression having n as first element and j as last element results in the asserted formula. □

Next the values of n will be determined for which $P(n,s)$ is even being necessary for the existence of a balanced Steinhaus triangle.

Theorem 2.2. *The number $P(n,s)$ is even if and only if*

(1) s *odd and*

(1.1) $n \equiv 0 \pmod 2$ *or*

(1.2) $n \equiv s, s+2, \ldots, 2s-3 \pmod{2s-2}$

or

(2) s *even and*

(2.1) $n \equiv 0, 2, \ldots, s-2 \pmod{4s-4}$ *or*

(2.2) $n \equiv 2s-1, 2s+1, \ldots, 3s-3 \pmod{4s-4}$ *or*

(2.3) $n \equiv 3s-2, 3s-1, \ldots, 4s-5 \pmod{4s-4}$.

Proof. Using $n = (t-1)(s-1)+j$, $1 \le j \le s-1$, we have

$$P = P(n,s) = \frac{t((t-1)(s-1)+2j)}{2}.$$

Let s be odd. If $t = 2i+1$ then $P = (2i+1)(i(s-1)+j)$ which is even if and only if j is even, that is, $j = 2, 4, \ldots, s-1$. Since $n = i(2s-2)+j$ it follows $n \equiv 2, 4, \ldots s-1 \pmod{2s-2}$. If $t = 2i$ then $P = i((2i-1)(s-1)+2j)$ which is always even, that is, for $j = 1, 2, \ldots s-1$. Since $n = (2i-1)((s-1)+j) = (i-1)(2s-2)+s+j-1$ it follows $n \equiv s, s+1, \ldots, 2s-2 \pmod{2s-2}$. Together both sets of values of n correspond to (1.1) and (1.2).

Let s be even. If $t = 4i+1$ then $P = (4i+1)(2i(s-1)+j)$ is even if and only if j is even, that is, $j = 2, 4, \ldots, s-2$. From $n = i(4s-4)+j$ we obtain $n \equiv 2, 4, \ldots, s-2 \pmod{4s-4}$. If $t = 4i+3$ then $P = (4i+3)((2i+1)(s-1)+j)$ is even if and only if j is odd, that is, $j = 1, 3, \ldots, s-1$. From $n = (4i+2)(s-1)+j = i(4s-4)+2s+j-2$ we have $n \equiv 2s-1, 2s+1, \ldots, 3s-3 \pmod{4s-4}$. If $t = 4i+2$ then $P = (2i+1)((4i+1)(s-1)+2j)$ which is always odd. If $t = 4i$ then $P = 2i((4i-1)(s-1)+2j)$ which is always even, that is, for $j = 1, 2, \ldots, s-1$. With $n = (4i-1)(s-1)+j = (i-1)(4s-4)+3s+j-3$ we obtain $n \equiv 3s-2, 3s-1, \ldots, 4s-4 \pmod{4s-4}$. Together these three sets of values of n correspond to (2.1) to (2.3). □

Theorem 2.3. *If $s \equiv 1 \pmod 4$ and $n = s$ or if $s \equiv 0 \pmod 2$, $s > 2$, and $n = 3s$ then balanced Steinhaus triangles do not exist although $P(n,s)$ is even.*

Proof. If $s \equiv 1 \pmod 4$ and $n = s$ then $P(n,s) = s+1 \equiv 2 \pmod 4$ follows since we have $a = (a_1, a_2, \ldots, a_n)$ and $a' = (a'_1)$ with $a'_1 = (a_1 + a_2 + \ldots + a_n) \bmod 2$. Now the total number of 1s is $a_1 + a_2 + \ldots + a_n + a'_1 \equiv 2(a_1 + a_2 + \ldots + a_n) \equiv 0 \pmod 2$ so that the triangle cannot be balanced since $P(n,s)/2 \equiv 1 \pmod 2$.

If $s \equiv 0 \pmod 2$, $s > 2$, and $n = 3s$ then $t = \lceil 3s/(s-1) \rceil = 3 + \lceil 3/(s-1) \rceil = 4$ and $j = 3$. This implies $P(n,s)/2 = 4(3s+3)/4 = 3(s+1) \equiv 1 \pmod 2$. Then the triangle cannot be balanced if we prove that the number of 1s is always even, that is, the total sum of the entries is even.

This is the case if the number of entries in which each a_i occurs an odd number of times is even, that is, in a triangle with only one 1 in the first row the total number of 1s is always even. We start with a triangle with 0s only (see italic values in Figure 2 for $s = 6$), thus we have an even number of 1s. Then we shift the single 1 from right to left through the first row.

$$\begin{array}{c}
\boldsymbol{\mathit{0}}\ \mathit{0\ 0\ \ 0\ 0\ 0\ 0\ 0\ \ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0}\ \underline{\mathbf{1}}\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \\
\boldsymbol{\mathit{0}}\ \mathit{0\ 0\ \ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0}\ \underline{1}\ 1\ 1\ 1\ 1\ \mathbf{1}\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \\
\boldsymbol{\mathit{0}}\ \mathit{0\ 0\ \ 0\ 0\ 0\ 0\ 0}\ \underline{1}\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ \mathbf{1}\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \\
\boldsymbol{\mathit{0}}\ \mathit{0\ 0}\ \underline{1}\ 1\ 0\ 0\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 0\ 0\ 1\ \mathbf{1}\ 0\ 0\ 0
\end{array}$$

FIGURE 2. Example for $s = 6$.

The corresponding derivatives are shifted simultaneously. In the first step the four underlined 1s are shifted into the triangle and at the same time the four bold 0s at the left border leave the triangle. Due to symmetry the four underlined 1s correspond to the four bold 1s. Figures 3 and 4 show a, a', a'', and a''' for general $s \equiv 0$ and $s \equiv 2 \pmod 4$ such that the four bold 1s and the four bold 0s are in one column each, that is, in columns $i = 0$ and $i = 3s$. In the

$$\begin{array}{rl}
i & = 0\ 1 \qquad \ldots \qquad s \quad \ldots \quad 2s \qquad \ldots \qquad 3s \\
a & = \mathbf{0}|0\ 0\ 0\ 0 \ldots 0\ 0\ 0\ 0|0\ 0 \ldots 0\ 0|0\ 0\ 0|0\ 0\ 0\ 0 \ldots 0\ 0\ 0\ 0|\mathbf{1} \\
a' & = \mathbf{0}|0\ 0\ 0\ 0 \ldots 0\ 0\ 0\ 0|0\ 0 \ldots 0\ 0|1\ 1\ 1|1\ 1\ 1\ 1 \ldots 1\ 1\ 1\ 1|\mathbf{1} \\
a'' & = \mathbf{0}|0\ 0\ 0\ 0 \ldots 0\ 0\ 0\ 0|0\ 1 \ldots 0\ 1|0\ 1\ 0|1\ 0\ 1\ 0 \ldots 1\ 0\ 1\ 0|\mathbf{1} \\
a''' & = \mathbf{0}|0\ 0\ 1\ 1 \ldots 0\ 0\ 1\ 1|0\ 0 \ldots 0\ 0|0\ 0\ 1|1\ 0\ 0\ 1 \ldots 1\ 0\ 0\ 1|\mathbf{1} \\
\sigma & = 0|0\ 0\ 1\ 1 \ldots 0\ 0\ 1\ 1|0\ 1 \ldots 0\ 1|1\ 0\ 0|1\ 1\ 0\ 0 \ldots 1\ 1\ 0\ 0|0
\end{array}$$

FIGURE 3. $s \equiv 0 \pmod 4$.

$$\begin{array}{rl}
i & = 0\ 1 \qquad \ldots \qquad s \quad \ldots \quad 2s \qquad \ldots \qquad 3s \\
a & = \mathbf{0}\ 0\ 0|0\ 0\ 0\ 0 \ldots 0\ 0\ 0\ 0|0\ 0 \ldots 0\ 0|0|0\ 0\ 0\ 0 \ldots 0\ 0\ 0\ 0|\mathbf{1} \\
a' & = \mathbf{0}\ 0\ 0|0\ 0\ 0\ 0 \ldots 0\ 0\ 0\ 0|0\ 0 \ldots 0\ 0|1|1\ 1\ 1\ 1 \ldots 1\ 1\ 1\ 1|\mathbf{1} \\
a'' & = \mathbf{0}\ 0\ 0|0\ 0\ 0\ 0 \ldots 0\ 0\ 0\ 0|0\ 1 \ldots 0\ 1|0|1\ 0\ 1\ 0 \ldots 1\ 0\ 1\ 0|\mathbf{1} \\
a''' & = \mathbf{0}\ 0\ 0|1\ 1\ 0\ 0 \ldots 1\ 1\ 0\ 0|1\ 1 \ldots 1\ 1|1|1\ 0\ 0\ 1 \ldots 1\ 0\ 0\ 1|\mathbf{1} \\
\sigma & = 0\ 0\ 0|1\ 1\ 0\ 0 \ldots 1\ 1\ 0\ 0|1\ 0 \ldots 1\ 0|0|1\ 1\ 0\ 0 \ldots 1\ 1\ 0\ 0|0
\end{array}$$

FIGURE 4. $s \equiv 2 \pmod 4$.

following steps the four entries in column $i = 3s - j$ are shifted into the triangle and at the same time the four entries in column $i = j$ leave the triangle for $j = 1, \ldots, 3s - 1$. The total number of 1s in the triangle remains even if the sum of the entering and leaving entries is even in every step. This can be checked in Figures 3 and 4 since for the sums σ_i of the columns modulo 2 it holds $\sigma_j = \sigma_{3s-j}$ where $\sigma = (\sigma_0, \sigma_1, \ldots, \sigma_{3s})$. □

Theorem 2.3 shows that some of the conditions for n in Theorem 2.2 are not sufficient. For $s = 2^u$ we conjecture in addition to Theorem 2.3 that there exist no balanced Steinhaus triangles for $n = s(4s^i - 1)$, $i = 1, 2, \ldots$, although $P(n, s)$ is even. This has been checked by computer for $s, n \leq 100000$. Moreover, we conjecture that balanced Steinhaus triangles exist in all other cases where $P(n, s)$ is even.

3. Odd values of s

For one half of the odd values of s the existence of balanced triangles can be proved completely.

Theorem 3.1. *For $s \equiv 3 \pmod 4$ balanced Steinhaus triangles exist for all n fulfilling the necessary conditions (1.1) or (1.2) in Theorem 2.2.*

Proof. If $a = 0101\ldots$ then $a' = 1010\ldots$, $a'' = 0101\ldots$ and thus every second row has the same pattern. For n even the triangle is balanced since all rows are of even length. For n odd all rows are of odd length and all pairs of consecutive rows are balanced. Then the triangle is balanced if the number t of rows is even which is the case for the odd values n in (1.2), see the proof of Theorem 2.2. $\square$

For the other half of the odd values of s one residue class remains open up to some small examples.

Theorem 3.2. *For $s \equiv 1 \pmod 4$ balanced Steinhaus triangles exist for all $n \equiv 0 \pmod 2$ and all $n \equiv s+2, s+4, \ldots, 2s-3 \pmod{2s-2}$.*

Proof. If $a = 0101\ldots$ it follows $a' = 0101\ldots$ so that every row has the same pattern and the triangle is balanced for all even values of n since then all rows are of even length.

For odd n we choose $a = 00(01)1$ to obtain $a' = 10(01)1$ and $a'' = 00(01)1$ where the part in brackets can be repeated arbitrarily often. It follows that all pairs of consecutive rows are balanced. Thus for $t = 2i+2$ the triangle is balanced, that is, for $n = (t-1)(s-1)+j = i(2s-2)+s-1+j$. With $j = 3, 5, \ldots, s-2$ we have obtained a solution for $n \equiv s+2, s+4, \ldots, 2s-3 \pmod{2s-2}$. $\square$

Note that the construction in the preceding proof is not possible for $j = 1$, that is, for $n \equiv s \pmod{2s-2}$. Due to Theorems 2.2 and 2.3 for $s \equiv 1 \pmod 4$, $n = s + i(2s-2)$, $i = 1, 2, \ldots$, balanced triangles may be possible. The following theorem guarantees such balanced triangles for some residue classes of n by general construction.

Theorem 3.3. *For $s \equiv 1 \pmod 4$ and $s \geq 1 + 2^c$, $c \geq 1$, balanced Steinhaus triangles exist for $n \equiv s + (s-1)2^c \pmod{(2s-2)2^c}$.*

Proof. We define $2^c \times 2^c$ matrices M_c, $c \geq 1$, recursively by

$$M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad M_{c+1} = \left(\begin{array}{c|c} M_c & M_c \\ \hline 0\ 1\ \ldots\ 0\ 1 & \\ \ldots & M_c \\ 0\ 1\ \ldots\ 0\ 1 & \end{array}\right).$$

For example,

$$M_2 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

If every row of M_c is augmented by an infinite sequence 0101... to the right hand side then for $s \equiv 1 \pmod 4$ and $s \geq 2^c$ the first row is the derivative of the last row and each of all other rows is the derivative of the preceding row. This can be seen by induction on c using that each but the last row of M_c has an even number of 1s for $c \geq 2$.

Now let P_c be the $2^{c+1} \times (1+2^c)$ matrix consisting of the $1+2^c$ rightmost columns of M_{c+1}. Note that the first column of P_c consists of 2^c consecutive 0s followed by 2^c consecutive 1s. Let Q_c be the $2^{c+1} \times 2^c$ matrix obtained from the 2^c rightmost columns of M_{c+1} by horizontal reflection and taking the complement. Then we define the trapezoid-like scheme T_c by

$$T_c = \left(\begin{array}{c|ccccc|c} & 0 & 1 & \dots & 0 & 1 & \\ P_c & & & \dots & & & Q_c \\ & 0 & 1 & \dots & 0 & 1 & \end{array} \right)$$

where each row has $(s-1)/2$ pairs 01 less that its preceding one in the central part. Note that the first row of T_c has the same pattern (only different numbers of pairs 01 in the central part) as the derivative of the last row and each of all other rows is the derivative of its preceding one. Moreover, T_c is balanced since the first column is balanced and each row of the rest of T_c is balanced, since the rest is the complement of its mirror image. For example, see Figure 5.

$$T_2 = \left(\begin{array}{ccccc|ccccc|cccc} 0 & 1 & 0 & 1 & 0 & 0 & 1 & \dots & 0 & 1 & 1 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 & \dots & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & \dots & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & \dots & 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & \dots & 0 & 1 & 1 & 0 & 1 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 1 & \dots & 0 & 1 & 1 & 1 & 1 & 1 \\ \hline 1 & 0 & 1 & 1 & 0 & 0 & 1 & \dots & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & \dots & 0 & 1 & 1 & 1 & 0 & 1 \end{array} \right)$$

FIGURE 5. T_2.

For $s \geq 1 + 2^{c+1}$ we choose from T_c the last $1 + 2^{c-1}$ rows beginning with 0 followed by the first 2^{c-1} rows beginning with 1 (see Figure 6 for $s = 9$ and 13 with the first and last horizontal line of Figure 5). If there are $(s - (1 + 2^{c+1}))/2$ pairs 01 in the last of these chosen rows then

$$\begin{array}{l} \hline 0\,0\,0\,0\,0|0\,1 \ \dots \ 0\,1|1\,1\,1\,1 \\ 0\,0\,1\,1\,0|0\,1\,0\,1\,0\,1\,0\,1|1\,0\,0\,1 \\ 0\,0\,1\,0\,0|0\,1\,0\,1|1\,1\,0\,1 \\ 1\,1\,0\,1\,0|1\,0\,1\,0 \\ \hline 1\,1\,0\,1\,0 \\ 1 \end{array} \qquad \begin{array}{l} \hline 0\,0\,0\,0\,0|0\,1 \ \dots \ 0\,1|1\,1\,1\,1 \\ 0\,0\,1\,1\,0|0\,1 \ \dots \ 0\,1|1\,0\,0\,1 \\ 0\,0\,1\,0\,0|0\,1 \ \dots \ 0\,1|1\,1\,0\,1 \\ 1\,1\,0\,1\,0|0\,1\,0\,1\,0\,1\,0\,1|1\,0\,1\,0 \\ 1\,0\,0\,0\,0|1\,1\,1\,1 \\ \hline 1 \end{array}$$

$$\begin{array}{l} \hline 0\,0\,0\,0\,0|0\,1 \qquad \dots \qquad 0\,1|1\,1\,1\,1 \\ 0\,0\,1\,1\,0|0\,1 \qquad \dots \qquad 0\,1|1\,0\,0\,1 \\ 0\,0\,1\,0\,0|0\,1 \qquad \dots \qquad 0\,1|1\,1\,0\,1 \\ 1\,1\,0\,1\,0|0\,1\,0\,1\,0\,1\,0\,1\,0\,1\,0\,1\,0\,1\,0\,1|1\,0\,1\,0 \\ 1\,0\,0\,0\,0|0\,1\,0\,1|1\,1\,1\,1 \\ \hline 1 \end{array}$$

FIGURE 6. $(s, n) = (5, 21)$, $(9, 41)$, and $(13, 61)$.

it has s entries and its derivative is a single 1. Thus we have constructed a Steinhaus triangle being balanced since the first column has as many 1s as 0s. If on top of this triangle $i2^{c+1}$ rows are added which are the cyclically preceding rows of T_c then we obtain balanced triangles for

$$\begin{aligned} n &= 1 + (s-1)(2^{c-1} + 2^{c-1} + 1 + i2^{c+1}) \\ &= s + (2^{c-1} + i2^c)(2s-2) \\ &\equiv s + (s-1)2^c \pmod{(2s-2)2^c}. \end{aligned}$$

For $1+2^c \leq s \leq 2^{c+1}$ we can proceed similar to the preceding case. From T_c we now choose the last $1+2^{c-1}$ rows beginning with 0 followed by the first $2^{c-1}-1$ rows beginning with 1 (see Figure 6 for $s=5$ with the first and second horizontal line of Figure 5). If there are $s-1-2^c$ pairs 01 in the last of the chosen rows then it has $1+2^{c+1}+2(s-1-2^c) = 2s-1$ entries and its derivative has length s and starts with 1. The rest of this row is the complement of its mirror image and thus it is balanced and has an even number of 1s. Then the derivative of this row is a single 1 and we have balanced triangles for

$$n = 1 + (s-1)(1 + 2^{c-1} - 1 + 2^{c-1} + 1 + i2^{c+1})$$

as above. □

For $s \equiv 1 \pmod 4$ and c such that $1+2^c \leq s \leq 2^{c+1}$ by Theorems 2.2, 2.3, 3.2, and 3.3 only the cases $n \equiv s \pmod{(2s-2)2^c}$ remain open. For $s=5$ and $s=9$ balanced Steinhaus triangles exist in these cases:

$$\begin{array}{lll} s=5: & n \equiv 5 \pmod{64},\ n \neq 5: & a = 11111111110(01), \\ & n \equiv 37 \pmod{64}: & a = 00100100010(01), \\ s=9: & n \equiv 9 \pmod{256},\ n \neq 9: & \\ & \quad a = 0101110111101000100(01)11011101000100010, & \\ & n \equiv 137 \pmod{256}: & \\ & \quad a = 00101011001010100(01)1101010110010101, & \end{array}$$

where the part between the brackets is repeated appropriately.

4. Even values of s

The case $s=2$ is solved in [8]. Moreover, for $s \equiv 0 \pmod 2$ we so far have found balanced Steinhaus triangles for some small values of s only:

$$\begin{array}{lll} s=4: & n \equiv 0 \pmod{24}: & a = 000100(001111), \\ & n \equiv 2 \pmod{12}: & a = 1000(011110)1000, \\ & n \equiv 7 \pmod{12}: & a = 100(010100)1000, \\ & n \equiv 9 \pmod{12}: & a = 1000(100010)1000, \\ & n \equiv 10 \pmod{12}: & a = 1010(010001), \\ & n \equiv 11 \pmod{12}: & a = 01010(010001), \end{array}$$

For $s=4$ the existence of balanced Steinhaus triangles remains open for $n \equiv 12 \pmod{24}$, $n \neq 12$.

$$\begin{array}{lll}
s=6: & n \equiv 0 \pmod{20}: & a = 0011100010(0010000010), \\
 & n \equiv 2 \pmod{20},\ n \neq 2: & a = 10111000001(0010000010), \\
 & n \equiv 4 \pmod{20}: & a = 0011(0010000010), \\
 & n \equiv 11 \pmod{20}: & a = 10111000001(0001000001), \\
 & n \equiv 13 \pmod{20}: & a = 0111010000010(0010000010), \\
 & n \equiv 15 \pmod{20}: & a = 00111(0010000010), \\
 & n \equiv 16 \pmod{20}: & a = 100110(0010000010), \\
 & n \equiv 17 \pmod{20}: & a = 1000111(0010000010), \\
 & n \equiv 19 \pmod{20}: & a = 100100110(0010000010), \\
 & n \equiv 38, 58, 78 \pmod{80}: & a = 000101101011100010(0010000010), \\
 & n \equiv 98 \pmod{160}: & a = 110001101010101111(0010000010),
\end{array}$$

For $s = 6$ the case $n \equiv 18 \pmod{160}$, $n \neq 18$, remains open.

5. Remarks

Summarizing, the existence of balanced Steinhaus triangles with generalized Pascal addition in the case of odd n remains open only for $s \equiv 1 \pmod 4$ and large values of n with $n \equiv s \pmod{2s-2}$. In the case of n even only examples for small values of s and some classes of n are known.

References

[1] M. Bartsch: *Steinhaus-Figuren modulo 2 und verallgemeinerte Steinhaus-Dreiecke*. Dissertation, TU Braunschweig, 1985, 84pp.

[2] B.A. Bondarenko: *Generalized Pascal Triangles and Pyramids, Their Fractals, Graphs and Applications*. Fibonacci Association 1993. `www.fq.math.ca/books.html`

[3] J.M. Brunat and M. Maureso: *Symmetries in Steinhaus triangles and in generalized Pascal triangles*. Integers **11** (2011), A1, 23pp.

[4] J. Chappelon: *On a problem of Molluzzo concerning Steinhaus triangles in finite cyclic groups*. Integers **8** (2008), A37, 29pp.

[5] J. Chappelon and S. Eliahou: *On the problem of Molluzzo for the modulus 4*. Integers **12** (2012), 723–739.

[6] W.M. Dymacek, M. Koerlin, and T. Whaley: *A survey of Steinhaus graphs*. In: Y. Alavi, D.R. Lick, and A. Schwenk, eds.: Combinatorics, Graph Theory, and Algorithms. Western Michigan University, Kalamazoo, Michigan, 1999. Vol. I, 313–323.

[7] W.M. Dymacek, J.-G. Speton, and T. Whaley: *Planar Steinhaus graphs*. Congr. Numer. **144** (2000), 193–206.

[8] H. Harborth: *Solutions of Steinhaus's problem with plus and minus signs*. J. Combin. Theory (A) **12** (1972), 253–259.

[9] H. Steinhaus: *One Hundred Problems in Elementary Mathematics*. Pergamon Press, 1963, pp. 47–48.

MSC2010: 05B30

Diskrete Mathematik, Technische Universität Braunschweig, 38023 Braunschweig, Germany
E-mail address: `jp.bode@tu-bs.de`

Diskrete Mathematik, Technische Universität Braunschweig, 38023 Braunschweig, Germany
E-mail address: `h.harborth@tu-bs.de`

GENERALIZING ZECKENDORF'S THEOREM: THE KENTUCKY SEQUENCE

MINERVA CATRAL, PARI FORD, PAMELA HARRIS, STEVEN J. MILLER, AND DAWN NELSON

Abstract. By Zeckendorf's theorem, an equivalent definition of the Fibonacci sequence (appropriately normalized) is that it is the unique sequence of increasing integers such that every positive number can be written uniquely as a sum of non-adjacent elements; this is called a legal decomposition. Previous work examined the distribution of the number of summands, and the spacings between them, in legal decompositions arising from the Fibonacci numbers and other linear recurrence relations with non-negative integral coefficients. These results were restricted to the case where the first term in the defining recurrence was positive. We study a generalization of the Fibonacci sequence with a simple notion of legality which leads to a recurrence where the first term vanishes. We again have unique legal decompositions, Gaussian behavior in the number of summands, and geometric decay in the distribution of gaps.

Contents

1. Introduction

One of the standard definitions of the Fibonacci numbers $\{F_n\}$ is that it is the unique sequence satisfying the recurrence $F_{n+1} = F_n + F_{n-1}$ with initial conditions $F_1 = 1$, $F_2 =$

2010 *Mathematics Subject Classification.* 60B10, 11B39, 11B05 (primary) 65Q30 (secondary).

Key words and phrases. Zeckendorf decompositions, Fibonacci numbers, Generacci numbers, positive linear recurrence relations, Gaussian behavior, distribution of gaps.

The fourth named author was partially supported by NSF grant DMS1265673. This research was performed while the third named author held a National Research Council Research Associateship Award at USMA/ARL. This work was begun at the 2014 REUF Meeting at AIM; it is a pleasure to thank them for their support, the participants there and at the 16$^{\text{th}}$ International Conference on Fibonacci Numbers and their Applications for helpful discussions, and the referee for comments on an earlier draft.

2. An interesting and equivalent definition is that it is the unique increasing sequence of positive integers such that every positive number can be written uniquely as a sum of non-adjacent elements of the sequence.[1] This equivalence is known as Zeckendorf's theorem [27], and frequently one says every number has a unique legal decomposition as a sum of non-adjacent Fibonacci numbers.

Past research regarding generalized Zeckendorf decompositions have involved sequences $\{G_n\}$ where the recurrence relation coefficients are non-negative integers, with the additional restriction being that the first and last terms are positive.[2] See for instance [22], where the authors call these **Positive Linear Recurrence** (PLR) Sequences. In this setting, much is known about the properties of the summands including that the distribution of the number of summands converges to a Gaussian, [9, 23]. There have also been recent results about gaps between summands, including a proof that the distribution of the longest gap converges to the same distribution one sees when looking at the longest run of heads in tosses of a biased coin, see [2, 3, 5]. There is a large set of literature addressing generalized Zeckendorf decompositions, these include[1, 8, 10, 11, 12, 13, 14, 15, 16, 17, 25, 26] among others.

However, all of these results only hold for PLR Sequences. In this paper, we extend the results on Gaussian behavior and average gap measure to recurrences that cannot be handled by existing techniques. To that end, we study a sequence arising from a notion of a legal decomposition whose recurrence has first term equal to zero.[3] While our sequence fits into the framework of an f-decomposition introduced in [9], their arguments only suffice to show that our decomposition rule leads to unique decompositions. The techniques in [9] do not address the distribution of the number of summands nor the behavior of the gaps between the summands for our particular sequence. We address these questions completely in Theorems 1.5 and 1.6, respectively.

We now describe our object of study. We can view the decomposition rule corresponding to the Fibonacci sequence by saying the sequence is divided into bins of length 1, and (i) we can use at most one element from a bin at most one time, and (ii) we cannot choose elements from adjacent bins. This suggests a natural extension where the bins now contain b elements and any two summands of a decomposition (i) cannot be members of the same bin and (ii) must be at least s bins away from each other. We call this the (s, b)**-Generacci sequence** (see Definition 5.2) and the Fibonacci numbers are the $(1, 1)$-Generacci sequence. In this paper we consider the case $s = 1, b = 2$. We give this special sequence a name: the **Kentucky** sequence, after the home state of one of our authors. Although we expect our results to extend in full generality, we have found that new techniques are needed for the two parameter family. See Section 5 for more details on the general case.

Definition 1.1. *Let an increasing sequence of positive integers $\{a_i\}_{i=1}^{\infty}$ be given and partition the elements into bins*

$$\mathcal{B}_k := \{a_{2k-1}, a_{2k}\}$$

for $k \geq 1$. We declare a decomposition of an integer

$$m = a_{\ell_1} + a_{\ell_2} + \cdots + a_{\ell_k}$$

where $\ell_1 < \ell_2 < \cdots < \ell_k$ and $\{a_{\ell_j}, a_{\ell_{j+1}}\} \not\subset \mathcal{B}_i \cup \mathcal{B}_{i-1}$ for any i, j to be a Kentucky legal decomposition.

---

[1] If we started the Fibonacci numbers with a zero, or with two ones, we would lose uniqueness of decompositions.

[2] Thus $G_{n+1} = c_1 G_n + \cdots + c_L G_{n-(L-1)}$ with $c_1 c_L > 0$ and $c_i \geq 0$.

[3] Thus $G_{n+1} = c_1 G_n + c_2 G_{n-1} + \cdots + C_L G_{n-(L-1)}$ with $c_1 = 0$ and $c_i \geq 0$.

This says that we cannot decompose a number using more than one summand from the same bin or two summands from adjacent bins.

Definition 1.2. *An increasing sequence of positive integers* $\{a_i\}_{i=1}^{\infty}$ *is called a* Kentucky sequence *if every* a_i $(i \geq 1)$ *is the smallest positive integer that does not have a Kentucky legal decomposition using the elements* $\{a_1, \ldots, a_{i-1}\}$.

From the definition of a Kentucky legal decomposition, the reader can see that the first five terms of the sequence must be $\{1, 2, 3, 4, 5\}$. We have $a_6 \neq 6$ as $6 = a_1 + a_5 = 1 + 5$ is a Kentucky legal decomposition. In the same way we find $a_6 \neq 7$, and this is the largest integer that could be legally decomposed using the first five entries in the sequence. Thus we must have $a_6 = 8$. Continuing we have the first few terms of the Kentucky sequence:

$$\underbrace{1,\ 2}_{\mathcal{B}_1},\ \underbrace{3,\ 4}_{\mathcal{B}_2},\ \underbrace{5,\ 8}_{\mathcal{B}_3},\ \underbrace{11,\ 16}_{\mathcal{B}_4},\ \underbrace{21,\ 32}_{\mathcal{B}_5},\ \underbrace{43,\ 64}_{\mathcal{B}_6},\ \underbrace{85,\ 128}_{\mathcal{B}_7},\ \underbrace{171,\ 256}_{\mathcal{B}_8},\ \ldots$$

We have a nice closed form expression for the elements of this sequence.

Theorem 1.3. *If* $\{a_n\}$ *is the Kentucky sequence, then*

$$a_{n+1} \ = \ a_{n-1} + 2a_{n-3},\ a_1 \ = \ 1,\ a_2 \ = \ 2,\ a_3 \ = \ 3,\ a_4 \ = \ 4,$$

which implies

$$a_{2n} \ = \ 2^n \qquad \text{and} \qquad a_{2n-1} \ = \ \frac{1}{3}\left(2^{n+1} + (-1)^n\right).$$

This is not a PLR Sequence as the leading coefficient (that of a_n) is zero, and this sequence falls outside the scope of many of the previous techniques. We prove the following theorems concerning the Kentucky Sequence.

Theorem 1.4 (Uniqueness of Decompositions)**.** *Every positive integer can be written uniquely as a sum of distinct terms from the Kentucky sequence where no two summands are in the same bin and no two summands belong to consecutive bins in the sequence.*

The above follows immediately from the work in [9] on f-decompositions. In Theorem 1.3 of [9] take $f(n) = 3$ if n is even and $f(n) = 2$ otherwise. For completeness we give an elementary proof in Appendix A. generalize the results on Gaussian behavior for the summands to this case.

Theorem 1.5 (Gaussian Behavior of Summands)**.** *Let the random variable* Y_n *denote the number of summands in the (unique) Kentucky decomposition of an integer picked at random from* $[0, a_{2n+1})$ *with uniform probability.*[4] *Normalize* Y_n *to* $Y_n' = (Y_n - \mu_n)/\sigma_n$*, where* μ_n *and* σ_n *are the mean and variance of* Y_n *respectively, which satisfy*

$$\begin{aligned}
\mu_n &= \frac{n}{3} + \frac{2}{9} + O\left(\frac{n}{2^n}\right) \\
\sigma_n^2 &= \frac{2n}{27} + \frac{8}{81} + O\left(\frac{n^2}{2^n}\right).
\end{aligned}$$

Then Y_n' *converges in distribution to the standard normal distribution as* $n \to \infty$.

[4] Using the methods of [4], these results can be extended to hold almost surely for a sufficiently large subinterval of $[0, a_{2n+1})$.

Our final results concern the behavior of gaps between summands. For the legal decomposition

$$m \;=\; a_{\ell_1} + a_{\ell_2} + \cdots + a_{\ell_k} \quad \text{with} \quad \ell_1 \;<\; \ell_2 \;<\; \cdots \;<\; \ell_k$$

and $m \in [0, a_{2n+1})$, we define the set of gaps as follows:

$$\mathrm{Gaps}_n(m) \;:=\; \{\ell_2 - \ell_1, \ell_3 - \ell_2, \dots, \ell_k - \ell_{k-1}\}.$$

Notice we do not include the wait to the first summand, $\ell_1 - 0$, as a gap. We could include this if we wish; one additional gap will not affect the limiting behavior.

In the theorem below we consider all the gaps between summands in Kentucky legal decompositions of all $m \in [0, a_{2n+1})$. We let $P_n(g)$ be the fraction of all these gaps that are of length g (i.e., the probability of a gap of length g among Kentucky legal decompositions of $m \in [0, a_{2n+1})$). For example, notice $m = a_1 + a_{11} + a_{15} + a_{22} + a_{26}$ contributes two gaps of length 4, one gap of length 7 and one gap of length 10.

Theorem 1.6 (Average Gap Measure)**.** *For $P_n(g)$ as defined above, the limit $P(g) := \lim_{n\to\infty} P_n(g)$ exists, and*

$$P(0) \;=\; P(1) \;=\; P(2) \;=\; 0, \quad P(3) \;=\; 1/8,$$

and for $g \geq 4$ we have

$$P(g) \;=\; \begin{cases} 2^{-j} & \text{if } g = 2j \\ \frac{3}{4}\, 2^{-j} & \text{if } g = 2j+1. \end{cases}$$

In §2 we derive the recurrence relation and explicit closed form expressions for the terms of the Kentucky sequence, as well as a useful generating function for the number of summands in decompositions. We then prove Theorem 1.5 on Gaussian behavior in §3, and Theorem 1.6 on the distribution of the gaps in §4. We end with some concluding remarks and directions for future research in §5.

2. Recurrence Relations and Generating functions

In the analysis below we constantly use the fact that every positive integer has a unique Kentucky legal decomposition; see [9] or Appendix A for proofs.

2.1. Recurrence Relations.

Proposition 2.1. *For the Kentucky sequence, $a_n = n$ for $1 \leq n \leq 5$ and for any $n \geq 5$ we have $a_n = a_{n-2} + 2a_{n-4}$. Further for $n \geq 1$ we have*

$$a_{2n} \;=\; 2^n \quad \text{and} \quad a_{2n-1} \;=\; \frac{1}{3}\left(2^{n+1} + (-1)^n\right). \tag{2.1}$$

Proof. We recall that the integers a_{2n+1} and a_{2n} in the Kentucky sequence are elements of the sequence as they are the smallest integers that cannot be legally decomposed using the members of $\{a_1, a_2, \dots, a_{2n}\}$ or $\{a_1, a_2, \dots, a_{2n-1}\}$ respectively:

$$\underbrace{1,\ 2}_{\mathcal{B}_1},\ \underbrace{3,\ 4}_{\mathcal{B}_2},\ \underbrace{5,\ 8}_{\mathcal{B}_3},\ \underbrace{11,\ 16}_{\mathcal{B}_4},\ \underbrace{21,\ 32}_{\mathcal{B}_5},\ \underbrace{43,\ 64}_{\mathcal{B}_6},\ \cdots,\ \underbrace{a_{2n-3},\ a_{2n-2}}_{\mathcal{B}_{n-1}},\ \underbrace{a_{2n-1},\ a_{2n}}_{\mathcal{B}_n}.$$

As a_{2n} is the largest entry in the bin $\mathcal{B}_n$, it is one more than the largest number we can legally decompose, and thus

$$a_{2n} \;=\; a_{2n-1} + a_{2(n-2)} + a_{2(n-4)} + \cdots + a_j + 1$$

where $a_j = a_2$ if n is odd and $a_j = a_4$ if n is even. By construction of the Kentucky sequence we have $a_{2(n-2)} + a_{2(n-4)} + \cdots + a_j + 1 = a_{2(n-2)+1} = a_{2n-3}$. Thus

$$a_{2n} \;=\; a_{2n-1} + a_{2n-3}. \tag{2.2}$$

Similarly a_{2n+1} is the smallest entry in bin $\mathcal{B}_{n+1}$, so

$$a_{2n+1} \;=\; a_{2n} + a_{2(n-2)} + a_{2(n-4)} + \cdots + a_j + 1$$

where $a_j = a_2$ if n is odd and $a_j = a_4$ if n is even. Thus

$$a_{2n+1} \;=\; a_{2n} + a_{2n-3}. \tag{2.3}$$

Substituting (2.2) into (2.3) yields

$$a_{2n+1} \;=\; a_{2n-1} + 2a_{2n-3}, \tag{2.4}$$

and thus for $m \geq 5$ odd we have $a_m = a_{m-2} + 2a_{m-4}$.

Now using (2.4) in (2.2), we have

$$a_{2n} \;=\; a_{2n-1} + a_{2n-3} \;=\; a_{2n-3} + 2a_{2n-5} + a_{2n-3} \;=\; 2(a_{2n-3} + a_{2n-5}).$$

Shifting the index in (2.2) gives

$$a_{2n} \;=\; 2a_{2n-2}. \tag{2.5}$$

Since $a_2 = 2$ and $a_4 = 4$, together with (2.5) we now have $a_{2n} = 2^n$ for all $n \geq 1$. A few algebraic steps then confirm $a_m = a_{m-2} + 2a_{m-4}$ for $m \geq 6$ even.

Finally, we prove that $a_{2n-1} = \frac{1}{3}(2^{n+1} + (-1)^n)$ for $n \geq 1$ by induction. The base case is immediate as $a_1 = 1$ and $\frac{1}{3}(2^{1+1} + (-1)^1) = \frac{1}{3}(4 - 1) = 1$. Assume for some $N \geq 1$, $a_{2N-1} = \frac{1}{3}(2^{N+1} + (-1)^N)$. By (2.4), we have

$$\begin{aligned}
a_{2(N+1)-1} &= a_{2N+1} \\
&= a_{2N-1} + 2a_{2N-3} \\
&= \frac{1}{3}(2^{N+1} + (-1)^N) + (2)\left(\frac{1}{3}\right)(2^{N-1+1} + (-1)^{N-1}) \\
&= \frac{1}{3}(2^{N+1} + (-1)^N + 2^{N+1} + (-1)^{N-1} + (-1)^{N-1}) \\
&= \frac{1}{3}(2^{N+2} + (-1)^{N+1}),
\end{aligned}$$

and thus for all $n \geq 1$ we have $a_{2n-1} = \frac{1}{3}(2^{n+1} + (-1)^n)$. □

2.2. **Counting Integers With Exactly k Summands.** In [18], Koloğlu, Kopp, Miller and Wang introduced a very useful combinatorial perspective to attack Zeckendorf decomposition problems. While many previous authors attacked related problems through continued fractions or Markov chains, they instead partitioned the $m \in [F_n, F_{n+1})$ into sets based on the number of summands in their Zeckendorf decomposition. We employ a similar technique here, which when combined with identities about Fibonacci polynomials allows us to easily obtain Gaussian behavior.

Let $p_{n,k}$ denote the number of $m \in [0, a_{2n+1})$ whose legal decomposition contains exactly k summands where $k \geq 0$. We have $p_{n,0} = 1$ for $n \geq 0$, $p_{0,k} = 0$ for $k > 0$, $p_{1,1} = 2$, and $p_{n,k} = 0$

if $k > \lfloor \frac{n+1}{2} \rfloor$. Also, by definition,

$$\sum_{k=0}^{\lfloor \frac{n+1}{2} \rfloor} p_{n,k} \;=\; a_{2n+1},$$

and we have the following recurrence.

Proposition 2.2. *For $p_{n,k}$ as above, we have*

$$p_{n,k} \;=\; 2p_{n-2,k-1} + p_{n-1,k}$$

for $n \geq 2$ and $k \leq \lfloor \frac{n+1}{2} \rfloor$.

Proof. We partition the Kentucky legal decompositions of all $m \in [0, a_{2n+1})$ into two sets, those that have a summand from bin $\mathcal{B}_n$ and those that do not.

If we have a legal decomposition $m = a_{\ell_1} + a_{\ell_2} + \cdots + a_{\ell_k}$ with $a_{\ell_k} \in \mathcal{B}_n$, then $a_{\ell_{k-1}} \leq a_{2(n-2)}$ and there are two choices for a_{ℓ_k}. The number of legal decompositions of the form $a_{\ell_1} + a_{\ell_2} + \cdots + a_{\ell_{k-1}}$ with $a_{\ell_{k-1}} \leq a_{2(n-2)}$ is $p_{n-2,k-1}$. Note the answer is independent of which value $a_{\ell_k} \in \mathcal{B}_n$ we have. Thus the number of legal decompositions of m containing exactly k summands with largest summand in bin $\mathcal{B}_n$ is $2p_{n-2,k-1}$.

If $m \in [0, a_{2n+1})$ does not have a summand from bin $\mathcal{B}_n$ in its decomposition, then $m \in [0, a_{2n-1})$, and by definition the number of such m with exactly k summands in a legal decomposition is $p_{n-1,k}$.

Combining these two cases yields

$$p_{n,k} \;=\; 2p_{n-2,k-1} + p_{n-1,k},$$

completing the proof. □

This recurrence relation allows us to compute a closed-form expression for $F(x, y)$, the generating function of the $p_{n,k}$'s.

Proposition 2.3. *Let*

$$F(x, y) \;:=\; \sum_{n,k \geq 0} p_{n,k} x^n y^k$$

be the generating function of the $p_{n,k}$'s arising from Kentucky legal decompositions. Then

$$F(x, y) \;=\; \frac{1 + 2xy}{1 - x - 2x^2 y}. \tag{2.6}$$

Proof. Noting that $p_{n,k} = 0$ if either $n < 0$ or $k < 0$, using explicit values of $p_{n,k}$ and the recurrence relation from Proposition 2.2, after some straightforward algebra we obtain

$$F(x, y) \;=\; 2x^2 y F(x, y) + xF(x, y) + 2xy + 1.$$

From this, (2.6) follows. □

While the combinatorial vantage of [18] has been fruitfully applied to a variety of recurrences (see [22, 23]), their proof of Gaussianity does not generalize. The reason is that for the Fibonacci numbers (which are also the $(1, 1)$-Generacci numbers) we have an explicit, closed form expression for the corresponding $p_{n,k}$'s, which greatly facilitates the analysis. Fortunately for us a similar closed form expression exists for Kentucky decompositions.

Proposition 2.4. *Let $p_{n,k}$ be the number of integers in $[0, a_{2n+1})$ that have exactly k summands in their Kentucky legal decomposition. For all $k \geq 1$ and $n \geq 1 + 2(k-1)$, we have*

$$p_{n,k} \;=\; 2^k \binom{n-(k-1)}{k}.$$

Proof. We are counting decompositions of the form $a'_{\ell_1} + \cdots + a'_{\ell_k}$ where $a'_{\ell_i} \in \mathcal{B}_{\ell_i} = \{a_{2\ell_i - 1}, a_{2\ell_i}\}$ and $\ell_i \leq n$. Define $x_1 := \ell_1 - 1$ and $x_{k+1} := n - \ell_k$. For $2 \leq i \leq k$, define $x_i := \ell_i - \ell_{i-1} - 1$. We have

$$x_1 + 1 + x_2 + 1 + x_3 + 1 + \cdots + x_k + 1 + x_{k+1} \;=\; n.$$

We change variables to rewrite the above. Essentially what we are doing is replacing the x's with new variables to reduce our Diophantine equation to a standard form that has been well-studied. As we have a legal decomposition, our bins must be separated by at least one and thus $x_i \geq 1$ for $2 \leq i \leq k-1$ and $x_1, x_k \geq 0$. We remove these known gaps in our new variables by setting $y_1 := x_1$, $y_{k+1} := x_{k+1}$ and $y_i := x_i - 1$ for $2 \leq i \leq k$, which gives

$$\begin{aligned} y_1 + y_2 + \cdots + y_k + y_{k+1} &= x_1 + (x_2 - 1) + \cdots + (x_k - 1) + x_{k+1} \\ &= n - k - (k-1). \end{aligned} \tag{2.7}$$

Finding the number of non-negative integral solutions to this Diophantine equation has many names (the Stars and Bars Problem, Waring's Problem, the Cookie Problem). As the number of solutions to $z_1 + \cdots + z_P = C$ is $\binom{C+P-1}{P-1}$ (see for example [21, 24], or [20] for a proof and an application of this identity in Bayesian analysis), the number of solutions to (2.7) is given by the binomial coefficient

$$\binom{n-k-(k-1)+k}{k} \;=\; \binom{n-(k-1)}{k}.$$

As there are two choices for each a'_{ℓ_i}, we have 2^k legal decompositions whose summands are from the bins $\{\mathcal{B}_{\ell_1}, \mathcal{B}_{\ell_2}, \ldots, \mathcal{B}_{\ell_k}\}$ and thus

$$p_{n,k} \;=\; 2^k \binom{n-(k-1)}{k}.$$

□

3. Gaussian Behavior

Before launching into our proof of Theorem 1.5, we provide some numerical support in Figure 1. We randomly chose 200,000 integers from $[0, 10^{600})$. We observed a mean number of summands of 666.899, which fits beautifully with the predicted value of 666.889; the standard deviation of our sample was 12.154, which is in excellent agreement with the prediction of 12.176.

We split Theorem 1.5 into three parts: a proof of our formula for the mean, a proof of our formula for the variance, and a proof of Gaussian behavior. We isolate the first two as separate propositions; we will prove these after first deriving some useful properties of the generating function of the $p_{n,k}$'s.

Proposition 3.1. *The mean number of summands in the Kentucky legal decompositions for integers in $[0, a_{2n+1})$ is*

$$\mu_n \;=\; \frac{n}{3} + \frac{2}{9} + O\left(\frac{n}{2^n}\right).$$

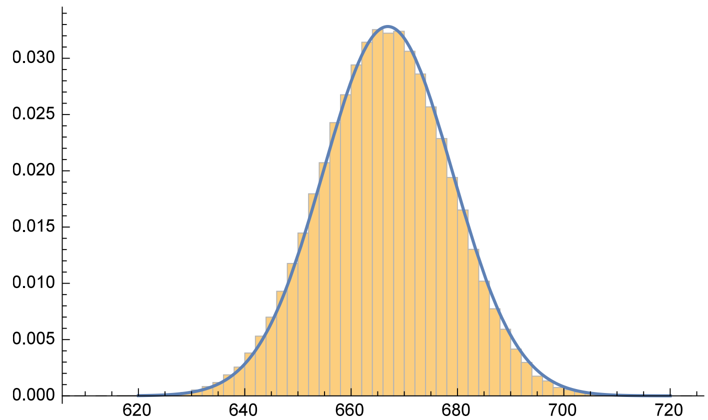


FIGURE 1. The distribution of the number of summands in Kentucky legal decompositions for 200,000 integers from $[0, 10^{600})$.

Proposition 3.2. *The variance σ_n^2 of Y_n (from Theorem 1.5) is*

$$\sigma_n^2 \;=\; \frac{2n}{27} + \frac{8}{81} + O\left(\frac{n^2}{2^n}\right).$$

3.1. **Mean and Variance.** Recall Y_n is the random variable denoting the number of summands in the unique Kentucky decomposition of an integer chosen uniformly from $[0, a_{2n+1})$, and $p_{n,k}$ denotes the number of integers in $[0, a_{2n+1})$ whose legal decomposition contains exactly k summands. The following lemma yields expressions for the mean and variance of Y_n using a generating function for the $p_{n,k}$'s; in fact, it is this connection of derivatives of the generating function to moments that make the generating function approach so appealing. The proof is standard (see for example [9]).

Lemma 3.3. [9, Propositions 4.7, 4.8] *Let $F(x,y) := \sum_{n,k\geq 0} p_{n,k}x^n y^k$ be the generating function of $p_{n,k}$, and let $g_n(y) := \sum_{k=0}^{n} p_{n,k}y^k$ be the coefficient of x^n in $F(x,y)$. Then the mean of Y_n is*

$$\mu_n \;=\; \frac{g_n'(1)}{g_n(1)},$$

and the variance of Y_n is

$$\sigma_n^2 \;=\; \frac{\frac{d}{dy}(y g_n'(y))|_{y=1}}{g_n(1)} - \mu_n^2.$$

In our analysis our closed form expression of $p_{n,k}$ as a binomial coefficient is crucial in obtaining simple closed form expressions for the needed quantities. We are able to express these needed quantities in terms of the **Fibonacci polynomials**, which are defined recursively as follows:

$$F_0(x) \;=\; 0,\;\; F_1(x) \;=\; 1,\;\; F_2(x) \;=\; x,$$

and for $n \geq 3$

$$F_n(x) \;=\; xF_{n-1}(x) + F_{n-2}(x).$$

For $n \geq 3$, the Fibonacci polynomial[5] $F_n(x)$ is given by

$$F_n(x) = \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-j-1}{j} x^{n-2j-1}, \tag{3.1}$$

and also has the explicit formula

$$F_n(x) = \frac{(x+\sqrt{x^2+4})^n - (x-\sqrt{x^2+4})^n}{2^n\sqrt{x^2+4}}. \tag{3.2}$$

The derivative of $F_n(x)$ is given by

$$F_n'(x) = \frac{2nF_{n-1}(x) + (n-1)xF_n(x)}{x^2+4}. \tag{3.3}$$

For a reference on Fibonacci polynomials and the formulas given above (which follow immediately from the definitions and straightforward algebra), see [19].

Proposition 3.4. *For $n \geq 3$*

$$g_n(y) = (\sqrt{2y})^{n+1} F_{n+2}\left(\frac{1}{\sqrt{2y}}\right). \tag{3.4}$$

Proof. By Proposition 2.4, we have

$$F(x,y) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} p_{n,k}x^n y^k = \sum_{n=0}^{\infty}\sum_{k=0}^{n} 2^k \binom{n-k+1}{k} x^n y^k.$$

Thus, using (3.1) we find

$$\begin{aligned} F(x,y) &= \frac{1}{x^2\sqrt{2y}} \sum_{n=0}^{\infty}\sum_{k=0}^{n+2} \binom{(n+2)-k-1}{k} \left(\frac{1}{\sqrt{2y}}\right)^{(n+2)-2k-1} (x\sqrt{2y})^{n+2} \\ &= \frac{1}{x^2\sqrt{2y}} \sum_{n=0}^{\infty} F_{n+2}\left(\frac{1}{\sqrt{2y}}\right)(x\sqrt{2y})^{n+2} = \sum_{n=0}^{\infty} F_{n+2}\left(\frac{1}{\sqrt{2y}}\right)(\sqrt{2y})^{n+1}x^n, \end{aligned}$$

completing the proof. □

In Appendix B we provide alternate proofs of Proposition 3.1, Proposition 3.2 and Theorem 1.5 using different methods. In doing so, we uncovered another formula for $g_n(y)$, the coefficient for x^n in $F(x,y)$ as given in Lemma 3.3, and this leads to a derivation of a formula for the Fibonacci polynomials.

[5]Note that $F_n(1)$ gives the standard Fibonacci sequence.

Proof of Proposition 3.1. By Lemma 3.3, the mean of Y_n is $g_n'(1)/g_n(1)$. Calculations of derivatives using equations (3.3) and (3.4) give

$$\begin{aligned}
\frac{g_n'(1)}{g_n(1)} &= \frac{(n+1)(\sqrt{2})^{n-1}F_{n+2}(\frac{1}{\sqrt{2}})}{F_{n+2}(\frac{1}{\sqrt{2}})(\sqrt{2})^{n+1}} - \frac{(\sqrt{2})^{n-2}F_{n+2}'(\frac{1}{\sqrt{2}})}{F_{n+2}(\frac{1}{\sqrt{2}})(\sqrt{2})^{n+1}}\\
&= \frac{n+1}{2} - \frac{1}{(\sqrt{2})^3}\frac{F_{n+2}'\left(\frac{1}{\sqrt{2}}\right)}{F_{n+2}\left(\frac{1}{\sqrt{2}}\right)}.\\
&= \frac{n+1}{2} - \frac{2(n+2)F_{n+1}\left(\frac{1}{\sqrt{2}}\right) + \frac{n+1}{\sqrt{2}}F_{n+2}\left(\frac{1}{\sqrt{2}}\right)}{9\sqrt{2}F_{n+2}\left(\frac{1}{\sqrt{2}}\right)}\\
&= \frac{4}{9}(n+1) - \frac{\sqrt{2}}{9}(n+2)\frac{F_{n+1}\left(\frac{1}{\sqrt{2}}\right)}{F_{n+2}\left(\frac{1}{\sqrt{2}}\right)}\\
&= \frac{4}{9}(n+1) - \frac{\sqrt{2}}{9}(n+2)\left(\frac{1}{\sqrt{2}} + O(2^{-n})\right) \;=\; \frac{n}{3} + \frac{2}{9} + O(n2^{-n}),
\end{aligned}$$

where in the next-to-last step we use (3.2) to approximate $F_{n+1}(1/\sqrt{2})/F_{n+2}(1/\sqrt{2})$. □

Proof of Proposition 3.2. By Lemma 3.3,

$$\sigma_n^2 \;=\; \frac{g_n''(1)}{g_n(1)} + \frac{g_n'(1)}{g_n(1)} - \mu_n^2 \;=\; \frac{g_n''(1)}{g_n(1)} + \mu_n(1-\mu_n).$$

Now,

$$\frac{g_n''(1)}{g_n(1)} \;=\; \frac{(-2n+1)}{4\sqrt{2}}\frac{F_{n+2}'(\frac{1}{\sqrt{2}})}{F_{n+2}(\frac{1}{\sqrt{2}})} + \frac{(n^2-1)}{4} + \frac{1}{8}\frac{F_{n+2}''(\frac{1}{\sqrt{2}})}{F_{n+2}(\frac{1}{\sqrt{2}})}.$$

Applying the derivative formula in (3.3) and using (3.2), we find

$$\begin{aligned}
\frac{F_{n+2}'(\frac{1}{\sqrt{2}})}{F_{n+2}(\frac{1}{\sqrt{2}})} &= \frac{4(n+2)}{9}\frac{F_{n+1}(\frac{1}{\sqrt{2}})}{F_{n+2}(\frac{1}{\sqrt{2}})} + \frac{\sqrt{2}(n+1)}{9}\\
&= \frac{4(n+2)}{9}\left[\frac{1}{\sqrt{2}} + O(2^{-n})\right] + \frac{\sqrt{2}(n+1)}{9}
\end{aligned}$$

and

$$\begin{aligned}
\frac{F_{n+2}''(\frac{1}{\sqrt{2}})}{F_{n+2}(\frac{1}{\sqrt{2}})} &= \frac{16(n^2+3n+2)}{81}\frac{F_n(\frac{1}{\sqrt{2}})}{F_{n+2}(\frac{1}{\sqrt{2}})} + \frac{4\sqrt{2}(2n^2+3n-2)}{81}\frac{F_{n+1}(\frac{1}{\sqrt{2}})}{F_{n+2}(\frac{1}{\sqrt{2}})} + \frac{2(n^2+9n+8)}{81}\\
&= \frac{16(n^2+3n+2)}{81}\left[\frac{1}{2} + O(2^{-n})\right] + \frac{4\sqrt{2}(2n^2+3n-2)}{81}\left[\frac{1}{\sqrt{2}} + O(2^{-n})\right]\\
&\quad + \frac{2(n^2+9n+8)}{81}.
\end{aligned}$$

Thus

$$\sigma_n^2 = \frac{(-2n+1)}{4\sqrt{2}}\left[\frac{\sqrt{2}}{9}(3n+5)+O(n2^{-n})\right]+\frac{(n^2-1)}{4}+\frac{1}{8}\left[\frac{2n^2}{9}+\frac{2n}{3}+\frac{8}{27}+O(n^2 2^{-n})\right]$$
$$+\left[\frac{n}{3}+\frac{2}{9}+O\left(\frac{n}{2^n}\right)\right]\left[1-\frac{n}{3}-\frac{2}{9}+O\left(\frac{n}{2^n}\right)\right] = \frac{2n}{27}+\frac{8}{81}+O\left(\frac{n^2}{2^n}\right),$$

completing the proof. □

3.2. **Gaussian Behavior.**

Proof of Theorem 1.5. We prove that Y_n' converges in distribution to the standard normal distribution as $n \to \infty$ by showing that the moment generating function of Y_n' converges to that of the standard normal (which is $e^{t^2/2}$). Following the same argument as in [9, Lemma 4.9], the moment generating function $M_{Y_n'}(t)$ of Y_n' is

$$M_{Y_n'}(t) = \frac{g_n(e^{t/\sigma_n})e^{-t\mu_n/\sigma_n}}{g_n(1)}.$$

Thus we have

$$M_{Y_n'}(t) = \frac{F_{n+2}\left(\frac{1}{\sqrt{2e^{t/\sigma_n}}}\right)e^{(\frac{n+1}{2}-\mu_n)t/\sigma_n}}{F_{n+2}\left(\frac{1}{\sqrt{2}}\right)},$$

and

$$\log(M_{Y_n'}(t)) = \log F_{n+2}\left(\frac{1}{\sqrt{2e^{t/\sigma_n}}}\right)+\frac{t}{\sigma_n}\left(\frac{n+1}{2}-\mu_n\right)-\log F_{n+2}\left(\frac{1}{\sqrt{2}}\right).$$

From (3.2),

$$F_{n+2}(x) = \frac{(x+\sqrt{x^2+4})^{n+2}}{2^{n+2}\sqrt{x^2+4}}\left[1-\left(\frac{x-\sqrt{x^2+4}}{x+\sqrt{x^2+4}}\right)^{n+2}\right].$$

Thus

$$\begin{aligned}\log F_{n+2}(x) &= (n+2)\log(x+\sqrt{x^2+4})-(n+2)\log 2\\ &\qquad -\frac{1}{2}\log(x^2+4)+\log(1-r(x)^{n+2})\\ &= (n+2)\log x+(n+2)\log\left(1+\sqrt{1+\frac{4}{x^2}}\right)-(n+2)\log 2\\ &\qquad -\frac{1}{2}\log(x^2+4) + O(r(x)^n),\end{aligned}$$

where for all x

$$r(x) = \left(\frac{x-\sqrt{x^2+4}}{x+\sqrt{x^2+4}}\right) \in (0,1].$$

Thus

$$\log F_{n+2}\left(\tfrac{1}{\sqrt{2}}\right) = \tfrac{1}{2}(n+3)\log 2-\log 3+O(2^{-n})$$

and

$$\begin{aligned}\log F_{n+2}\left(\frac{1}{\sqrt{2e^{t/\sigma_n}}}\right) &= -\frac{(n+2)}{2}\log 2 - \frac{(n+2)}{2\sigma_n}t - (n+2)\log 2\\ &+ (n+2)\alpha_n(t) - \frac{1}{2}\beta_n(t) + O(r^n),\end{aligned}$$

where

$$\alpha_n(t) = \log(1+\sqrt{1+8e^{t/\sigma_n}}), \quad \beta_n(t) = \log\left(\frac{1}{2}e^{-t/\sigma_n}+4\right),$$

and

$$r = r\left(\frac{1}{\sqrt{2e^{t/\sigma_n}}}\right) < 1.$$

The Taylor series expansions for $\alpha_n(t)$ and $\beta_n(t)$ about $t = 0$ are given by

$$\alpha_n(t) = \log 4 + \frac{1}{3\sigma_n}t + \frac{1}{27\sigma_n^2}t^2 + O(n^{-3/2})$$

and

$$\beta_n(t) = \log\left(\frac{9}{2}\right) - \frac{1}{9\sigma_n}t + \frac{4}{81\sigma_n^2}t^2 + O(n^{-3/2}).$$

Going back to $\log(M_{Y_n'}(t))$ we now have

$$\begin{aligned}\log(M_{Y_n'}(t)) &= -\frac{3}{2}(n+2)\log 2 - \frac{(n+2)}{2\sigma_n}t + (n+2)\left[2\log 2 + \frac{1}{3\sigma_n}t + \frac{1}{27\sigma_n^2}t^2 + O(n^{-3/2})\right]\\ &\quad -\frac{1}{2}\left[2\log 3 - \log 2 + O(n^{-1/2})\right] + \frac{(n+1-2\mu_n)}{2\sigma_n}t - \frac{1}{2}(n+3)\log 2 + \log 3\\ &\quad +O(2^{-n}) + O(r^n)\\ &= -\frac{(2\mu_n+1)}{2\sigma_n}t + \frac{(n+2)}{3\sigma_n}t + \frac{(n+2)}{27\sigma_n^2}t^2 + O(n^{-1/2}) + O(2^{-n}) + O(r^n).\end{aligned}$$

Since $\mu_n \sim \frac{n}{3}$ and $\sigma_n^2 \sim \frac{2n}{27}$, it follows that $\log(M_{Y_n'}(t)) \to \frac{1}{2}t^2$ as $n \to \infty$. As this is the moment generating function of the standard normal, our proof is completed. □

4. Average Gap Distribution

In this section we prove our results about the behavior of gaps between summands in Kentucky decompositions. The advantage of studying the average gap distribution is that, following the methods of [2, 5], we reduce the problem to a combinatorial one involving how many $m \in [0, a_{2n+1})$ have a gap of length g starting at a given index i. We then write the gap probability as a double sum over integers m and starting indices i, interchange the order of summation, and invoke our combinatorial results.

While the calculations are straightforward once we adopt this perspective, they are long. Additionally, it helps to break the analysis into different cases depending on the parity of i and g, which we do first below and then use those results to determine the probabilities.

Proof of Theorem 1.6. Let $I_n := [0, a_{2n+1})$ and let $m \in I_n$ with the legal decomposition

$$m = a_{\ell_1} + a_{\ell_2} + \cdots + a_{\ell_k},$$

with $\ell_1 < \ell_2 < \cdots < \ell_k$. For $1 \leq i, g \leq n$ we define $X_{i,g}(m)$ as an indicator function which denotes whether the decomposition of m has a gap of length g beginning at i. Formally,

$$X_{i,g}(m) \;=\; \begin{cases} 1 & \text{if } \exists\ j,\ 1 \leq j \leq k \text{ with } i = \ell_j \text{ and } i+g = \ell_{j+1} \\ 0 & \text{otherwise.} \end{cases}$$

Notice when $X_{i,g}(m) = 1$, this implies that there exists a gap between a_i and a_{i+g}. Namely m does not contain $a_{i+1}, \ldots, a_{i+g-1}$ as summands in its legal decomposition.

As the definition of the Kentucky sequence implies $P(g) = 0$ for $0 \leq g \leq 2$, we assume below that $g \geq 3$. Hence if a_j is a summand in the legal decomposition of m and $a_j < a_i$, then the admissible j are at most $i-4$ if and only if i is even, whereas the admissible j are at most $i-3$ if and only if i is odd. We are interested in computing the fraction of gaps (arising from the decompositions of all $m \in I_n$) of length g. This probability is given by

$$P_n(g) \;=\; c_n \sum_{m=0}^{a_{2n+1}-1} \sum_{i=1}^{2n-g} X_{i,g}(m),$$

where

$$c_n \;=\; \frac{1}{(\mu_n - 1)a_{2n+1}}. \tag{4.1}$$

To compute the above-mentioned probability we argue based on the parity of i. We find the contribution of gaps of length g from even i and odd i separately and then add these two. The case when $g = 3$ is a little simpler, as only even i contribute. If i were odd and $g = 3$ we would violate the notion of a Kentucky legal decomposition.

Part 1 of the Proof, Gap Preliminaries:

Case 1, i is even: Suppose that i is even. This means that a_i is the largest entry in its bin. Thus the largest possible summand less than a_i would be a_{i-4}. First we need to know the number of legal decompositions that only contain summands from $\{a_1, \ldots, a_{i-4}\}$, but this equals the number of integers that lie in $[0, a_{2(\frac{i-4}{2})+1}) \;= [0, a_{i-3})$. By (2.1), this is given by

$$a_{2(\frac{i-4}{2})+1} = a_{i-3} \;=\; \frac{1}{3}(2^{\frac{i}{2}} + (-1)^{\frac{i-2}{2}}).$$

Next we must consider the possible summands between a_{i+g} and a_{2n+1}. There are two cases to consider depending on the parity of $i+g$.

Subcase (i), g is even: Notice that in this case $i+g$ is even and if a_j is a summand in the legal decomposition of m with $a_{i+g} < a_j$, then $j \geq i+g+3$. In this case the number of legal decompositions only containing summands from the set $\{a_{i+g+3}, a_{i+g+4}, \ldots, a_{2n}\}$ is the same as the number of integers that lie in $[0, a_{(2n-(i+g+2))+1})$, which equals

$$a_{(2n-(i+g+2))+1} \;=\; a_{2\left(\frac{2n-(i+g+2)}{2}+1\right)-1} \;=\; \frac{1}{3}\left(2^{\frac{2n-(i+g)}{2}+1} + (-1)^{\frac{2n-(i+g)}{2}}\right).$$

So for fixed i and g both even, the number of $m \in I_n$ that have a gap of length g beginning at i is

$$\frac{1}{9}\left(2^{\frac{i}{2}} + (-1)^{\frac{i-2}{2}}\right)\left(2^{\frac{2n-(i+g)}{2}+1} + (-1)^{\frac{2n-(i+g)}{2}}\right).$$

Hence in this case we have that

$$\sum_{m=0}^{a_{2n+1}-1} \sum_{\substack{i=1 \\ i,g \text{ even}}}^{2n-g} X_{i,g}(m) \;=\; \frac{1}{9} \sum_{\substack{i=1 \\ i,g \text{ even}}}^{2n-g} \left(2^{\frac{i}{2}} + (-1)^{\frac{i-2}{2}}\right)\left(2^{\frac{2n-(i+g)}{2}+1} + (-1)^{\frac{2n-(i+g)}{2}}\right).$$

Subcase (ii), g is odd: In the case when i is even and g is odd, any legal decomposition of an integer $m \in I_n$ with a gap from i to $i+g$ that contains summands $a_j > a_{i+g}$ must have $j \geq i+g+4$. The number of legal decompositions achievable only with summands in the set $\{a_{i+g+4}, a_{i+g+5}, \dots, a_{2n}\}$ is the same as the number of integers in the interval $[0, a_{2n-(i+g+2)})$, which is given by

$$a_{2n-(i+g+2)} \;=\; a_{2\left(\frac{2n-(i+g+1)}{2}\right)-1} \;=\; \frac{1}{3}\left(2^{\frac{2n-(i+g+1)}{2}+1} + (-1)^{\frac{2n-(i+g+1)}{2}}\right).$$

Hence when i is even and g is odd we have that

$$\sum_{m=0}^{a_{2n+1}-1} \sum_{\substack{i=1 \\ i \text{ even}, g \text{ odd}}}^{2n-g} X_{i,g}(m) \;=\; \frac{1}{9} \sum_{\substack{i=1 \\ i \text{ even}, g \text{ odd}}}^{2n-g} (2^{\frac{i}{2}} + (-1)^{\frac{i-2}{2}})\left(2^{\frac{2n-(i+g+1)}{2}+1} + (-1)^{\frac{2n-(i+g+1)}{2}}\right).$$

Subcase (iii), $g = 3$: As remarked above, there are no gaps of length 3 when i is odd, and thus the contribution from i even is the entire answer and we can immediately find that

$$\begin{aligned}
P_n(3) \;&=\; c_n \sum_{m=0}^{a_{2n+1}-1} \sum_{\substack{i=1 \\ i \text{ even}}}^{2n-3} X_{i,3}(m) \\
&=\; \frac{1}{9} c_n \sum_{\substack{i=1 \\ i \text{ even}}}^{2n-3} \left(2^{\frac{i}{2}} + (-1)^{\frac{i-2}{2}}\right)\left(2^{\frac{2n-(i+4)}{2}+1} + (-1)^{\frac{2n-(i+4)}{2}}\right) \\
&=\; \frac{1}{9} c_n \sum_{\substack{i=1 \\ i \text{ even}}}^{2n-3} \left(2^{n-1} + 2^{\frac{i}{2}}(-1)^{\frac{2n-(i+4)}{2}} + 2^{\frac{2n-(i+4)}{2}+1}(-1)^{\frac{i-2}{2}} + (-1)^{n-3}\right).
\end{aligned}$$

As the largest term in the above sum is 2^{n-1}, we have

$$P_n(3) = \frac{c_n}{9}\left[(n-1)2^{n-1} + O(2^n)\right].$$

Since $\mu_n \sim \frac{n}{3}$ and $a_{2n+1} \sim \frac{1}{3}(4)(2^n)$, using (4.1) we find that up to lower order terms which vanish as $n \to \infty$ we have

$$c_n \;\sim\; \frac{9}{n2^{n+2}}. \tag{4.2}$$

Therefore

$$P_n(3) \;\sim\; \frac{1}{n2^{n+2}}\left[(n-1)2^{n-1} + O(2^n)\right] \;=\; \left(\frac{1}{8}\right)\left(\frac{n-1}{n}\right) + O\left(\frac{1}{n}\right).$$

Now as n goes to infinity we see that $P(3) = 1/8$.

Case 2, i is odd: Suppose now that i is odd. The largest possible summand less than a_i in a legal decomposition is a_{i-3}. As before we now need to know the number of integers that lie in $[0, a_{2(\frac{i-3}{2})+1})$, but this equals

$$a_{2(\frac{i-3}{2})+1} \;=\; a_{2(\frac{i-1}{2})-1} \;=\; \frac{1}{3}\left(2^{\frac{i-1}{2}+1} + (-1)^{\frac{i-1}{2}}\right).$$

We now need to consider the parity of $i+g$.

Subcase (i), g is odd: When i and g are odd, we know $i+g$ is even and therefore the first possible summand greater than a_{i+g} is a_{i+g+3}. Like before, the number of legal decompositions using summands from the set $\{a_{i+g+3}, a_{i+g+4}, \dots, a_{2n}\}$ is the same as the number of m with legal decompositions using summands from the set $\{a_1, a_2, \dots, a_{2n-(i+g+2)}\}$, which is $\frac{1}{3}\left(2^{\frac{2n-(i+g)}{2}+1} + (-1)^{\frac{2n-(i+g)}{2}}\right)$. This leads to

$$\sum_{m=0}^{a_{2n+1}-1} \sum_{\substack{i=1 \\ i \text{ odd}, g \text{ odd}}}^{2n-g} X_{i,g}(m) \;=\; \frac{1}{9} \sum_{\substack{i=1 \\ i \text{ odd}, g \text{ odd}}}^{2n-g} \left(2^{\frac{i-1}{2}+1} + (-1)^{\frac{i-1}{2}}\right)\left(2^{\frac{2n-(i+g)}{2}+1} + (-1)^{\frac{2n-(i+g)}{2}}\right).$$

Subcase (ii), g is even: Following the same line of argument we see that if i is odd and g is even, then

$$\sum_{m=0}^{a_{2n+1}-1} \sum_{\substack{i=1 \\ i \text{ odd}, g \text{ even}}}^{2n-g} X_{i,g}(m) \;=\; \frac{1}{9} \sum_{\substack{i=1 \\ i \text{ odd}, g \text{ even}}}^{2n-g} \left(2^{\frac{i-1}{2}+1} + (-1)^{\frac{i-1}{2}}\right)\left(2^{\frac{2n-(i+g+1)}{2}+1} + (-1)^{\frac{2n-(i+g+1)}{2}}\right).$$

Using these results, we can combine the various cases to determine the gap probabilities for different g.

Part 2 of the Proof, Gap Probabilities:

Case 1, g is even: As g is even, we have $g = 2j$ for some positive integer j. Therefore

$$\begin{aligned}
P_n(2j) &= c_n \sum_{m=0}^{a_{2n+1}-1} \sum_{i=1}^{2n-2j} X_{i,2j}(m) \\
&= c_n \sum_{m=0}^{a_{2n+1}-1} \sum_{\substack{i=1 \\ i \text{ even}}}^{2n-2j} X_{i,2j}(m) + c_n \sum_{m=0}^{a_{2n+1}-1} \sum_{\substack{i=1 \\ i \text{ odd}}}^{2n-2j} X_{i,2j}(m) \\
&= c_n \left[\frac{1}{9} \sum_{\substack{i=1 \\ i \text{ even}}}^{2n-2j} (2^{\frac{i}{2}} + (-1)^{\frac{i-2}{2}})(2^{\frac{2n-(i+2j)}{2}+1} + (-1)^{\frac{2n-(i+2j)}{2}}) \right] \\
&\quad + c_n \left[\frac{1}{9} \sum_{\substack{i=1 \\ i \text{ odd}}}^{2n-2j} (2^{\frac{i-1}{2}+1} + (-1)^{\frac{i-1}{2}})(2^{\frac{2n-(i+2j+1)}{2}+1} + (-1)^{\frac{2n-(i+2j+1)}{2}}) \right] \\
&= \frac{1}{9} c_n \sum_{\substack{i=1 \\ i \text{ even}}}^{2n-2j} (2^{n-j+1} + 2^{\frac{i}{2}}(-1)^{\frac{2n-(i+2j)}{2}} + 2^{\frac{2n-(i+2j)}{2}+1}(-1)^{\frac{i-2}{2}} + (-1)^{n-j-1}) \\
&\quad + \frac{1}{9} c_n \sum_{\substack{i=1 \\ i \text{ odd}}}^{2n-2j} (2^{n-j+1} + 2^{\frac{i-1}{2}+1}(-1)^{\frac{2n-(i+2j+1)}{2}} + 2^{\frac{2n-(i+2j+1)}{2}+1}(-1)^{\frac{i-1}{2}} + (-1)^{n-j-1}).
\end{aligned}$$

Notice that the largest terms in the above sums/expressions are given by 2^{n-j+1} and 2^{n-j+1}, the sum of which gives $4(n-j)2^{n-j}$. The rest of the terms are of lower order and are dominated as $n \to \infty$. Using (4.2) for c_n we find

$$P_n(2j) \sim \frac{c_n}{9} 4(n-j)2^{n-j} \sim \left(\frac{1}{n2^{n+2}} \right) 4(n-j)2^{n-j} = \frac{n-j}{n2^j},$$

and thus as n goes to infinity we see that $P(2j) = 1/2^j$.

Case 2, g is odd: As g is odd we may write $g = 2j+1$. Thus

$$\begin{aligned}
P_n(2j+1) &= c_n \sum_{m=0}^{a_{2n+1}-1} \sum_{i=1}^{2n-2j-1} X_{i,2j+1}(m) \\
&= c_n \sum_{m=0}^{a_{2n+1}-1} \sum_{\substack{i=1 \\ i \text{ even}}}^{2n-2j-1} X_{i,2j+1}(m) + c_n \sum_{m=0}^{a_{2n+1}-1} \sum_{\substack{i=1 \\ i \text{ odd}}}^{2n-2j-1} X_{i,2j+1}(m) \\
&= c_n \left[\frac{1}{9} \sum_{\substack{i=1 \\ i \text{ even}}}^{2n-2j-1} (2^{\frac{i}{2}} + (-1)^{\frac{i-2}{2}}) \left(2^{\frac{2n-(i+2j+2)}{2}+1} + (-1)^{\frac{2n-(i+2j+2)}{2}} \right) \right] \\
&\quad + c_n \left[\frac{1}{9} \sum_{\substack{i=1 \\ i \text{ odd}}}^{2n-2j-1} (2^{\frac{i-1}{2}+1} + (-1)^{\frac{i-1}{2}})(2^{\frac{2n-(i+2j+1)}{2}+1} + (-1)^{\frac{2n-(i+2j+1)}{2}}) \right] \\
&= \frac{1}{9} c_n \sum_{\substack{i=1 \\ i \text{ even}}}^{2n-2j-1} \left(2^{n-j} + 2^{\frac{i}{2}} (-1)^{\frac{2n-(i+2j+2)}{2}} + 2^{\frac{2n-(i+2j+2)}{2}+1} (-1)^{\frac{i-2}{2}} + (-1)^{n-j-2} \right) \\
&\quad + \frac{1}{9} c_n \sum_{\substack{i=1 \\ i \text{ odd}}}^{2n-2j-1} \left(2^{n-j+1} + 2^{\frac{i-1}{2}+1} (-1)^{\frac{2n-(i+2j+1)}{2}} + 2^{\frac{2n-(i+2j+1)}{2}+1} (-1)^{\frac{i-1}{2}} + (-1)^{n-j-1} \right).
\end{aligned}$$

Notice that the largest terms in the above sums/expressions are given by 2^{n-j} and 2^{n-j+1}, the sum of which gives $3(n-j)2^{n-j}$. The rest of the terms are of lower order and are dominated as $n \to \infty$. Using (4.2) for c_n we find

$$P_n(2j+1) \sim \frac{c_n}{9} 3(n-j)2^{n-j} \sim \left(\frac{1}{n 2^{n+2}} \right) 3(n-j) 2^{n-j} = \left(\frac{3}{4} \right) \left(\frac{n-j}{n 2^j} \right),$$

and thus as n goes to infinity we see that $P(2j+1) = \frac{3}{4}\left(1/2^j\right)$. □

5. Conclusion and Future Work

Our results generalize Zeckendorf's theorem to an interesting new class of recurrence relations, specifically to a case where the first coefficient is zero. While we still have uniqueness of decompositions in the Kentucky sequence, that is not always the case for this class of recurrences. In a future work [6] we study another example with first coefficient zero, the recurrence $a_{n+1} = a_{n-1} + a_{n-2}$. This leads to what we call the Fibonacci Quilt, and there uniqueness of decomposition fails. The non-uniqueness gives rise to new interesting discussions, for example the handling of the question of Gaussian behavior for the distribution of the number of summands given that we now have multiple decompositions for most integers; we address these issues in [6].

Additionally, the Kentucky sequence is but one of infinitely many (s, b)-Generacci sequences; in a future work [7] we hope to give a detailed study of these sequences and to extend the results of this paper to arbitrary (s, b). The difficulty is that many of the arguments in the paper here crucially use explicit formulas available for quantities associated to the Kentucky sequence, which are not known for general sequences. This difficulty mirrors the difference between [18] (which used binomial coefficient expressions from the Zeckendorf decompositions) and [22] (the general case required many technical arguments).

Definition 5.1. *Let an increasing sequence of positive integers $\{a_i\}_{i=1}^{\infty}$ and a family of subsequences $\mathcal{B}_n = \{a_{b(n-1)+1}, \ldots, a_{bn}\}$ be given. (We call these subsequences* bins*.) We declare a decomposition of an integer $m = a_{\ell_1} + a_{\ell_2} + \cdots + a_{\ell_k}$ where $a_{\ell_i} < a_{\ell_{i+1}}$ to be a (s, b)-*Generacci decomposition *provided $\{a_{\ell_i}, a_{\ell_{i+1}}\} \not\subset \mathcal{B}_{j-s} \cup \mathcal{B}_{j-s+1} \cup \cdots \cup \mathcal{B}_j$ for all i, j. (We say $\mathcal{B}_j = \emptyset$ for $j \leq 0$.)*

This says that for all $a_{\ell_i} \in \mathcal{B}_j$, no other $a_{\ell_{i'}}$ is also in the jth bin nor in any of the adjacent s bins preceding $\mathcal{B}_j$ nor the s bins succeeding $\mathcal{B}_j$.

Definition 5.2. *An increasing sequence of positive integers $\{a_i\}_{i=1}^{\infty}$ is called an (s, b)-*Generacci sequence *if every a_i for $i \geq 1$ is the smallest positive integer that does not have a (s, b)-Generacci legal decomposition using the elements $\{a_1, \ldots, a_{i-1}\}$.*

Note that we still have uniqueness of decompositions as in Theorem 1.4; this follows from Theorem 1.3 of [9]. Numerical simulations suggest that the number of summands in the unique (s, b)-Generacci decomposition of a positive integer exhibits Gaussian behavior. The Fibonacci polynomial approach in Section 3 extends nicely for general b, thus proving Gaussianity for all $(1, b)$-Generacci sequences. The technique however fails to generalize for $s > 1$. We are investigating methods to attack the general case.

Appendix A. Unique Decompositions

Proof of Theorem 1.4. Our proof is constructive. We build our sequence by only adjoining terms that ensure that we can *uniquely* decompose a number while never using more than one summand from the same bin or two summands from adjacent bins. The sequence begins:

$$\underbrace{1,\ 2}_{\mathcal{B}_1},\ \underbrace{3,\ 4}_{\mathcal{B}_2},\ \underbrace{5,\ 8}_{\mathcal{B}_3},\ \ldots.$$

Note we would not adjoin 9 because then 9 would legally decompose two ways, as $9 = 9$ and as $9 = 8 + 1$. The next number in the sequence must be the smallest integer that cannot be decomposed legally using the current terms.

We proceed with proof by induction. The base case follows from a direct calculation. Notice that if $i \leq 5$ then $i = a_i$. Also $6 = a_5 + a_1$.

The sequence continues:

$$\ldots,\ \underbrace{a_{2n-5},\ a_{2n-4}}_{\mathcal{B}_{n-2}},\ \underbrace{a_{2n-3},\ a_{2n-2}}_{\mathcal{B}_{n-1}},\ \underbrace{a_{2n-1},\ a_{2n}}_{\mathcal{B}_n},\ \underbrace{a_{2n+1},\ a_{2n+2}}_{\mathcal{B}_{n+1}},\ \ldots$$

By induction we assume that there exists a unique decomposition for all integers $m \leq a_{2n} + w$, where w is the maximum integer that legally can be decomposed using terms in the set $\{a_1, a_2, a_3, \ldots, a_{2n-4}\}$. By construction we know that $w = a_{2n-3} - 1$, as this was the reason we adjoined a_{2n-3} to the sequence.

Now let y be the maximum integer that can be legally decomposed using terms in the set $\{a_1, a_2, a_3, \ldots, a_{2n}\}$. By construction we have

$$y \ = \ a_{2n} + w \ = \ a_{2n} + a_{2n-3} - 1.$$

Similarly, let x be the maximum integer that legally can be decomposed using terms in the set $\{a_1, a_2, a_3, \ldots, a_{2n-2}\}$. Note $x = a_{2n-1} - 1$ as this is why we include a_{2n-1} in the sequence.

Claim: $a_{2n+1} = y + 1$ and this decomposition is unique.

By induction we know that y was the largest value that we could legally make using only terms in $\{a_1, a_2, \dots, a_{2n}\}$. Hence we choose $y+1$ as a_{2n+1} and $y+1$ has a unique decomposition.

Claim: All $N \in [y+1, y+1+x] = [a_{2n+1}, a_{2n+1} + x]$ have a unique decomposition.

We can legally and uniquely decompose all of $1, 2, 3, \dots, x$ using elements in the set $\{a_1, a_2, \dots, a_{2n-2}\}$. Adding a_{2n+1} to the decomposition is still legal since a_{2n+1} is not a member of any bins adjacent to $\{\mathcal{B}_1, \mathcal{B}_2, \dots, \mathcal{B}_{n-1}\}$. The uniqueness follows from the fact that if we do not include a_{2n+1} as a summand, then the decomposition does not yield a number big enough to exceed $y + 1$.

Claim: $a_{2n+2} = y + 1 + x + 1 = a_{2n+1} + x + 1$ and this decomposition is unique.

By construction the largest integer that legally can be decomposed using terms $\{a_1, a_2, \dots, a_{2n+1}\}$ is $y + 1 + x$.

Claim: All $N \in [a_{2n+2}, a_{2n+2} + x]$ have a unique decomposition.

First note that the decomposition exists as we can legally and uniquely construct $a_{2n+2} + v$, where $0 \le v \le x$. For uniqueness, we note that if we do not use a_{2n+2}, then the summation would be too small.

Claim: $a_{2n+2} + x$ is the largest integer that legally can be decomposed using terms $\{a_1, a_2, \dots, a_{2n+2}\}$.

This follows from construction. □

Appendix B. Generating Function Proofs

In §3 we proved that the distribution of the number of summands in a Kentucky decomposition exhibits Gaussian behavior by using properties of Fibonacci polynomials. This approach was possible because we had an explicit, tractable form for the $p_{n,k}$'s (Proposition 2.4) that coincided with the explicit sum formulas associated with the Fibonacci polynomials. Below we present a second proof of Gaussian behavior using a more general approach, which might be more useful in addressing the behavior of the number of summands when dealing with general (s, b)-Generacci sequences.

As in the first proof, we are interested in $g_n(y)$, the coefficient of the x^n term in $F(x, y)$.

Lemma B.1. *We have*

$$\begin{aligned} g_n(y) \quad &= \quad \frac{1}{2^{n+1}\sqrt{1+8y}} \left[4y\left(1+\sqrt{1+8y}\right)^n - 4y\left(1-\sqrt{1+8y}\right)^n \right. \\ &\quad + \left. \left(1+\sqrt{1+8y}\right)^{n+1} - \left(1-\sqrt{1+8y}\right)^{n+1} \right]. \end{aligned} \tag{B.1}$$

Proof. For brevity set $x_1 = x_1(y)$ and $x_2 = x_2(y)$ for the roots of x in $x^2 + \frac{1}{2y}x - \frac{1}{2y}$. In particular, we find

$$x_1 \;=\; -\frac{1}{4y}\left(1+\sqrt{1+8y}\right) \qquad x_2 \;=\; -\frac{1}{4y}\left(1-\sqrt{1+8y}\right). \tag{B.2}$$

Since x_1 and x_2 are unequal for all $y > 0$, we can decompose $F(x, y)$ using partial fractions:

$$F(x,y) \;=\; \frac{1+2xy}{-2y(x-x_1)(x-x_2)} \;=\; \frac{1+2xy}{-2y}\frac{1}{x_1-x_2}\left[\frac{1}{x-x_1} - \frac{1}{x-x_2}\right].$$

Using the geometric series formula, after some algebra we obtain

$$F(x,y) \;=\; \frac{1+2xy}{-2y}\frac{1}{x_1-x_2}\sum_{i\geq 0}\left[\frac{1}{x_1}\left(\frac{x}{x_1}\right)^i - \frac{1}{x_2}\left(\frac{x}{x_2}\right)^i\right].$$

From here we find that that the coefficient of x^n is

$$g_n(y) \;=\; \frac{1}{-2y(x_1-x_2)}\left[\frac{1}{x_1^{n+1}} - \frac{1}{x_2^{n+1}} + \frac{2y}{x_1^n} - \frac{2y}{x_2^n}\right].$$

Substituting the functions from (B.2) and simplifying we obtain the desired result. □

As we mentioned in §3.1, we have the following corollary.

Corollary B.2. *Let $F_n(x)$ be a Fibonacci polynomial. Then*

$$F_n(x) \;=\; \frac{(x+\sqrt{x^2+4})^n - (x-\sqrt{x^2+4})^n}{2^n\sqrt{x^2+4}}.$$

Proof. Set the right hand sides of equations (3.4) and (B.1) equal and let $x = 1/\sqrt{2y}$. □

Proof of Proposition 3.1. Straightforward, but somewhat tedious, calculations give

$$\begin{aligned} g_n(1) \;&=\; \frac{1}{3}\left((-1)^{n+1}+2^{n+2}\right) \\ g_n'(1) \;&=\; \frac{n}{9}\left(2^{n+2}+2(-1)^{n+1}\right) + \frac{2}{27}\left(2^{n+2}\right) + o(1). \end{aligned}$$

Dividing these two quantities and using Lemma 3.3 gives the desired result. □

Proof of Proposition 3.2. Another straightforward (and again somewhat tedious) calculation yields

$$\begin{aligned} \sigma_n^2 \;&=\; \frac{2^{2n+5}(4+3n) - 2(8+3n) - 2^{n+2}(-1)^n(28+36n+9n^2)}{81(2^{n+2}-(-1)^n)^2} \\ &=\; \frac{n\left[(6)2^{2n+4} - 18(-1)^n2^{n+3} - 6\right] + \left[(8)2^{2n+4} - 14(-1)^n2^{n+3} - 16\right] - 4.5(-1)^n n^2 2^{n+3}}{81\left[2^{2n+4} - (-1)^n2^{n+3}+1\right]}. \end{aligned}$$

□

Proof of Theorem 1.5. As in our earlier proof, we show that the moment generating function of Y_n' converges to that of the standard normal. Following the same argument as in [9, Lemma 4.9], the moment generating function $M_{Y_n'}(t)$ of Y_n' is

$$M_{Y_n'}(t) \;=\; \frac{g_n(e^{t/\sigma_n})e^{-t\mu_n/\sigma_n}}{g_n(1)}.$$

Taking logarithms yields

$$\log M_{Y_n'}(t) \;=\; \log[g_n(e^{t/\sigma_n})] - \log[g_n(1)] - \frac{t\mu_n}{\sigma_n}. \tag{B.3}$$

We tackle the right hand side in pieces.

Let $r_n = t/\sigma_n$. Since $\sigma_n^2 = \frac{2n}{27} + \frac{8}{81} + O\left(\frac{n^2}{2^n}\right)$, as n goes to infinity r_n goes to 0. This allows us to use Taylor series expansions.

First we rewrite $g_n(e^{r_n})$

$$\begin{aligned} g_n(e^{r_n}) =& \frac{1}{\sqrt{1+8e^{r_n}}}\left[\frac{(1+\sqrt{1+8e^{r_n}})^n(4e^{r_n}+1+\sqrt{1+8e^{r_n}})}{2^{n+1}}\right. \\ & \left. -\frac{4e^{r_n}(1-\sqrt{1+8e^{r_n}})^n}{2^{n+1}} - \frac{(1-\sqrt{1+8e^{r_n}})^{n+1}}{2^{n+1}}\right]. \end{aligned}$$

Using Taylor series expansions of the exponential and square root functions we obtain

$$e^{r_n} = 1 + o(1) \quad \text{and} \quad \frac{1-\sqrt{1+8e^{r_n}}}{2} = -1 + o(1).$$

Thus

$$\begin{aligned} \frac{4e^{r_n}(1-\sqrt{1+8e^{r_n}})^n}{2^{n+1}} + \frac{(1-\sqrt{1+8e^{r_n}})^{n+1}}{2^{n+1}} &= 2(-1)^n + o(1) - (-1)^n + o(1) \\ &= (-1)^n + o(1). \end{aligned}$$

Hence

$$g_n(e^{r_n}) = \frac{1}{\sqrt{1+8e^{r_n}}}\left[\frac{(1+\sqrt{1+8e^{r_n}})^n(4e^{r_n}+1+\sqrt{1+8e^{r_n}})}{2^{n+1}} - (-1)^n + o(1)\right].$$

So

$$\begin{aligned} \log(g_n(e^{r_n})) &= -\tfrac{1}{2}\log(1+8e^{r_n}) + n\log(1+\sqrt{1+8e^{r_n}}) \\ &\quad + \log(4e^{r_n}+1+\sqrt{1+8e^{r_n}}) - (n+1)\log 2 + o(1). \end{aligned}$$

Continuing to use Taylor series expansions

$$\begin{aligned} \log(g_n(e^{r_n})) =& -\tfrac{1}{2}\left[\log 9 + \frac{8}{9}r_n + \frac{4}{81}r_n^2\right] + n\left[\log 4 + \frac{1}{3}r_n + \frac{1}{27}r_n^2\right] \\ & + \left[\log 8 + \frac{2}{3}r_n + \frac{2}{27}r_n^2\right] + O(r_n^3) - (n+1)\log 2 + o(1). \end{aligned} \tag{B.4}$$

Finally, recall $g_n(1) = \frac{1}{3}[(-1)^{n+1} + 2^{n+2}]$ so

$$\log[g_n(1)] = -\log 3 + (n+2)\log 2 + o(1). \tag{B.5}$$

To finish we plug values into (B.3). In particular, plug in $\log(g_n(e^{r_n}))$ from (B.4), $\log[g_n(1)]$ from (B.5), μ_n from Proposition 3.1, σ_n from Proposition 3.2, and $r_n = t/\sigma_n$. This gives

$$\log M_{Y_n'}(t) = \frac{t^2}{2} + o(1).$$

Thus, $M_{Y_n'}(t)$ converges to the moment generating function of the standard normal distribution. Which according to probability theory, implies that the distribution of Y_n' converges to the standard normal distribution. □

References

[1] H. Alpert, *Differences of multiple Fibonacci numbers*, Integers: Electronic Journal of Combinatorial Number Theory **9** (2009), 745–749.

[2] O. Beckwith, A. Bower, L. Gaudet, R. Insoft, S. Li, S. J. Miller and P. Tosteson, *The Average Gap Distribution for Generalized Zeckendorf Decompositions*, Fibonacci Quarterly **51** (2013), 13–27.

[3] I. Ben-Ari and S. J. Miller, *A Probabilistic Approach to Generalized Zeckendorf Decompositions*, preprint. `http://arxiv.org/pdf/1405.2379`

[4] A. Best, P. Dynes, X. Edelsbrunner, B. McDonald, S. J. Miller, K. Tor, C. Turnage-Butterbaugh, M. Weinstein, *Gaussian Distribution of Number of Summands in Zeckendorf Decompositions in Small Intervals*, `http://arxiv.org/pdf/1501.06833`

[5] A. Bower, R. Insoft, S. Li, S. J. Miller and P. Tosteson, *The Distribution of Gaps between Summands in Generalized Zeckendorf Decompositions*, to appear in the Journal of Combinatorial Theory, Series A. `http://arxiv.org/abs/1402.3912`.

[6] M. Catral, P. Ford, P. Harris, S. J. Miller and D. Nelson, *The Fibonacci Quilt and Zeckendorf Decompositions*, in preparation.

[7] M. Catral, P. Ford, P. Harris, S. J. Miller and D. Nelson, *The Generacci Recurrences and Zeckendorf's Theorem*, in preparation.

[8] D. E. Daykin, *Representation of Natural Numbers as Sums of Generalized Fibonacci Numbers*, J. London Mathematical Society **35** (1960), 143–160.

[9] P. Demontigny, T. Do, A. Kulkarni, S. J. Miller, D. Moon and U. Varma, *Generalizing Zeckendorf's Theorem to f-decompositions*, Journal of Number Theory **141** (2014), 136–158.

[10] P. Demontigny, T. Do, A. Kulkarni, S. J. Miller and U. Varma, *A Generalization of Fibonacci Far-Difference Representations and Gaussian Behavior*, to appear in the Fibonacci Quarterly. http://arxiv.org/pdf/1309.5600v2.

[11] M. Drmota and J. Gajdosik, *The distribution of the sum-of-digits function*, J. Théor. Nombrés Bordeaux **10** (1998), no. 1, 17–32.

[12] P. Filipponi, P. J. Grabner, I. Nemes, A. Pethö, and R. F. Tichy, *Corrigendum to: "Generalized Zeckendorf expansions"*, Appl. Math. Lett., **7** (1994), no. 6, 25–26.

[13] P. J. Grabner and R. F. Tichy, *Contributions to digit expansions with respect to linear recurrences*, J. Number Theory **36** (1990), no. 2, 160–169.

[14] P. J. Grabner, R. F. Tichy, I. Nemes, and A. Pethö, *Generalized Zeckendorf expansions*, Appl. Math. Lett. **7** (1994), no. 2, 25–28.

[15] T. J. Keller, *Generalizations of Zeckendorf's theorem*, Fibonacci Quarterly **10** (1972), no. 1 (special issue on representations), 95–102.

[16] C. G. Lekkerkerker, *Voorstelling van natuurlyke getallen door een som van getallen van Fibonacci*, Simon Stevin **29** (1951-1952), 190–195.

[17] T. Lengyel, *A Counting Based Proof of the Generalized Zeckendorf's Theorem*, Fibonacci Quarterly **44** (2006), no. 4, 324–325.

[18] M. Koloğlu, G. Kopp, S. J. Miller and Y. Wang, *On the number of summands in Zeckendorf decompositions*, Fibonacci Quarterly **49** (2011), no. 2, 116–130.

[19] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, Wiley-Interscience, New York, 2001.

[20] S. J. Miller, E. Bradlow and K. Dayaratna, *Closed-Form Bayesian Inferences for the Logit Model via Polynomial Expansions*, Quantitative Marketing and Economics **4** (2006), no. 2, 173–206.

[21] S. J. Miller and R. Takloo-Bighash, *An Invitation to Modern Number Theory*, Princeton University Press, Princeton, NJ, 2006.

[22] S. J. Miller and Y. Wang, *From Fibonacci numbers to Central Limit Type Theorems*, Journal of Combinatorial Theory, Series A **119** (2012), no. 7, 1398–1413.

[23] S. J. Miller and Y. Wang, *Gaussian Behavior in Generalized Zeckendorf Decompositions*, to appear in the conference proceedings of the 2011 Combinatorial and Additive Number Theory Conference. http://arxiv.org/pdf/1107.2718v1.

[24] M. Nathanson, *Additive Number Theory: The Classical Bases*, Graduate Texts in Mathematics, Springer-Verlag, New York, 1996.

[25] W. Steiner, *Parry expansions of polynomial sequences*, Integers **2** (2002), Paper A14.

[26] W. Steiner, *The Joint Distribution of Greedy and Lazy Fibonacci Expansions*, Fibonacci Quarterly **43** (2005), 60–69.

[27] E. Zeckendorf, *Représentation des nombres naturels par une somme des nombres de Fibonacci ou de nombres de Lucas*, Bulletin de la Société Royale des Sciences de Liége **41** (1972), 179–182.

Department of Mathematics and Computer Science, Xavier University, Cincinnati, OH 45207
E-mail address: catralm@xavier.edu

Department of Mathematics and Statistics, University of Nebraska at Kearney, Kearney, NE 68849
E-mail address: fordpl@unk.edu

Department of Mathematical Sciences, United States Military Academy, West Point, NY 10996
E-mail address: Pamela.Harris@usma.edu

Department of Mathematics and Statistics, Williams College, Williamstown, MA 01267
E-mail address: sjm1@williams.edu, Steven.Miller.MC.96@aya.yale.edu

Department of Mathematics, Saint Peter's University, Jersey City, NJ 07306
E-mail address: dnelson1@saintpeters.edu

ALGEBRAIC STATEMENTS SIMILAR TO THOSE IN RAMANUJAN'S "LOST NOTEBOOK"

CURTIS COOPER

Abstract. Ramanujan's "lost notebook" contains algebraic statements

$$\text{if } g^4 = 5, \text{ then } \frac{\sqrt[5]{3+2g} - \sqrt[5]{4-4g}}{\sqrt[5]{3+2g} + \sqrt[5]{4-4g}} = 2 + g + g^2 + g^3,$$

and

$$\text{if } g^5 = 2, \text{ then } \sqrt{1+g^2} = \frac{g^4 + g^3 + g - 1}{\sqrt{5}}.$$

In this paper we will discover algebraic statements similar to those in Ramanujan's "lost notebook". For example, we will prove algebraic statements like

$$\text{if } g^3 = 2, \text{ then } \frac{\sqrt[4]{111-87g} + \sqrt[4]{g-1}}{\sqrt[4]{111-87g} - \sqrt[4]{g-1}} = 2 + g + g^2,$$

and

$$\text{if } g^5 = 2, \text{ then } \sqrt{-3g^2 + 4g + 5} = g^4 - g^3 + g + 1.$$

1. Introduction

Page 344 of Ramanujan's "lost notebook" [2] contains twelve algebraic statements. Recently, Hirschhorn [1] gave simple proofs of these statements. Here are some of the statements.

If $g^5 = 3$, then

$$\frac{\sqrt{g^2+1} + \sqrt{5g-5}}{\sqrt{g^2+1} - \sqrt{5g-5}} = \frac{1}{g} + g + g^2 + g^3. \tag{1.1}$$

If $g^5 = 2$, then

$$\sqrt{1+g^2} = \frac{g^4 + g^3 + g - 1}{\sqrt{5}}. \tag{1.2}$$

If $g^5 = 2$, then

$$\sqrt{4g-3} = \frac{g^9 + g^7 - g^6 - 1}{\sqrt{5}}. \tag{1.3}$$

If $g^5 = 2$, then

$$\sqrt[5]{1+g+g^3} = \frac{\sqrt{1+g^2}}{\sqrt[10]{5}}, \tag{1.4}$$

If $g^4 = 5$, then

$$\frac{\sqrt[5]{3+2g} - \sqrt[5]{4-4g}}{\sqrt[5]{3+2g} + \sqrt[5]{4-4g}} = 2 + g + g^2 + g^3. \tag{1.5}$$

In this paper we will discover algebraic statements similar to the above statements in Ramanujan's "lost notebook". The paper is organized as follows. Section 2 gives algebraic statements similar to (1.1). Section 3 gives algebraic statements similar to (1.2) and (1.3).

Section 4 gives algebraic statements similar to (1.4) and Section 5 gives algebraic statements similar to (1.5). Finally, Section 6 gives other algebraic statements in the spirit of Ramanujan.

One technique we will use throughout the paper is *componendo et dividendo*, which states that

$$\frac{a+b}{a-b}=\frac{c+d}{c-d} \text{ if and only if } \frac{a}{b}=\frac{c}{d}.$$

2. Algebraic Statement Similar to (1.1)

We wish to find an algebraic statement similar to Ramanujan's (1.1). The following theorem is similar to (1.1).

Theorem 2.1. *If $g^5=2$, then*

$$\frac{\sqrt{4g^2+g+2}+\sqrt{8g^2+41g-54}}{\sqrt{4g^2+g+2}-\sqrt{8g^2+41g-54}}=\frac{1}{g}+g+g^2+g^3. \tag{2.1}$$

Proof. Equation (2.1) can be written as

$$\frac{\sqrt{4g^2+g+2}+\sqrt{8g^2+41g-54}}{\sqrt{4g^2+g+2}-\sqrt{8g^2+41g-54}}=\frac{1+g^2+g^3+g^4}{g}.$$

Thus, by *componendo et dividendo*, we need to show that

$$\sqrt{\frac{4g^2+g+2}{8g^2+41g-54}}=\frac{1+g+g^2+g^3+g^4}{1-g+g^2+g^3+g^4}.$$

This is equivalent to showing that

$$(1+g+g^2+g^3+g^4)^2(8g^2+41g-54)=(1-g+g^2+g^3+g^4)^2(4g^2+g+2). \tag{2.2}$$

Expanding both sides of (2.2) and using the fact that $g^5=2$, the left- and right-hand sides of (2.2) are equal and the theorem is proved. □

3. Algebraic Statements Similar to (1.2) and (1.3)

We wish to find an algebraic statement similar to Ramanujan's (1.2) and (1.3). The following theorem is similar to (1.2).

Theorem 3.1. *If $g^5=8$, then*

$$\sqrt{2g^2-3}=\frac{g^4+2g^3-2g^2-2}{2\sqrt{5}}.$$

Proof. Using the fact that $g^5=8$, we have the following equalities.

$$\begin{aligned}(g^4+2g^3-2g^2-2)^2&=g^8+4g^7-8g^5-8g^3+8g^2+4\\&=8g^3+32g^2-64-8g^3+8g^2+4\\&=40g^2-60=20(2g^2-3).\end{aligned}$$

This proves the theorem. □

To find more algebraic identities similar to (1.2), we wrote a C++ program to search for solutions to

$$(R+Sg+Tg^2+Ug^3+Vg^4)^2=Cg^2+E.$$

We discovered the following (two) theorems.

Theorem 3.2. *If $g^5 = 18$, then*

$$\sqrt{g^2-3} = \frac{g^4+3g^3-6g^2-3g-9}{15}.$$

Proof.

$$\begin{aligned}(g^4+3g^3-6g^2-3g-9)^2 &= g^8+6g^7-3g^6-42g^5-18g^3+117g^2+54g+81\\ &= 18g^3+108g^2-54g-756-18g^3+117g^2+54g+81\\ &= 225g^2-675 = 225(g^2-3).\end{aligned}$$

□

Theorem 3.3. *If $g^5 = 49$, then*

$$\sqrt{8g^2-7} = \frac{6g^4+14g^3-14g^2+14g-49}{35}.$$

Proof.

$$\begin{aligned}&(6g^4+14g^3-14g^2+14g-49)^2\\ &= 36g^8+168g^7+28g^6-224g^5-1764g^3+1568g^2-1372g+2401\\ &= 1764g^3+8232g^2+1372g-10976-1764g^3+1568g^2-1372g+2401\\ &= 9800g^2-8575 = 1225(8g^2-7).\end{aligned}$$

□

The following theorem is similar to Ramanujan's (1.3).

Theorem 3.4. *If $g^5 = 8$, then*

$$\sqrt{g+2} = \frac{g^4-g^3+4g+4}{2\sqrt{10}}.$$

Proof. Using the fact that $g^5 = 8$, we have the following equalities.

$$\begin{aligned}(g^4-g^3+4g+4)^2 &= g^8-2g^7+g^6+8g^5-8g^3+16g^2+32g+16\\ &= 8g^3-16g^2+8g+64-8g^3+16g^2+32g+16\\ &= 40g+80 = 40(g+2).\end{aligned}$$

This proves the theorem. □

To find more algebraic identities similar to (1.3), we wrote a C++ program to search for solutions to

$$(R+Sg+Tg^2+Ug^3+Vg^4)^2 = Dg+E.$$

We discovered the following (three) theorems. The proofs are similar to the proof above.

Theorem 3.5. *If $g^5 = 12$, then*

$$\sqrt{11g-7} = \frac{g^4-g^3+2g^2-8g-10}{2\sqrt{5}}.$$

Theorem 3.6. *If $g^5 = 2$, then*

$$\sqrt{4g-3} = \frac{2g^4 + 2g^2 - 2g - 1}{\sqrt{5}}.$$

Theorem 3.7. *If $g^5 = 4$, then*

$$\sqrt{g+1} = \frac{g^4 + g^3 + 2g^2 - 2}{2\sqrt{5}}.$$

Theorem 3.8. *If $g^5 = 7$, then*

$$\sqrt{-g+8} = \frac{2g^4 - g^3 - 2g^2 + 6g + 2}{5}.$$

Theorem 3.9. *If $g^5 = 24$, then*

$$\sqrt{-g+2} = \frac{g^4 - g^3 - 4g^2 + 4g - 4}{10\sqrt{2}}.$$

We discovered the following (two) theorems similar to Ramanujan's (1.2) and (1.3).

Theorem 3.10. *If $g^5 = 2$, then*

$$\sqrt{8g^2 - 20g + 17} = g^9 - g^7 + g^6 - 1.$$

Proof. Using the fact that $g^5 = 2$, we have the following equalities.

$$\begin{aligned}
(g^9 - g^7 + g^6 - 1)^2 &= (2g^4 - 2g^2 + 2g - 1)^2 \\
&= 4g^8 - 8g^6 + 8g^5 - 8g^3 + 8g^2 - 4g + 1 \\
&= 8g^3 - 16g + 16 - 8g^3 + 8g^2 - 4g + 1 \\
&= 8g^2 - 20g + 17.
\end{aligned}$$

This proves the theorem. □

Theorem 3.11. *If $g^5 = 2$, then*

$$\sqrt{-3g^2 + 4g + 5} = g^4 - g^3 + g + 1.$$

4. Algebraic Statements Similar to (1.4)

We wish to find an algebraic statement similar to Ramanujan's (1.4). The following theorem is similar to (1.4).

Theorem 4.1. *If $g^5 = 8$, then*

$$\sqrt[5]{2 + 2g + g^2} = \frac{\sqrt{2+g}}{\sqrt[10]{10}}.$$

Proof. Using the fact that $g^5 = 8$, we have the following equalities.

$$\begin{aligned}
(2+g)^5 &= 32 + 80g + 80g^2 + 40g^3 + 10g^4 + g^5 \\
&= 32 + 80g + 80g^2 + 40g^3 + 10g^4 + 8 \\
&= 40 + 80g + 80g^2 + 40g^3 + 10g^4 \\
&= 10(4 + 8g + 8g^2 + 4g^3 + g^4) \\
&= 10(2 + 2g + g^2)^2.
\end{aligned}$$

This proves the theorem. □

To find more algebraic identities similar to (1.4), we wrote a C++ program to search for solutions to

$$(R+Sg+Tg^2)^5=(A+Bg+Cg^2+Dg^3)^2.$$

The next theorem is one that we discovered. The proof is similar to the proof above.

Theorem 4.2. *If $g^5=8$, then*

$$\sqrt[5]{1+g}=\frac{\sqrt{2+g}}{\sqrt[10]{40}}.$$

5. Algebraic Statements Similar to (1.5)

We wish to find an algebraic statement similar to Ramanujan's (1.5). To construct a result similar to (1.5), we want to find integers h, A, B, C, D, and E such that if $g^4=h+1$, then

$$\sqrt[5]{\frac{Ag+B}{Cg+D}}=\frac{E+1+g+g^2+g^3}{1+g+g^2+g^3}. \tag{5.1}$$

Then, by *componendo et dividendo*, we would have

$$\frac{\sqrt[5]{Ag+B}+\sqrt[5]{Cg+D}}{\sqrt[5]{Ag+B}-\sqrt[5]{Cg+D}}=\frac{E+2+2g+2g^2+2g^3}{E}.$$

Simplifying the RHS of (5.1) and using the fact that $g^4=h+1$, we have

$$\begin{aligned}\sqrt[5]{\frac{Ag+B}{Cg+D}}&=\frac{E+1+g+g^2+g^3}{1+g+g^2+g^3}\\&=\frac{E+\frac{h}{g-1}}{\frac{h}{g-1}}=\frac{Eg+h-E}{h}.\end{aligned}$$

Thus,

$$\frac{Ag+B}{Cg+D}=\frac{(Eg+h-E)^5}{h^5}$$

and so

$$h^5(Ag+B)=(Cg+D)(Eg+h-E)^5. \tag{5.2}$$

Expanding the polynomial on the RHS of (5.2), we have

$$\begin{aligned}&CE^5g^6+(DE^5-5CE^5+5ChE^4)g^5\\&+(10Ch^2E^3+5DhE^4-20ChE^4+10CE^5-5DE^5)g^4\\&+(10Ch^3E^2+30ChE^4+10DE^5+10Dh^2E^3-30Ch^2E^3-10CE^5-20DhE^4)g^3\\&+(5Ch^4E-20ChE^4-30Dh^2E^3-10DE^5+30Ch^2E^3-20Ch^3E^2+5CE^5\\&\qquad+30DhE^4+10Dh^3E^2)g^2\\&+(-10Ch^2E^3+10Ch^3E^2+5Dh^4E-20Dh^3E^2+30Dh^2E^2+30Dh^2E^3+5DE^5\\&\qquad+5ChE^4-20DhE^4-5Ch^4E-CE^5+Ch^5)g\\&-10Dh^2E^3-DE^5-5Dh^4E+Dh^5+5DhE^4+10Dh^3E^2.\end{aligned}$$

Assuming $g^4 = h+1$, we continue to simplify the polynomial on the RHS of (5.2) by defining

$$\begin{aligned}
&CE^5g^6 = CE^5(h+1)g^2 = Fg^2\\
&(DE^5 - 5CE^5 + 5ChE^4)g^5 = (DE^5 - 5CE^5 + 5ChE^4)(h+1)g = Gg\\
&(10Ch^2E^3 + 5DhE^4 - 20ChE^4 + 10CE^5 - 5DE^5)g^4\\
&\qquad = (10Ch^2E^3 + 5DhE^4 - 20ChE^4 + 10CE^5 - 5DE^5)(h+1) = H.
\end{aligned}$$

In addition we define

$$\begin{aligned}
&10Ch^3E^2 + 30ChE^4 + 10DE^5 + 10Dh^2E^3 - 30Ch^2E^3\\
&\qquad - 10CE^5 - 20DhE^4 = I\\
&5Ch^4E - 20ChE^4 - 30Dh^2E^3 - 10DE^5 + 30Ch^2E^3 - 20Ch^3E^2\\
&\qquad + 5CE^5 + 30DhE^4 + 10Dh^3E^2 = J\\
&-10Ch^2E^3 + 10Ch^3E^2 + 5Dh^4E - 20Dh^3E^2 + 30Dh^2E^2 + 30Dh^2E^3\\
&\qquad + 5DE^5 + 5ChE^4 - 20DhE^4 - 5Ch^4E - CE^5 + Ch^5 = K\\
&-10Dh^2E^3 - DE^5 - 5Dh^4E + Dh^5 + 5DhE^4 + 10Dh^3E^2 = L.
\end{aligned}$$

Thus, the expanded polynomial on the RHS of (5.2) is

$$Ig^3 + (F+J)g^2 + (G+K)g + (H+L).$$

To simplify this polynomial, we want $I = 0$, $F = -J$, $h \neq 0$, $h \neq -1$, $C \neq 0$, and $E \neq 0$. Equation (5.2) becomes

$$Ah^5g + Bh^5 = (G+K)g + (H+L).$$

Therefore, we wrote a C++ program to search for integers C, D, E, and h with the above constraints and with A and B integers. We found the following solutions.

C	D	E	h	A	B
4	-4	2	4	2	3
1	2	-79	79	512	-1536
4	12	-202	404	486	-2187

The first line of the table is Ramanujan's algebraic statement (1.5). Here are the other two theorems.

Theorem 5.1. *If $g^4 = 80$, then*

$$\frac{\sqrt[5]{512g-1536} + \sqrt[5]{g+2}}{\sqrt[5]{512g-1536} - \sqrt[5]{g+2}} = \frac{77 - 2g - 2g^2 - 2g^3}{79}. \tag{5.3}$$

Theorem 5.2. *If $g^4 = 405$, then*

$$\frac{\sqrt[5]{486g-2187} + \sqrt[5]{4g+12}}{\sqrt[5]{486g-2187} - \sqrt[5]{4g+12}} = \frac{200 - 2g - 2g^2 - 2g^3}{202}.$$

But, Theorems 5.1 and 5.2 are equivalent to Ramanujan's equation (1.5). To see this, we start by rewriting Ramanujan's equation (1.5), using some algebra, as

$$\frac{\sqrt[5]{2g+3} + \sqrt[5]{4g-4}}{\sqrt[5]{2g+3} - \sqrt[5]{4g-4}} = 2 + g + g^2 + g^3. \tag{5.4}$$

Changing equation (5.4) using *componendo et dividendo* results in the following equation.

$$\sqrt[5]{\frac{2g+3}{4g-4}} = \frac{3+g+g^2+g^3}{1+g+g^2+g^3}. \tag{5.5}$$

Changing equation (5.3) using *componendo et dividendo* results in the following algebraic equation.

$$\sqrt[5]{\frac{512g-1536}{g+2}} = \frac{-78+g+g^2+g^3}{1+g+g^2+g^3}. \tag{5.6}$$

Now we show equation (5.5) and equation (5.6) are equivalent. Start with equation (5.6) under the assumption that $g^4 = 80$. Substituting $g = -2f$ into equation (5.6), we have the following equation under the assumption that $f^4 = 5$.

$$\sqrt[5]{\frac{-1024f-1536}{-2f+2}} = \frac{-78-2f+4f^2-8f^3}{1-2f+4f^2-8f^3}. \tag{5.7}$$

Simplifying equation (5.7), we obtain

$$\sqrt[5]{\frac{2f+3}{4f-4}} = \frac{-78-2f+4f^2-8f^3}{4-8f+16f^2-32f^3}. \tag{5.8}$$

But, if $f^4 = 5$, we have that

$$\frac{-78-2f+4f^2-8f^3}{4-8f+16f^2-32f^3} = \frac{3+f+f^2+f^3}{1+f+f^2+f^3}. \tag{5.9}$$

We can prove this by showing that if $f^4 = 5$, then

$$(-78-2f+4f^2-8f^3)(1+f+f^2+f^3) = (4-8f+16f^2-32f^3)(3+f+f^2+f^3).$$

But equations (5.8) and (5.9) produce equation (5.5). Thus, we have shown that Theorem 5.1 is equivalent to Ramanujan's equation (1.5). Following the same procedure with $g = -3f$ shows that Theorem 5.2 is equivalent to Ramanujan's equation (1.5).

6. More Algebraic Statements

We state and prove some theorems that are similar to some of Ramanujan's algebraic statements.

Theorem 6.1. *If $g^3 = 2$, then*

$$\frac{\sqrt[4]{111-87g}+\sqrt[4]{g-1}}{\sqrt[4]{111-87g}-\sqrt[4]{g-1}} = 2+g+g^2. \tag{6.1}$$

Proof. Using *componendo et dividendo* and the fact that

$$1 = g^3 - 1 = (1+g+g^2)(g-1)$$

we rewrite (6.1) as

$$\begin{aligned}\sqrt[4]{\frac{111-87g}{g-1}} &= \frac{3+g+g^2}{1+g+g^2} = \frac{1+g+g^2+2}{1+g+g^2}\\ &= \frac{\frac{1}{g-1}+2}{\frac{1}{g-1}} = 2g-1.\end{aligned}$$

But, since $g^3 = 2$, we have that

$$\begin{aligned}(2g-1)^4 &= (2g)^4 - 4(2g)^3 + 6(2g)^2 - 4(2g) + 1 = 16g^4 - 32g^3 + 24g^2 - 8g + 1\\ &= 32g - 64 + 24g^2 - 8g + 1 = 24g^2 + 24g - 63\\ &= 24g^2 + 24g + 24 - 87 = \frac{24}{g-1} - 87 = \frac{111-87g}{g-1}.\end{aligned}$$

□

We can generalize this result in the following way.

Theorem 6.2. *Let A and h be given integers and let*

$$\begin{aligned}B &= 6A^2(h-A)^2\\ D &= A^4(h+1) + 4A(h-A)^3\\ C &= 4A^3(h-A)(h+1) + (h-A)^4.\end{aligned}$$

Then if $B = D$ and $B \neq 0$, we have the following result. If $g^3 = h+1$, then

$$\pm\sqrt[4]{\frac{(C-B)g + Bh + B - C}{h^4 g - h^4}} = \frac{A+1+g+g^2}{1+g+g^2}. \tag{6.2}$$

We will choose the plus or minus sign depending on the real value of the RHS of equation (6.2).

Proof. We wish to find an unknown function f of A and h $(g^3 = h+1)$ such that

$$\pm\sqrt[4]{f} = \frac{A+1+g+g^2}{1+g+g^2} = \frac{\frac{h}{g-1} + A}{\frac{h}{g-1}} = \frac{1}{h}\left(Ag + h - A\right).$$

So,

$$\begin{aligned}f &= \frac{1}{h^4}\left(Ag+h-A\right)^4\\ &= \frac{1}{h^4}\left(A^4g^4 + 4A^3g^3(h-A) + 6A^2g^2(h-A)^2 + 4Ag(h-A)^3 + (h-A)^4\right)\\ &= \frac{1}{h^4}\left(A^4(h+1)g + 4A^3(h+1)(h-A) + 6A^2(h-A)^2g^2 + 4A(h-A)^3g + (h-A)^4\right)\\ &= \frac{1}{h^4}\left(6A^2(h-A)^2g^2 + \left(A^4(h+1) + 4A(h-A)^3\right)g + 4A^3(h+1)(h-A) + (h-A)^4\right).\end{aligned}$$

Now the last expression for f is equal to

$$\begin{aligned}f = \frac{1}{h^4}\left(Bg^2 + Bg + C\right) &= \frac{1}{h^4}\left(B(g^2+g+1) + C - B\right)\\ &= \frac{1}{h^4}\left(\frac{Bh}{g-1} + C - B\right)\\ &= \frac{1}{h^4}\left(\frac{Bh + (C-B)g + B - C}{g-1}\right)\\ &= \frac{1}{h^4}\frac{(C-B)g + Bh + B - C}{g-1}.\end{aligned}$$

So

$$\pm\sqrt[4]{f} = \sqrt[4]{\frac{(C-B)g + Bh + B - C}{h^4(g-1)}} = \frac{A+1+g+g^2}{1+g+g^2}.$$

□

The following table gives A and h which apply to the theorem. We omitted the values of h where $h+1$ is a perfect cube. The range of A and h which was searched is -30000 to 30000 for both variables.

A	h	B	C
-16384	-24576	108086391056891904	148618787703226368
-8192	-12288	6755399441055744	9288674231451648
-511	4599	40910505984600	-11864046735534000
-188	846	226729497984	-22134467240688
-185	-1665	449798640000	-57574225920000
-117	351	17989317216	-1007401764096
-55	55	219615000	-3953070000
-22	-99	17217816	-286246191
2	1	24	-63
8192	12288	6755399441055744	9288674231451648
16384	24576	108086391056891904	148618787703226368

The third line from the bottom of the table is Theorem 6.1. If we construct an algebraic statement from the fifth line from the bottom of the table with $A = -55$, $h = 55$, $B = 219615000$ and $C = -3953070000$, we obtain the following theorem.

Theorem 6.3. *If $g^3 = 56$, then*

$$\frac{\sqrt[4]{1776-456g} + \sqrt[4]{g-1}}{\sqrt[4]{1776-456g} - \sqrt[4]{g-1}} = \frac{55}{53-2g-2g^2}. \tag{6.3}$$

However, this statement can be simplified to the following statement.

Theorem 6.4. *If $h^3 = 7$, then*

$$\frac{\sqrt[4]{111-57h} + \sqrt[4]{2h-1}}{\sqrt[4]{111-57h} - \sqrt[4]{2h-1}} = \frac{7+h+h^2}{5-h-h^2}. \tag{6.4}$$

To prove these two theorems are equivalent, we will show that their alternate forms using *componendo et dividendo* are equivalent. The alternate form of Theorem 6.3 is if $g^3 = 56$, then

$$\sqrt[4]{\frac{1776-456g}{g-1}} = \frac{54-g-g^2}{1+g+g^2} \tag{6.5}$$

and the alternate form of Theorem 6.4 is if $h^3 = 7$, then

$$\sqrt[4]{\frac{111-57h}{2h-1}} = \frac{6}{1+h+h^2}. \tag{6.6}$$

We start with equation (6.5). Substituting $g = 2h$ in equation (6.5) and simplifying, we obtain the following algebraic statement.

If $h^3 = 7$, then

$$\sqrt[4]{\frac{111-57h}{2h-1}} = \frac{27-h-2h^2}{1+2h+4h^2}. \tag{6.7}$$

Finally, if $h^3 = 7$, we have

$$\frac{27-h-2h^2}{1+2h+4h^2} = \frac{6}{1+h+h^2}. \tag{6.8}$$

But equations (6.7) and (6.8) give equation (6.6). Therefore, since their alternate statements are equivalent, Theorem 6.3 and Theorem 6.4 are equivalent.

We give another algebraic statement theorem and its proof.

Theorem 6.5. *If $g^5 = 4$, then*

$$\frac{3\sqrt{3g^2+4g+6}+\sqrt{55g^2+40g-50}}{3\sqrt{3g^2+4g+6}-\sqrt{55g^2+40g-50}} = \frac{6+g^2-g^3}{-g^2+g^3}. \tag{6.9}$$

Proof. Using the equality

$$\frac{6+g^2-g^3}{-g^2+g^3} = \frac{3+3+g^2-g^3}{3-3-g^2+g^3},$$

we rewrite (6.9) using *componendo et dividendo* as

$$3\sqrt{\frac{3g^2+4g+6}{55g^2+40g-50}} = \frac{3}{3-g^2+g^3}.$$

But, since $g^5 = 4$, we have that

$$\begin{aligned}
&(3g^2+4g+6)(3+g^2-g^3)^2 \\
&= 54+36g+63g^2-12g^3-26g^5+g^6-2g^7+3g^8 \\
&= 54+36g+63g^2-12g^3-104+4g-8g^2+12g^3 \\
&= 55g^2+40g-50
\end{aligned}$$

and the theorem is proved. □

Here is another algebraic statement and its proof.

Theorem 6.6. *If $g^5 = 2$, then*

$$\frac{\sqrt[3]{5g^2+1}+\sqrt[3]{35g^2+g-43}}{\sqrt[3]{5g^2+1}-\sqrt[3]{35g^2+g-43}} = \frac{2+g-g^2}{-g+g^2}. \tag{6.10}$$

Proof. Using the equality

$$\frac{2+g-g^2}{-g+g^2} = \frac{1+1+g-g^2}{1-1-g+g^2},$$

we rewrite (6.10) using *componendo et dividendo* as

$$\sqrt[3]{\frac{5g^2+1}{35g^2+g-43}} = \frac{1}{1+g-g^2}.$$

But, since $g^5 = 2$, we have that

$$\begin{aligned}
&(1+g-g^2)^3(5g^2+1) \\
&= -5g^8 + 15g^7 - g^6 - 22g^5 + 10g^3 + 5g^2 + 3g + 1 \\
&= -10g^3 + 30g^2 - 2g - 44 + 10g^3 + 5g^2 + 3g + 1 \\
&= 35g^2 + g - 43
\end{aligned}$$

and the theorem is proved. □

7. Acknowledgement

The author wishes to thank an anonymous referee for the very helpful suggestions which improved the paper.

References

[1] M. D. Hirschhorn and V. Sinescu, *Elementary algebra in Ramanujan's lost notebook*, The Fibonacci Quarterly, **51.2** (2013), 123–129.

[2] S. Ramanujan, *The Lost Notebook and Other Unpublished Papers*, New Delhi, Narosa, 1988, p. 344.

MSC2010: 11A55

Department of Mathematics and Computer Science, University of Central Missouri, Warrensburg, Mo 64093

E-mail address: cooper@ucmo.edu

ENUMERATING DISTINCT CHESSBOARD TILINGS

DARYL DEFORD

Abstract. Counting the number of distinct colorings of various discrete objects, via Burnside's Lemma and Pólya Counting, is a traditional problem in combinatorics. Motivated by a method for proving upper bounds on the order of the minimal recurrence relation satisfied by a set of tiling instances, we address a related problem in a more general setting. Given an $m \times n$ chessboard and a fixed set of (possibly colored) tiles, how many distinct tilings exist, up to symmetry?

More specifically, we are interested in the sequences formed by counting the number of distinct tilings of boards of size $(m \times 1), (m \times 2), (m \times 3) \ldots$, for a fixed set of tiles and some natural number m. We present explicit results and closed forms for several well known classes of tiling problems as well as a general result showing that all such sequences satisfy some linear, homogeneous, constant–coefficient recurrence relation. Additionally, we give a characterization of all $1 \times n$ distinct tiling problems in terms of the generalized Fibonacci tilings.

1. Introduction

1.1. **Background.** Enumerating the number of ways to cover a rectangular chessboard with a fixed set of tiles is a motivating problem for many interesting recurrence relations and integer sequences. Many examples of these problems and their associated solution methods can be found in [6, 7, 12, 14, 15]. A complete and informative treatment of the one–dimensional case is contained in Benjamin and Quinn's wonderful book [2]. Often, restrictions are made on the types and orientations of the permissible tiles in order to model a particular combinatorial problem. For example, it is well known that the number of ways to tile a $1 \times n$ board with 1×1 squares and 1×2 dominoes is the n^{th} combinatorial Fibonacci number f_n, while generalized domino tilings have deep connections to questions in statistical mechanics [9, 13, 22].

The particular case when the tiles are restricted to be square was considered by Brigham et al. [3] and Hare [10]. In 1999, Heubach used the combinatorial method of counting indecomposable blocks to generalize these earlier results [11]. More recently, Calkin et al. showed that when the square tiles are restricted in dimension, the number of tilings can be calculated as the sum of the entries in the n^{th} power of a recursively defined matrix [4]. This solution is based on a method of Calkin and Wilf for counting grid tilings [5]. This problem is a specific case of the forbidden sub–matrix problem. Furthermore, Webb has shown that such problems always have a recurrence solution [23].

Date: April 20, 2015.

Key words and phrases. Tilings; Recurrence Relations; Integer Sequences.

Aside from their intrinsic interest and applications, tilings are also an effective combinatorial technique for proving identities. While most of the identities concerning combinatorial objects have straightforward proofs through mathematical induction or algebraic manipulation with Binet forms, these approaches do not provide intuition for the results or suggest avenues for further investigation. Thus, bijective proofs utilizing tilings and other combinatorial models are preferred. Indeed, many of the most common Fibonacci and Lucas identities have simple and elegant proofs using the $1\times n$ tiling model mentioned above.

1.2. **Notation.** In this paper, the Fibonacci sequence will be indexed combinatorially as $f_0 = 1$ and $f_1 = 1$, in order to have a direct connection with the tiling interpretation. The primary object of interest in this paper are the sequences formed by counting the number of legitimate tilings of rectangular boards by some fixed sets of tiles. In particular, for any arbitrary fixed set of tiles T (note that we do not require that the tiles be connected) and fixed board height m we will let the sequence $\{T_n\}$ be the number of ways to tile a $m\times n$ board with tiles in T. More generally, we will also be interested in the collection of sequences $\{\{T_n\}^{(m)}\}$ as m ranges over the natural numbers. Throughout, d will represent the length of the longest tile in T.

We will frequently need to consider the number of ways to tile boards where some subset of the initial squares have been deleted. In these examples the set of tiles will be clear from context and we will use capital letters to represent the boards and lower case letters to represent the number of ways to tile the board (see Figure 1 in Section 2.1).

Following DeTemple and Webb, we will denote the successor operator on sequences by E. That is, for any sequence a_n we have $E(a_n) = a_{n+1}$. The successor operator offers an elegant way to express and prove many combinatorial identities [6]. Finally, throughout this paper the phrase "recurrence relation" will be used to refer to a linear, homogeneous, constant–coefficient recurrence relation.

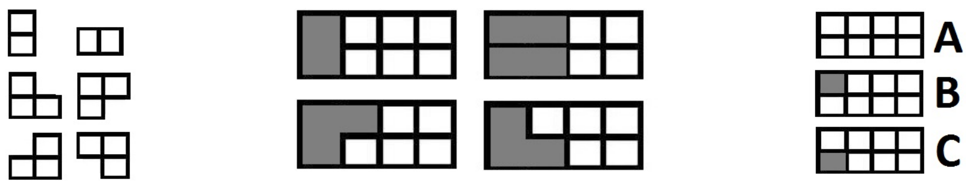
1.3. **Contributions.** In this paper we consider enumerating distinct tilings up to symmetry. These problems arise when trying to prove recurrence order bounds for standard tiling problems. We give a general formula for all $1\times n$ tiling problems generalized from the standard Fibonacci tiling model. Finally, we show that for any fixed tile set T and number of rows m the sequence of distinct tilings of $m \times n$ boards satisfies a recurrence relation and give examples incorporating the Fibonacci numbers.

2. Tilings and Recurrence Relations

As discussed in [2], if we permit ourselves to consider weighted tilings with initial phases, we can realize any sequence satisfying a recurrence relation as tiling problem on a $1\times n$ board. In 2004, Webb, Criddle, and DeTemple proved an interesting converse to this statement by showing that for any fixed set of tiles, T and any fixed board height, m, the sequence $\{T_n\}$ satisfies a recurrence relation, by conditioning on the number of ways to cover the leftmost column [24]. This proof and its generalizations rely on an algebraic lemma proved in [6] that any collection of arbitrary sequences that satisfy a homogeneous linear system in E are recurrent sequences annihilated by the determinant of that system.

Before proceeding, we provide a simple example using this methodology:

2.1. **Example: Tilings of a $2 \times n$ board with Dominoes and L–shaped Trominoes.** The tiles, endings, and necessary sub–boards are shown below in Figure 1:



(a) Tiles (b) Endings (c) Subboards

FIGURE 1. Figures for Example 1

Considering the number of ways to fill the initial column of board A, we see that we can either use one vertical domino, two horizontal dominoes, or an L–shaped tile in either orientation. The remaining boards, B and C, are simpler, because each tile may only be placed in one orientation. This leads to the following system of sequences:

$$a_n = a_{n-1} + a_{n-2} + b_{n-1} + c_{n-1} \tag{2.1}$$
$$b_n = c_{n-1} + a_{n-2} \tag{2.2}$$
$$c_n = b_{n-1} + a_{n-2}. \tag{2.3}$$

As an example of how these equations are obtained, consider (2.3). In order to tile a C board of length n we may either place a horizontal domino in the top row, leaving a B board of length $n-1$, or we may place a tromino that covers the remaining squares in the first two columns, leaving an A board of length $n-2$. Rewriting these as a linear system in E we obtain:

$$\begin{bmatrix} E^2 - E - 1 & -E & -E \\ -1 & E^2 & -E \\ -1 & -E & E^2 \end{bmatrix} \begin{bmatrix} a_n \\ b_n \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{2.4}$$

The determinant of this matrix $E^2(E^4 - E^3 - 2E^2 - E - 1) = E^2(E+1)(E^3 - 2E^2 - 1)$, and indeed we can check the initial conditions, $T_1 = 1$, $T_2 = 2$, $T_3 = 5$, $T_4 = 11$ and $T_5 = 24$ to see that our desired sequence satisfies the recurrence relation $T_n = 2T_{n-1} + T_{n-3}$ corresponding to the irreducible cubic factor. This appears in the OEIS as A052980, although this tiling interpretation is not yet included [19]. In a sense that will be made precise later, the matrix obtained in (2.4) is typical of such problems. The fact that the determinant is a degree six polynomial highlights the important fact that this method does not always directly return the polynomial corresponding to the minimal recurrence relation satisfied by the sequence, which we will discuss in the next section.

3. Recurrence Order Bounds

3.1. **Motivation.** The order of a sequence is defined as the degree of the characteristic polynomial of the minimal recurrence relation that the sequence satisfies. As discussed in Chapter 7 of [6] knowing the order or even an upper bound for the order of a sequence can allow us to prove identities and results without actually computing the coefficients themselves. In the case of tiling problems, where all of the sequences we are interested in satisfy some recurrence relation, having a bound on the recurrence order is particularly valuable.

The mechanical method for proving identities contained in [6] shows that the upper bound on sequence order describes how many initial conditions are necessary to compute in order to prove a desired identity. Additionally, it is possible compute the coefficients from initial conditions by solving a simple linear system of size equal to twice the order bound. Thus, providing a better upper bound limits the amount of computation necessary to make use of a particular tiling model. This is particularly important, because it has been shown that enumerating the number of tilings can be #P–complete in some cases [20].

3.2. **Sequences of Sequences.** For a given tile set T, we can form a family of sequences $T_n^{(m)}$, each of which satisfies some recurrence relation, by letting the number of rows, m, range over the positive integers. It is natural to investigate the relationships between these sequences. For example, matrix methods of Calkin and Wilf [5] as well as those of Anderson [1], show that for some fixed sets of tiles, the recurrence relations can be calculated for any n by constructing a particular recursively constructed matrix. Similarly, families of tilings with dominoes or with the tiles restricted to be square can generate divisibility sequences [25].

In this paper, we are particularly interested in the growth rate of the order of the sequences. That is, let $\mathcal{O}(T_n)$ be the order of the minimal recurrence relation that T_n satisfies. Then, we can construct a sequence $\{\mathcal{O}(T_n^{(m)})\}$ of these orders, and in particular, consider the growth rate of the sequence. As discussed previously, this measure provides important information about the sequence without excessive computation.

3.3. **Trivial Bounds.** The proof that every tiling sequence satisfies a recurrence relation proceeds by constructing the characteristic polynomial of such a recurrence relation, as the determinant of a matrix whose entries are polyomials in E. As noted above, this recurrence relation is rarely minimal, but does provide an upper bound on the order. In order to compute this bound in general, we can consider the number of rows in the matrix and the maximum degree in E of the entries in each row.

Consider the case for a fixed number of rows m and maximum tile length d. Because we are considering arbitrary, possibly disconnected tiles, there are $(2^m-1)2^{m(d-1)}$ legitimate tiles to choose from, and a maximum of $2^{m(d-1)}$ states remaining of the board after the initial column is tiled. Thus, our matrix could have up to $2^{m(d-1)}+1$ rows, one for each board state and one for the initial board. Each of these rows can have exponent at most d, which can always be achieved along the main diagonal when all of the tiles are used. The product of the main diagonal entries is one summand of the determinant and thus, we obtain our first trivial upper bound on $\mathcal{O}(T_n^{(m)}) \leq d2^{m(d-1)+1} \sim \mathsf{O}(d2^{md})$.

There is some additional structure of the constructed matrix that can be used to reduce this bound. For example, even in the worst case where T contains all $(2^m-1)2^{m(d-1)}$ tiles, the E^d factors will only occur along the main diagonal and the only polynomials with non–zero constant term will appear in the initial column as in the matrix in (2.4). Hence, expanding down the initial column shows that there will be extraneous factors of E corresponding to sequence eigenvalues of 0 that may be discarded. Moreover, some of the states are translates of each other, and could thus be combined in order to further reduce the order. However, these improvements do not significantly impact the asymptotic behavior of the upper bound.

In general, this bound grows much too fast to be useful either combinatorially or computationally. For example, even for tilings with dominoes and squares the bound grows like $2 \cdot 2^{m(2-1)} = 2^{m+1}$. However, the actual recurrence orders are much smaller, as can be seen in Table 1 below. Thus, the trivial bound obtained from the proof is too inefficient for practical use.

m	OEIS	Upper Bound	Observed Order
1	A000045	4	2
2	A030186	8	3
3	A033506	16	6
4	A033507	32	9
5	A033508	64	20
6	A033509	127	36

Table 1. Enumerating tilings with squares and dominoes. The data in column 4 is from the OEIS [19]. Most of the computations were performed by Lundow [16]. The observed orders may not be minimal in all cases.

4. Motivating Example

In this section, we present a simple and well–studied counting problem as a case study suggesting some approaches to obtaining more reasonable recurrence order bounds for fixed sets of tiles. For the remainder of this section T will consist of 1×1 and 2×2 squares with m arbitrary. We will let A_n be the whole $m \times n$ board and hence the sequence a_n is equivalent to the desired sequence T_n. It is well known that the number of ways to tile a $2 \times n$ strip with 1×1 and 2×2 squares is equal to f_n [7]. Thus, there are f_m possible beginnings for a tiling of A_n.

This implies that the associated successor matrix has size bounded by $f_m \times f_m$. The maximum exponent of E in each row is one, except for the row corresponding to a_n which has a quadratic term from the all 2×2 tiling, balanced by the all 1×1 ending which is counted by a_{n-1}. Combined, this analysis provides us with f_m as an upper bound on the order of the recurrence. This is an asymptotic improvement, since $f_m \sim \varphi^m$. Moreover, we note that we can further restrict the size of the matrix by only considering the distinct endings up to symmetry. This was also true in Example 1, as the sequences b_n and c_n are clearly identical as C can be obtained from B by a reflection.

Thus, we need to compute the number of distinct Fibonacci tilings, relying on Burnside's Lemma[1] (see Theorem 8.7 in [6]). The next result is a specific case of the general formula presented in the next section, with $a_1 = a_2 = 1$ and $a_j = 0$ for $j > 2$. Similarly, Lemma 4.3 corresponds to $a_1 = 0, a_2 = a_3 = 1$ and $a_j = 0$ for $j > 3$.

Lemma 4.1. *The number of distinct Fibonacci tilings of order n up to symmetry is equal to $\frac{1}{2}(f_{2k} + f_{k+1})$ when $n = 2k$ and $\frac{1}{2}(f_{2k+1} + f_k)$ when $n = 2k + 1$.*

Proof. Let $n = 2k$ and consider the tilings of an $1 \times n$ board with squares and dominoes. Any reflection of a tiling across the line of symmetry between the k^{th} and $(k+1)^{st}$ squares produces another legitimate tiling. However, some tilings are self–similar under reflection, hence we cannot simply take $\frac{1}{2}f_n$ as our answer. The number of self–similar tilings can be computed by considering that the line of symmetry may either be covered by a domino, or uncovered. There are f_{k-1} self–similar tilings whose center tile is a domino and f_k self–similar tilings where the line of symmetry is uncovered.

Thus, there are $f_{k-1} + f_k = f_{k+1}$ self–similar tilings. Figure 2 shows examples of tilings with this property. By adding this quantity to the total number of tilings of length n, we have exactly twice the number of distinct classes of tilings up to symmetry. Hence, the number of classes of tilings is $\frac{1}{2}(f_{2k} + f_{k+1})$ and this case is complete.

When $n = 2k + 1$ we can apply a similar argument. In this case however, the line of symmetry passes directly through the $(k+1)^{st}$ square and thus must be covered by a square to create a self–similar tiling. Hence, there are exactly f_k self–similar tilings, and by applying Burnside's lemma as above we see that there must be exactly $\frac{1}{2}(f_{2k+1} + f_k)$ distinct tilings which completes the proof. $\square$

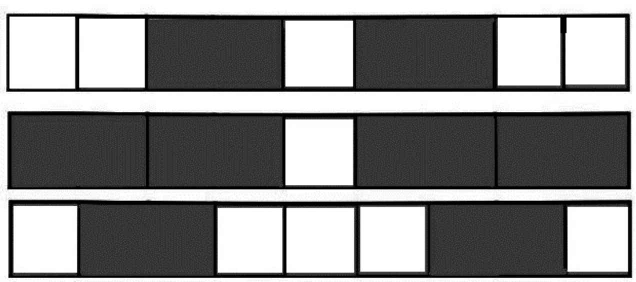


FIGURE 2. Self–Similar Fibonacci Tilings

The number of distinct classes of tilings provides a better bound on the order of our recurrence by limiting the number of rows in our successor operator matrix. However, we can offer another improvement by noticing that any ending that contains no consecutive 1×1 squares has exactly as many remaining tilings as a_{n-2} since the remaining un–tiled squares in the second column must also be covered by 1×1 squares. This implies that we can subtract the number of such endings, since they do not need to be represented

[1]Or rather, the lemma that is not Burnside's [26].

in our successor matrix. The number of endings that satisfy this condition is given by P_{n+2}, where P_n is the n^{th} Padovan number, which counts the number of ways to tile a $1 \times n$ board with 1×2 dominoes and 1×3 trominoes, satisfying the recurrence relation $P_n = P_{n-2} + P_{n-3}$. More interpretations of the Padovan sequence are given in the OEIS as sequence A000931 [19]. We prove this statement as the following lemma.

Lemma 4.2. *The number of endings with no consecutive* 1×1 *tiles is equal to* P_{n+2}.

Proof. We may construct a bijection between endings and tilings by associating every 2×2 square followed by a 1×1 square with a tromino in the Padovan tiling, while each 2×2 square not followed by a 1×1 square is associated with a domino. Then, since we need to count separately the cases when the tiling begins with a square or a domino, we have that the number of endings with no consecutive 1×1 squares is equal to $P_n + P_{n-1} = P_{n+2}$, by the Padovan recurrence. This completes our proof. □

Thus, we may subtract the number of distinct Padovan tilings from our previous bound to obtain a better order approximation. In order to calculate the number of distinct Padovan tilings we follow the methodology introduced in Lemma 1.

Lemma 4.3. *The number of distinct Padovan tilings of order* n *up to symmetry is equal to* $\frac{1}{2}(P_{2k} + P_{k+2})$ *when* $n = 2k$ *and* $\frac{1}{2}(P_{2k+1} + P_{k-1})$ *when* $n = 2k + 1$.

Proof. We may argue as in Lemma 4.1. Notice that we again have exactly one odd length and one even length tile, so the cases proceed exactly as in Lemma 4.1. Replacing the square by a tromino gives a third order recurrence, which now satisfies the defining relation of the Padovan numbers. It is then a straightforward calculation to verify the result.

□

The preceding discussion suffices to prove the following theorem:

Theorem 4.4. *The number of tilings of an* $m \times n$ *chessboard with* 1×1 *and* 2×2 *squares when* m *is fixed and* n *varies is not greater than:*

$$\frac{1}{2}(f_{2k} + f_{k+1} - P_{2k+2} - P_{k+3}) + 1$$

when $m = 2k$, *and*

$$\frac{1}{2}(f_{2k+1} + f_k - P_{2k+3} - P_k) + 1$$

when $m = 2k + 1$.

Table 2 below shows the differences between the bound and the actual order of the computed recurrence for the first several cases. Neither of these sequences appear to be contained in the OEIS. The values in the table row labelled $\mathcal{O}(a_n)$ are the orders of recurrences given in the OEIS for the solutions of these problems [19]. Computing the order of recurrences for other sets of tiles can be done in a similar fashion. For any particular case, analyzing the symmetry classes of the tiling endings can lead to greatly improved upper bounds.

n	2	3	4	5	6	7	8	9	10
$\mathcal{O}(a_n)$	2	2	3	4	6	8	14	19	32
Bound	2	2	3	4	7	10	17	26	44

TABLE 2. Comparison between the derived bound and the actual order

This theorem demonstrates the usefulness of our contributions. Using the successor operator method we may bound the order of the recurrence for a tiling problem by decomposing its endings into separate smaller problems of determining the number of distinct tilings of a simpler tile set. Since every chessboard tiling problem has an associated recurrence relation this is a very general method. Figure 3 below shows the possible endings of a $5 \times n$ board grouped in rows by equivalence class.

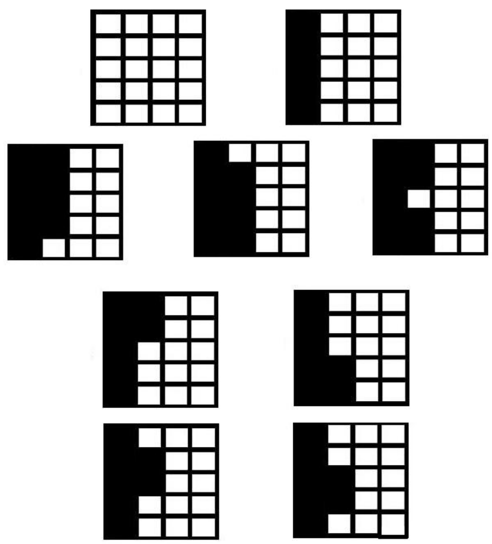


FIGURE 3. The $5 \times n$ endings

5. ONE DIMENSIONAL TILINGS

In this section we provide a complete characterization of the number of distinct tilings of a $1 \times n$ rectangle with colored tiles of fixed lengths. We also use the Pòlya Enumeration Theorem (see Theorem 8.15 in [6]) to prove a similar result for $1 \times n$ bracelet tilings.

5.1. **Generalized Fibonacci Tilings.** Tilings of $1 \times n$ rectangles have been inextricably linked to the Fibonacci numbers by Benjamin and Quinn's classic book [2]. They give an interpretation of every (positive) linear homogeneous constant coefficient recurrence relation in terms of a generalization of the standard Fibonacci tiling model. Here, we prove a complementary theorem counting the number of distinct tilings for any possible collection of colors and tiles.

We begin by defining some convenient notation. Since we are covering boards of dimension $1 \times n$ we will consider tiling sets consisting of colored k–dominoes. We will represent the tile set as a vector, $T = (a_1, a_2, a_3, \ldots)$, where a_k represents the number of distinctly colored k–length dominoes available. Further, we will let T_n represent the number of ways to tile a $1 \times n$ rectangle with the tiles in T.

Finally, let α_j represent the number of self–symmetric $1 \times n$ T–tilings where the line of symmetry is covered by a j–domino. This gives the following piecewise definition for a_j:

$$a_j = \begin{cases} T_{\frac{n-j}{2}} & j \equiv n \equiv 0 \pmod 2 \\ 0 & j \equiv 0, n \equiv 1 \pmod 2 \\ 0 & j \equiv 1, n \equiv 0 \pmod 2 \\ T_{\frac{n-j}{2}} & j \equiv n \equiv 1 \pmod 2 \end{cases}. \tag{5.1}$$

Methods for calculating numerical values for T_k and by extension α_k are given in [2]. Now we may give the statement of our theorem.

Theorem 5.1. *Let T be any set of colored k–length dominoes. Then the number of distinct tilings up to symmetry of a $1 \times n$ rectangle is equal to*

$$\frac{1}{2}\left(T_n + \sum_{i=1}^{\infty} a_i \alpha_i + \frac{T_{\frac{n}{2}}}{2} + \frac{(-1)^n T_{\frac{n}{2}}}{2}\right). \tag{5.2}$$

Proof. We proceed again using Burnside's lemma. Since our board is one–dimensional the only symmetry we are concerned with is the reflection across the vertical line of symmetry. Notice that the set of tilings is closed under reflection which implies that it is sufficient to add the number of self–symmetric tilings to T_n to obtain the number of distinct tilings.

As in the proof of Lemma 1, we begin with the even case so let $n = 2k$. Since n is even the symmetry line falls between two units of our board. Thus, there are T_k tilings where the line is uncovered. This accounts for the final two terms in our sum. Additionally, it is easy to see that when n is odd these terms annihilate leaving us a single closed–form expression instead of a piecewise representation. This fits the combinatorial interpretation since when n is odd the line of symmetry bisects some unit square and must be covered by some tile.

Finally, for each j–length domino in T we must consider the case where the line of symmetry is covered by a tile of length j. These cases separately naturally into four parts, conditioning on the parity of j and n, as represented in Figure 4.

I) Both j and n are even:
In this case the line of symmetry must pass through the center of the j–domino. This leaves $\frac{j}{2}$ units covered in each half of the board. In order to construct a self–symmetric tiling, we must have both halves equivalent. Since there are no other restrictions on the tiling, there are $T_{\frac{n-j}{2}}$ such coverings and this case is complete.

II) When j is even and n is odd:
In this case, the center of the j–domino does not correspond to the line of symmetry. Hence, there can be no self–symmetric tilings with these conditions.

III) When j is odd and n is even:
As in case II we are unable to construct such a self–symmetric tiling since the domino covers a different number of squares on each half of the board.

IV) Both j and k are odd:
Here we may place the j–domino such that exactly $\frac{j-1}{2}$ squares are covered on each side. As in case I this implies that there are $T_{\frac{n-j}{2}}$ such coverings and no more.

Since for each j there are a_j colors, summing over $a_j\alpha_j$ for all $j \in \mathbb{N}$ counts all self–symmetric tilings where the line of symmetry is covered. Since all self–symmetric tilings have the line of symmetry either covered or uncovered, this completes the proof. $\square$

This result is particularly valuable in light of our work presented in the previous section. Notice, that to produce the bounds on our recurrence relation we only needed to apply this theorem twice, even though the number of rows, m, could be selected arbitrarily. This is because using the successor operator method, we need only consider the initial columns, and frequently a bijection can be constructed between tilings of the initial columns and colored $1 \times n$ tilings. Thus, this theorem is sufficient to provide recurrence order bounds on most traditional tiling problems.

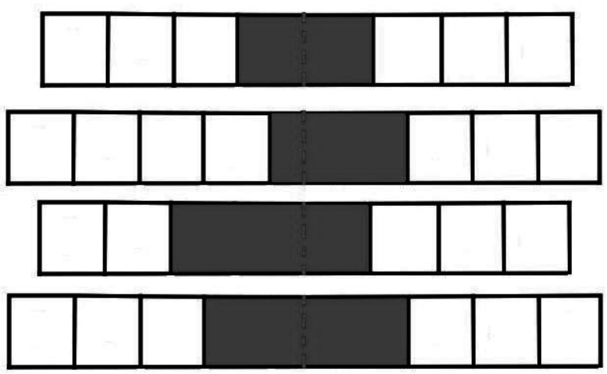


FIGURE 4. $1 \times n$ Self Symmetric Centers

5.2. **Distinct Lucas Tilings.** In addition to considering generalized Fibonacci relations, Benjamin and Quinn also provide a combinatorial interpretation of the Lucas numbers as tilings of a $1 \times n$ bracelet. We now show that the number of distinct Lucas tilings can be given by a number–theoretic formula, using the Pòlya Enumeration Theorem. The sequence generated by (5.3) occurs in the OEIS as A032190 [19].

Theorem 5.2. *The number of distinct Lucas tilings of a $1 \times n$ bracelet up to symmetry is:*

$$\sum_{i=0}^{\lceil \frac{n-1}{2} \rceil} \left[\frac{1}{n-i} \sum_{d|(i,n-i)} \varphi(d) \binom{\frac{n-i}{d}}{\frac{i}{d}} \right]. \tag{5.3}$$

Proof. In order to apply the Pòlya Enumeration Theorem, we must first calculate c_k for each bracelet B_n. Since the group acting on each bracelet is the n^{th} cyclic group we have that $c_k(B_n) = \varphi((n,k))$ elements of order k where φ represents the Euler totient function [21]. With this representation in hand, it follows that by the Pòlya Enumeration Theorem there are exactly

$$f(n,k) = \frac{1}{n} \sum_{d|(n,k)} \varphi(d) \binom{\frac{n}{d}}{\frac{k}{d}} \tag{5.4}$$

binary colorings of a n–bracelet with exactly k black units [14].

In order to enumerate the Lucas tilings we must condition on the number of dominoes in each tiling. Let each black unit in a distinct bracelet coloring represent a domino, and let each white unit in a distinct bracelet coloring represent a square. There can be at most $\lceil \frac{n}{2} \rceil$ dominoes in such a covering, since each domino covers two units. Replacing each domino with two squares, increases the number of available units by one up to n.

Each of these different combinations of tiles represents a unique distribution of the colors in a binary bracelet coloring of order $n-d$, with d representing the number of dominoes. Summing over all possible values for d gives:

$$\sum_{i=0}^{\lceil \frac{n}{2} \rceil} f(n-i,i). \tag{5.5}$$

Finally, substituting (5.4) for $f(n,k)$ gives the desired result completing this proof. $\square$

This result demonstrates the difficulties and complexities involved in employing the techniques of Burnside and Pòlya in more complex domains. While the number of distinct bracelet colorings has a convenient closed form expression [15], the techniques needed to catalog even the simplest cases of distinct Lucas tilings are much more significant. Consider extending Theorem 5.2 by adding curved trominoes to the tile–set. The resulting expression is a triple sum over multinomial coefficients. Similarly, adding colored dominoes or squares again increases the complexity of the expression exponentially.

6. Larger Rectangular Tilings

6.1. **Recurrence Relations for Distinct Tilings.** In this section we consider more generally the problem of enumerating the number of distinct tilings of an $m \times n$ chessboard. We prove a complementary result to the result of Webb et al. showing that every such sequence satisfies a recurrence relation and conclude with some examples of this method applied to some well known tiling problems.

Theorem 6.1. *Let T be a fixed set of tiles with maximum length d, and $m > 0$ be a fixed number of rows. The sequence $\{D_n\}$ of distinct tilings of an $m \times n$ board satisfies a recurrence relation.*

Proof. Since our chessboards are rectangular, the group of symmetries is isomorphic to the Klein group. We will use the notation $G = \{e, h, v, r\}$, where e is the identity element, h and v represent the horizontal and vertical reflections respectively, and r is the 180° rotation. Then, letting e_n, h_n, v_n, and r_n be the number of tilings of a $m \times n$ board with tiles in T fixed by each respective group element, by Burnside's Lemma, we have that

$$D_n = \frac{1}{4}(e_n + h_n + v_n + r_n). \tag{6.1}$$

Since the set of all sequences satisfying some recurrence relation is a vector space, any finite linear combination of such sequences also satisfies a recurrence relation. Thus, it suffices to show that e_n, h_n, v_n, and r_n are all recurrent sequences. Note that this together with Theorem 5.1 imply the case for $m = 1$ since the even and odd cases each separately are a finite linear combination of recurrent sequences (the T_i). In the general case, the theorem of Webb guarantees that e_n satisfies a recurrence relation since e fixes all T_n tilings.

We consider the remaining three cases in turn, following the idea in [24]. The case of h_n is simplest after the identity. Let $\mathcal{S}$ be the set of all possible boards formed from A by deleting some (possibly empty) collection of squares in the first $d-1$ columns and let $\mathcal{S}_h^*$ represent the corresponding sequences counting the number of ways to distinctly tile a $m \times n$ board with initial columns in $\mathcal{S}$. Note that we actually need only include those endings that are fixed under h in $\mathcal{S}$ since the corresponding sequences are 0 for all other endings.

For each board $B \in \mathcal{S}$ we may form a linear equation in E for b_n in terms of sequences in $\mathcal{S}_h^*$ by considering the number of distinct ways, up to symmetry, to tile the initial column of the board, since any such covering of the initial squares will leave another board in $\mathcal{S}$ of shorter length. Hence, each sequence in $\mathcal{S}_h^*$ (including h_n) can be represented as a linear combination in E of other sequences. Then, the determinant of this system is the characteristic polynomial of a recurrence that annihilates h_n.

We may proceed similarly for v and r, defining $\mathcal{S}_v^*$ and $\mathcal{S}_r^*$ to enumerate corresponding sequences counting the number of ways to distinctly tile a $m \times n$ board with initial columns in $\mathcal{S}$ fixed by v and r respectively. Again, by considering the number of ways to distinctly cover the initial column of each board in $\mathcal{S}$ we may form linear systems whose determinants give recurrences annihilating the sequences v_n and r_n. Hence, we have shown that D_n is a linear combination of sequences satisfying recurrence relations and so D_n must also be a recurrent sequence as desired.

□

6.2. **Examples.** We conclude this paper by presenting some simple, discrete examples of enumerating distinct two–dimensional tilings. These examples are meant to be representative of the solution methods necessary to approach more general problems.

6.2.1. *Tilings with Dominoes.* In this example we consider the distinct tilings of a $2 \times n$ rectangle with dominoes. Recall that the total number of ways to tile a $2 \times n$ rectangle with dominoes is f_n. For m up to 9 these distinct domino tiling values have been computed numerically by Mathar [17]. In [18], Mathar computes generating functions for several generalizations of this problem, including using larger dominoes and three dimensional tilings, using the transfer matrix method on a digraph constructed to represent possible endings. The sequence presented in the following example occurs in the OEIS as A060312 [19].

Proposition 6.2. *The number of distinct tilings of a* $2\times n$ *rectangle with* 1×2 *dominoes is*

$$\frac{1}{2}\left(f_{2k}+f_{k+1}\right) \tag{6.2}$$

when $n=2k$ *and*

$$\frac{1}{2}\left(f_{2k+1}+f_{k}\right) \tag{6.3}$$

when $n=2k+1$.

Proof. In order to apply Burnside's Lemma, we must count the number of elements fixed by each group action.

Since the identity element e fixes all tilings, it contributes f_n to the sum regardless of the parity of n. To see that h accounts for f_n regardless of parity, consider the bijection between $1\times n$ squares and dominoes and the Fibonacci recurrence [7]. Since applying h to a $2\times n$ board leaves a $1\times n$ board this is sufficient.

The last two group actions are parity dependent, so first let $n=2k$ and consider the actions of r and v. In both cases either the line of symmetry is covered by two horizontal dominoes or it is not covered at all. These observations add the final terms to the even case: $2f_k$ and $2f_{k-1}$ respectively. This completes the example when n is even. Figure 5 shows the symmetric centers under r and v for both parities.

When $n=2k+1$ is odd, under both v and r in order for a tiling to be self–similar the symmetric line must be covered by a single vertical domino leaving only $2f_k$ remaining tilings fixed by these actions. Since we have considered all of the elements of V and $|V|=4$ by Burnside's Lemma we have that the number of distinct tilings is equal to:

$$\frac{1}{4}\left(f_{2k}+f_{2k}+2f_k+2f_{k-1}\right) \tag{6.4}$$

when $n=2k$ and

$$\frac{1}{4}\left(f_{2k+1}+f_{2k+1}+2f_k\right) \tag{6.5}$$

when $n=2k+1$. Simplifying with the Fibonacci recurrence then gives the result. □

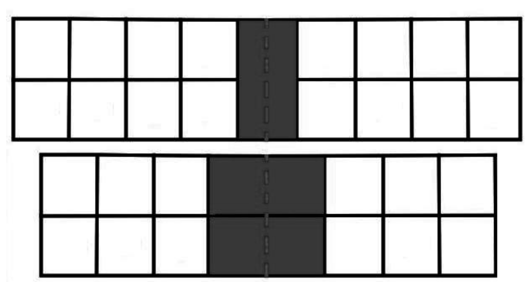


FIGURE 5. Legitimate Symmetric Centers for $2\times n$ Domino Tilings

6.2.2. *Tilings with Squares.* In this final example we extend the motivating example of Section 4, tiling with 1×1 squares and 2×2 squares.

Proposition 6.3. *The number of distinct tilings of a $3 \times n$ rectangle with squares of size 1×1 and 2×2 is*

$$\frac{1}{3}\left(2^{2n-1} + 2^n + 2^{n-1} + \frac{1+(-1)^n}{2}\right) \tag{6.6}$$

when n is odd, and

$$\frac{1}{3}\left(2^{2n} + 2^n + 2^{n-1} + 1\right) \tag{6.7}$$

when n is even.

Proof. Since our group of symmetric actions again has four elements, by Burnside's Lemma we need only compute the tilings that are fixed by each symmetric transformation. Using the notation of Heubach [11], let $T_{3,a}$ represent the number of traditional tilings of a $3 \times a$ rectangle.

The identity transformation fixes every tiling, which contributes a term of $T_{3,n}$. Similarly, the horizontal reflection fixes only the tiling with all 1×1 squares since any 2×2 square cannot be centered across the horizontal line of symmetry.

A rotation of $180°$ fixes exactly $T_{3,\lfloor \frac{n}{2} \rfloor}$ tilings since when n is odd the center column must be covered with 1×1 tiles and when n is even the center two columns must be covered with 1×1 tiles. If a 2×2 square infringes on one of these areas, it would overlap itself under r and hence cannot be self–symmetric.

The vertical line of symmetry separates the two parities. When $n = 2k + 1$ the symmetric line crosses the central units and must be covered by 1×1 squares contributing $T_{3,k}$ to the final sum. When $n = 2k$ is even the line of symmetry may be covered in one of two ways by a single 2×2 square or be surrounded but not covered by squares on both sides. These terms are $2T_{3,k-1}$ and $T_{3,k}$ respectively which completes the even case.

Applying Burnside's Lemma to these terms gives a representation of the number of tilings in terms of Heubach's recurrence relation:

$$\frac{1}{4}\left(T_{3,2k+1} + 1 + 2T_{3,k}\right) \tag{6.8}$$

when $n = 2k + 1$ and

$$\frac{1}{4}\left(T_{3,2n} + 1 + 2T_{3,n} + 2T_{3n+1}\right) \tag{6.9}$$

when $n = 2k$.

Constructing a generalized power sum for $T_{3,a}$ gives the following closed form expression [19],

$$T_{3,a} = \frac{2^{a+1} - (-1)^{a+1}}{3}. \tag{6.10}$$

Substituting (6.10) into (6.7) and (6.8) respectively gives the desired result and completes this example. □

References

[1] P. Anderson: *Tiling Rectangles, Cylinders, and Mobius Strips*, Proceedings of the Fourteenth International Conference on Fibonacci Numbers and their Applications, (2010).

[2] A. Benjamin and J. Quinn: *Proofs that Really Count*, MAA, Washington D.C., 2003.

[3] R. Brigham, R. Caron, P Chinn, and R. Grimaldi: *A Tilings Scheme for the Fibonacci Numbers*, Journal of Recreational Mathematics, **28 (1)**, (1997), 10–17.

[4] N. Calkin, K. James, S. Purvis, S. Race, K. Schneider, and M. Yancey: *Counting Kings: Explicit Formulas, Recurrence Relations, and Generating Functions! Oh My!*, Congressus Numerantium **182**, (2006), 41–51.

[5] N. Calkin and H. Wilf: *The Number of Independent Sets in a Grid Graph*, SIAM J. Discrete Math, **11** (1998), 54–60.

[6] D. DeTemple and W. Webb: *Combinatorial Reasoning: An Interoduction to the Art of Counting*, Wiley, Hoboken, 2014.

[7] R. Graham, D. Knuth, and O. Patashnik: *Concrete Mathematics*, Addison-Wesley, Reading, 1994.

[8] F. Harary: *Determinants, Permanents and Bipartite Graphs*, Mathematics Magazine, **42(3)**, (1969), 146-148.

[9] F. Harary: *Graph Theory and Theoretical Physics*, Academic P., New York, (1967).

[10] E. Hare: *Tiling a $3\times n$ Area with Cuisinaire Rods of Length Less Than or Equal to k*, Congressus Numerantium **105**, (1994), 34–55.

[11] S. Heubach: *Tiling an m–by–n Area with Squares of Size up to k–by–k* ($m \leq 5$), Congrssus Numerantium **140**, (1999), 43–64.

[12] R. Honsberger: *In Pólya's Footsteps*, MAA, New York, 1997.

[13] P. Kasteleyn: *The Statistics of Dimers on a Lattice: I. The Number of Dimer Arrangements on a Quadratic Lattice*, Physica, **27(12)**, (1961), 1209–1225.

[14] J. van Lint and R. Wilson: *A Course in Combinatorics*, Cambridge University Press, Cambridge, 2001.

[15] N. Loehr: *Bijective Combinatorics*, CRC Press, Boca Raton, 2011.

[16] P. Lundow: *Computation of matching polynomials and the number of 1-factors in polygraphs*, Research Reports Umeå, **12**, (1996).

[17] R. Mathar: *Paving Rectangular Region With Rectangular Tiles: Tatami And Non-Tatami Tilings*, arXiv:1311.6135 [math.CO], (2013).

[18] R. Mathar: *Tilings of Rectangular Regions by Rectangular Tiles: Counts Derived from Transfer Matrices*, arXiv:1406.7788 [math.CO], (2014).

[19] OEIS Foundation Inc.: *The On-Line Encyclopedia of Integer Sequences*, http://oeis.org, (2012).

[20] I. Pak and J. Yang: *The complexity of generalized domino tilings*, Electron. J. Combin., 20(4):Paper 12, 23, (2013).

[21] A. Slomson: *An Introduction to Combinatorics*, Chapman and Hall, London, 1991.

[22] H. Temperley and M. Fischer: *Dimer Problem in Statistical Mechanics-An Exact Result*, Philosophical Magazine, **6(68)**, (1961), 1061–1063.

[23] W. Webb: *Matrices with Forbidden Submatrics*, Aportaciones Mathemàticas, **20**, (2011), 275–278.

[24] W. Webb, N. Criddle, and D. DeTemple: *Combinatorial Chessboard Tilings*, Congressus Numerantium **194**, (2004), 257–262.

[25] H. Williams and R. Guy: *Some Fourth–Order Linear Divisibility Sequences*, Int. J. Number Theory **07**, 1255 (2011).

[26] E. Wright *Burnside's lemma: A Historical Note*, Journal of Combinatorial Theory, Series B **30(1)**, (1981), 89–90.

Department of Mathematics, Dartmouth College, Hanover, NH, 03755, USA
E-mail address: `ddeford@math.dartmouth.edu`

ON THE $q-$SEIDEL MATRIX

M. CETIN FIRENGIZ AND NAIM TUGLU

Abstract. Clarke and et. al recently introduced the q-Seidel matrix, and obtained some properties. In this article, we define a different form of q-Seidel matrix by $a_n^k(x,q) = xq^{n+2k-3}a_n^{k-1}(x,q) + a_{n+1}^{k-1}(x,q)$ with $k \geq 1,\ n \geq 0$ for an initial sequence $a_n^0(x,q) = a_n(x,q)$. By using our definition, we obtain several properties of the q-analogues of generalized Fibonacci and Lucas polynomials.

1. Introduction

The q-analogues of generalized Fibonacci and Lucas polynomials were investigated by many authors [3, 5, 7]. Carlitz [10] defined the q-Fibonacci polynomials by

$$\phi_{n+1}(a) - a\phi_n(a) = q^{n-1}\phi_{n-1}(a) \qquad (n > 1), \tag{1.1}$$

where $\phi_1(a) = 1,\ \phi_2(a) = a$.

The sequence of polynomials $S_n(x,q)$ is defined by the recurrence relation

$$S_{n+1}(x,q) = S_n(x,q) + xq^{n-2}S_{n-1}(x,q) \qquad (n \geq 1), \tag{1.2}$$

where $S_0(x,q) = a$ and $S_1(x,q) = b$. For $a = 0$ and $b = 1$, $S_n(x;q) = U_{n-1}\left(1; 0, -xq^{-1}\right)$, $S_n(x;q)$ is a special case Al-Salam and Ismail polynomials $U_n(x;a,b)$ introduced in [13]. Also the sequence of polynomials $S_n(x,q)$ is a special case $F_n(x;s,q)$ which is studied by Cigler in [7]. In particular, if we take $x = 1$, $q \to 1^-$ in (1.2), we get the classical Fibonacci and Lucas numbers for initial values $a = 0, b = 1$ and $a = 2, b = 1$ respectively.

q-Calculus started with L. Euler in the eighteenth century. q-Analogue of the binomial coefficients play important role in number theory, combinatorics, linear algebra and finite geometry. Now we mention some definitions of q-calculus [1]. Given value of $q > 0$, the q-integer $[n]_q$ is defined by

$$[n]_q = \begin{cases} \dfrac{1-q^n}{1-q} & \text{if } q \neq 1 \\ n & \text{if } q = 1, \end{cases}$$

and the q-factorial $[n]_q!$ is defined by

$$[n]_q! = \begin{cases} [n]_q \cdot [n-1]_q \cdots [1]_q & \text{if } n = 1, 2, ... \\ 1 & \text{if } n = 0 \end{cases}$$

for $n \in \mathbb{N}$. The $q-$binomial coefficients are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[n-k]_q!\ [k]_q!}, \qquad n \geq k \geq 0$$

with $\begin{bmatrix} n \\ 0 \end{bmatrix}_q = 1$ and $\begin{bmatrix} n \\ k \end{bmatrix} = 0$ for $n < k$. Note that the q-binomial coefficient satisfies the recurrence equations

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_q = q^k \begin{bmatrix} n \\ k \end{bmatrix}_q + \begin{bmatrix} n \\ k-1 \end{bmatrix}_q \tag{1.3}$$

and

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ k \end{bmatrix}_q + q^{n-k+1} \begin{bmatrix} n+1 \\ k-1 \end{bmatrix}_q . \tag{1.4}$$

In [9] Clarke and et. al give a kind of the generalization of a Seidel matrix, and obtain some properties by using the following relation:

$$\begin{aligned} &a_n^0(x,q) = a_n(x,q) && (n \geq 0), \\ &a_n^k(x,q) = xq^n a_n^{k-1}(x,q) + a_{n+1}^{k-1}(x,q) && (k \geq 1,\ n \geq 0). \end{aligned} \tag{1.5}$$

Here $(a_n(x,q))$ is a sequence of elements in a commutative ring. We can write $a_n^k(x,q)$ in terms of the initial sequence as

$$a_n^k(x,q) = \sum_{i=0}^{k} (xq^n)^{k-i} \begin{bmatrix} k \\ i \end{bmatrix}_q a_{n+i}^0(x,q). \tag{1.6}$$

Moreover $\left(a_n^0(x,q)\right)$ is called the initial sequence and $(a_0^n(x,q))$ the final sequence of the q-Seidel matrix. By using the Gauss inversion formula, we obtain relations between the initial sequence and final sequence:

$$a_0^n(x,q) = \sum_{i=0}^{n} x^{n-i} \begin{bmatrix} n \\ i \end{bmatrix}_q a_i^0(x,q), \tag{1.7}$$

$$a_n^0(x,q) = \sum_{i=0}^{n} (-x)^{n-i} q^{\binom{n-i}{2}} \begin{bmatrix} n \\ i \end{bmatrix}_q a_0^i(x,q). \tag{1.8}$$

Define the generating functions as follows:

$$a(t) = \sum_{n\geq 0} a_n^0(x,q)\, t^n, \qquad \overline{a}(t) = \sum_{n\geq 0} a_0^n(x,q)\, t^n \tag{1.9}$$

and

$$A(t) = \sum_{n\geq 0} a_n^0(x,q) \frac{t^n}{[n]_q!}, \qquad \overline{A}(t) = \sum_{n\geq 0} a_0^n(x,q) \frac{t^n}{[n]_q!}. \tag{1.10}$$

Thus the generating functions of the initial and final sequences are related by following equations:

$$\overline{a}(t) = \sum_{n\geq 0} a_n^0(x,q) \frac{t^n}{(xt;q)_{n+1}}, \tag{1.11}$$

$$\overline{A}(t) = e_q(xt)\, A(t). \tag{1.12}$$

Define $(t;q)_n = (1-t)(1-qt)\dots\left(1-q^{n-1}t\right)$ and $(t;q)_\infty = \lim_{n\to\infty}(t;q)_n$. Then

$$e_q(t) = \sum_{n\geq 0} \frac{t^n}{[n]_q!} = \frac{1}{((1-q)\,t;q)_\infty}. \tag{1.13}$$

Also

$$\frac{1}{(t;q)_{n+1}} = \sum_{k=0}^{\infty} \begin{bmatrix} n+k \\ k \end{bmatrix}_q t^k. \tag{1.14}$$

In this paper, we define a generalization of the q-Seidel matrix and obtain some properties for the generalized q-Seidel matrix. Furthermore we consider the q-analogues of generalized Fibonacci and Lucas polynomials $S_n(t,q)$ and give several properties of the sequence of polynomials $S_n(t,q)$ by using the generalized q-Seidel matrix method.

2. The Generalized q-Seidel Matrix

Let $(a_n(x,q))$ be a given real or complex sequence. The generalized $q-$Seidel matrix associated with $\left(a_n^0(x,q)\right)$ is defined recursively by the formula

$$\begin{aligned} a_n^0(x,q) &= a_n(x,q) \qquad (n \geq 0), \\ a_n^k(x,q) &= xq^{n+2k-3}a_n^{k-1}(x,q) + a_{n+1}^{k-1}(x,q) \qquad (n \geq 0,\ k \geq 1), \end{aligned} \tag{2.1}$$

where $a_n^k(x,q)$ represent the entry in the kth row and nth column.

We note that for $q \to 1^-$ and $x = 1$, the q-Seidel matrix turns into the usual Euler-Seidel matrix [2, 4, 6].

Lemma 2.1. *Let $\left(a_n^k(x,q)\right)$ satisfy equation (2.1) with initial sequence $\left(a_n^0(x,q)\right)$. Then*

$$a_n^k(x,q) = \sum_{i=0}^{k} x^{k-i} q^{(n+k-2)(k-i)} \begin{bmatrix} k \\ i \end{bmatrix}_q a_{n+i}^0(x,q). \tag{2.2}$$

Proof. We use induction to prove the proposition. The equation clearly holds for $k = 1$. Now, suppose that the equation is true for k. By (1.3) and (2.1) we have

$$\begin{aligned} a_n^{k+1}(x,q) &= xq^{n+2k-1}a_n^k(x,q) + a_{n+1}^k(x,q) \\ &= xq^{n+2k-1}\sum_{i=0}^{k} x^{k-i} q^{(n+k-2)(k-i)} \begin{bmatrix} k \\ i \end{bmatrix}_q a_{n+i}^0(x,q) \\ &\quad + \sum_{i=0}^{k} x^{k-i} q^{(n+k-1)(k-i)} \begin{bmatrix} k \\ i \end{bmatrix}_q a_{n+1+i}^0(x,q) \\ &= \sum_{i=0}^{k} x^{k+1-i} q^{(n+k-1)(k+1-i)} \begin{bmatrix} k \\ i \end{bmatrix}_q a_{n+i}^0(x,q) \\ &\quad + \sum_{i=1}^{k+1} x^{k+1-i} q^{(n+k-1)(k+1-i)} \begin{bmatrix} k \\ i-1 \end{bmatrix}_q a_{n+i}^0(x,q) \\ &= x^{k+1} q^{(n+k-1)(k+1)} a_n^0(x,q) \\ &\quad + \sum_{i=1}^{k} x^{k+1-i} q^{(n+k-1)(k+1-i)} \left\{ q^i \begin{bmatrix} k \\ i \end{bmatrix}_q + \begin{bmatrix} k \\ i-1 \end{bmatrix}_q \right\} a_{n+i}^0(x,q) + a_{n+k+1}^0(x,q) \\ &= \sum_{i=0}^{k+1} x^{k+1-i} q^{(n+k-1)(k+1-i)} \begin{bmatrix} k+1 \\ i \end{bmatrix}_q a_{n+i}^0(x,q). \end{aligned}$$

Hence, the equation is true for $n = k + 1$, which completes the proof. □

If we take $q \to 1^-$, $x = 1$ for (2.2), we get the well-known formula for the classical Euler-Seidel matrix [4].

The first row and column of the generalized q-Seidel matrix are defined by the inverse relation as in following corollary.

Corollary 2.2. *Let $a_n^0(x,q)$ and $a_0^n(x,q)$ be the first row and column in the generalized q-Seidel matrix. Then $a_n^0(x,q)$ and $a_0^n(x,q)$ have the inverse relation*

$$a_0^n(x,q) = \sum_{i=0}^{n} x^{n-i} q^{(n-2)(n-i)} \begin{bmatrix} n \\ i \end{bmatrix}_q a_i^0(x,q) \tag{2.3}$$

and

$$a_n^0(x,q) = \sum_{i=0}^{n} (-x)^{n-i} q^{\frac{(n-i)(n-3+i)}{2}} \begin{bmatrix} n \\ i \end{bmatrix}_q a_0^i(x,q). \tag{2.4}$$

Proposition 2.3. *Let $a_n^0(x,q)$ and $a_0^n(x,q)$ be the first row and column in the generalized q-Seidel matrix. Then $a_n^0(x,q)$ and $a_0^n(x,q)$ have the orthogonality relation*

$$\sum_{j=i}^{n} (-1)^{j-i} q^{(n-2)(n-j)} q^{\frac{(j-i)(j-3+i)}{2}} \begin{bmatrix} n \\ j \end{bmatrix}_q \begin{bmatrix} j \\ i \end{bmatrix}_q = \delta_{ni}. \tag{2.5}$$

Proof. We prove this by induction on n. A similar proof can be seen in [8, 11]. □

2.1. **Generating Functions.**

Proposition 2.4. *Let*

$$a(t) = \sum_{n=0}^{\infty} a_n^0(x,q) t^n$$

be the generating function of the initial sequence $\left(a_n^0(x,q)\right)$. Then the generating function of the sequence $\left(a_0^n(x,q)\right)$ is

$$\overline{a(t)} = \sum_{n=0}^{\infty} a_n^0(x,q) t^n \sum_{k=0}^{\infty} \begin{bmatrix} n+k \\ k \end{bmatrix}_q (xt)^k q^{k(k-2+n)}. \tag{2.6}$$

Proof. Considering (2.3) we write

$$\begin{aligned} \overline{a(t)} &= \sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} x^{n-i} q^{(n-2)(n-i)} \begin{bmatrix} n \\ i \end{bmatrix}_q a_i^0(x,q) \right) t^n \\ &= \sum_{n,k=0}^{\infty} \begin{bmatrix} n+k \\ k \end{bmatrix}_q x^k t^{n+k} q^{k(k-2+n)} a_n^0(x,q). \end{aligned}$$

Hence we obtain

$$\overline{a(t)} = \sum_{n=0}^{\infty} a_n^0(x,q) t^n \sum_{k=0}^{\infty} \begin{bmatrix} n+k \\ k \end{bmatrix}_q (xt)^k q^{k(k-2+n)}.$$

□

Proposition 2.5. *Let*

$$A(t) = \sum_{n=0}^{\infty} a_n^0(x,q) \frac{t^n}{[n]_q!}$$

be the exponential generating function of the initial sequence $\left(a_n^0(x,q)\right)$. Then the exponential generating function of the sequence $\left(a_0^n(x,q)\right)$ is

$$\overline{A(t)} = \sum_{n=0}^{\infty} a_n^0(x,q) \frac{t^n}{[n]_q!} \sum_{k=0}^{\infty} q^{k(k-2+n)} \frac{(xt)^k}{[k]_q!}. \tag{2.7}$$

Proof. The proof follows from equation (2.3). □

3. Applications of Generalized $q-$Seidel Matrices

In this section, we show that the generalized q-Seidel matrix is quite applicable for the q-analogues of generalized Fibonacci and Lucas polynomials. First we give the relationship between $S_{n+2k}(x,q)$ and the initial sequence $S_n(x,q)$ by using the generalized q-Seidel matrix.

Corollary 3.1. *The q-analogues of generalized Fibonacci and Lucas polynomials satisfy the following relation:*

$$S_{n+2k}(x,q) = \sum_{i=0}^{k} x^{k-i} q^{(n+k-2)(k-i)} \begin{bmatrix} k \\ i \end{bmatrix}_q S_{n+i}(x,q). \tag{3.1}$$

Proof. Let $a_n^0 = S_n(x,q)$, $n \geq 0$ be initial sequence. By using induction on k, (1.2) and (2.1), we have

$$a_n^k = S_{n+2k}(x,q).$$

Using (2.2) and applying $a_n^0 = S_n(x,q)$, we obtain

$$a_n^k = \sum_{i=0}^{k} x^{k-i} q^{(n+k-2)(k-i)} \begin{bmatrix} k \\ i \end{bmatrix}_q S_{n+i}(x,q).$$

This completes the proof. □

Corollary 3.2. *We have*

$$S_{2n}(x,q) = \sum_{i=0}^{n} x^{n-i} q^{(n-2)(n-i)} \begin{bmatrix} n \\ i \end{bmatrix}_q S_i(x,q), \tag{3.2}$$

$$S_n(x,q) = \sum_{i=0}^{n} (-x)^{n-i} q^{\frac{(n-i)(n-3+i)}{2}} \begin{bmatrix} n \\ i \end{bmatrix}_q S_{2i}(x,q) \tag{3.3}$$

and

$$S_{2n+1}(x,q) = \sum_{i=0}^{n} x^{n-i} q^{(n-1)(n-i)} \begin{bmatrix} n \\ i \end{bmatrix}_q S_{i+1}(x,q). \tag{3.4}$$

The following remark show that the well-known formulas [12] of Fibonacci numbers can be easily seen by using the properties of q-analogues of generalized Fibonacci and Lucas polynomials.

Remark 3.3. *Setting $a=0, b=1$ and $x=1$, $q \to 1^-$ in (3.1), we get the following equation of the Fibonacci numbers*

$$F_{n+2k} = \sum_{i=0}^{k} \binom{k}{i} F_{n+i}.$$

By taking $a = 0, b = 1$ and $x = 1$, $q \to 1^-$ as a special case of the equations (3.2), (3.3) and (3.4) we have the following identities for Fibonacci numbers:

$$F_{2n} = \sum_{i=0}^{n} \binom{n}{i} F_i,$$

$$F_n = \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} F_{2i},$$

$$F_{2n+1} = \sum_{i=0}^{n} \binom{n}{i} F_{i+1}$$

respectively. Also it is easily obtain similar formulas for the Lucas numbers.

Proposition 3.4. *The generating function of the polynomials $S_n(t,q)$ is*

$$\sum_{n=0}^{\infty} S_n(x,q)\, t^n = \frac{a + (b-a)\,t}{1 - t - xq^{-1}t^2\mu_t}, \tag{3.5}$$

where μ_t is the Fibonacci operator which is $\mu_t f(t) = f(tq)$ for any given function $f(t)$.

Proof. Let $g(x) = \sum\limits_{n=0}^{\infty} S_n(x,q)\, t^n$. We need to show the following equation:

$$g(x)\left(1 - t - xq^{-1}t^2\mu_t\right) = a + (b-a)\,t.$$

We have

$$\begin{aligned} g(x)\left(1 - t - xq^{-1}t^2\mu_t\right) &= a + bt + \sum_{n=2}^{\infty} S_n(x,q)\, t^n - \sum_{n=0}^{\infty} S_n(x,q)\, t^{n+1} - \sum_{n=0}^{\infty} S_n(x,q)\, xq^{n-1}t^{n+2} \\ &= a + bt - at + \sum_{n=2}^{\infty} \left\{S_n(x,q) - S_{n-1}(x,q) - xq^{n-3}S_{n-3}(x,q)\right\} t^n. \end{aligned}$$

This completes the proof. □

Corollary 3.5. *The generating function of $S_{2n}(x,q)$ is*

$$\sum_{n=0}^{\infty} S_{2n}(x,q)\, t^n = \frac{a + (b-a)\,t}{1 - t - xq^{-1}t^2\mu_t} \sum_{k=0}^{\infty} \begin{bmatrix} n+k \\ n \end{bmatrix}_q (xt)^k\, q^{k(k-2+n)}. \tag{3.6}$$

Proof. If we want to obtain the generating function of $S_{2n}(x,q)$ by using equation (2.6), we realize that by setting $a_n^0(x,q) = S_n(x,q)$ in (2.1). We obtain $a_0^n(x,q) = S_{2n}(x,q)$. By considering (2.6), we find

$$\overline{a(t)} = \sum_{n=0}^{\infty} a_0^n(x,q)\, t^n = \sum_{n=0}^{\infty} a_n^0(x,q)\, t^n \sum_{k=0}^{\infty} \begin{bmatrix} n+k \\ n \end{bmatrix}_q (xt)^k\, q^{k(k-2+n)}.$$

Therefore

$$\sum_{n=0}^{\infty} S_{2n}(x,q)\, t^n = \sum_{n=0}^{\infty} S_n(x,q)\, t^n \sum_{k=0}^{\infty} \begin{bmatrix} n+k \\ n \end{bmatrix}_q (xt)^k\, q^{k(k-2+n)}.$$

From (3.5) we have

$$\sum_{n=0}^{\infty} S_{2n}(x,q)\, t^n = \frac{a + (b-a)\,t}{1 - t - xq^{-1}t^2\mu_t} \sum_{k=0}^{\infty} \begin{bmatrix} n+k \\ n \end{bmatrix}_q (xt)^k\, q^{k(k-2+n)}.$$

□

This corollary points out that the generating functions of the first row and column of the generalized q-Seidel matrix are useful to obtain the generating function of $S_{2n}(x,q)$.

References

[1] A. Aral, V. Gupta and R.P. Agarwal, Applications of q-calculus in operator theory, New York, Springer, 2013.

[2] A. Dil and I. Mező, *A symmetric algorithm for hyperharmonic and Fibonacci numbers*, Appl. Math. Comput., **206** (2008), 942–951.

[3] C. Z. Jia , H. M. Liu and T. M. Wang, *q-Analogs of generalized Fibonacci and Lucas polynomials*, Fibonacci Quart., **45** (2007), 26–34.

[4] D. Dumont, *Matrices d'Euler-Seidel*, Séminaire Lotharingien de Combinatorie, 1981.

[5] G. E. Andrews, *Fibonacci numbers and the Rogers-Ramanujan identities*, Fibonacci Quart., **42** (2004), 3–19.

[6] I. Mező and A. Dil, *Euler-Seidel method for certain combinatorial numbers and a new characterization of Fibonacci sequence*, Cent. Eur. J. Math., **7** (2009), 310–321.

[7] J. Cigler, *q-Fibonacci polynomials*, Fibonacci Quart., **41** (2003), 31–40.

[8] J. Riordan, *Inverse relations and combinatorial identities*, Amer. Math. Monthly, **71** (1964), 485–498.

[9] R. J. Clarke, G. N. Han, J. Zeng, *A combinatorial interpretation of the Seidel generation of q-derangement numbers*, Ann. Comb., **4** (1997), 313–327.

[10] L. Carlitz, *Fibonacci notes 4: q-Fibonacci Polynomials,* Fibonacci Quart., **13** (1975), 97-102.

[11] R. B. Corcino, *On p,q-binomial coefficients*, Integers, **8** (2008), 1–16.

[12] S. Vajda, Fibonacci and Lucas numbers, and the Golden Section: Theory and Applications, Halsted Press, 1989.

[13] W. A. Al-Salam and M. E. H. Ismail, *Orthogonal Polynomials Associated with the Rogers-Ramanujan Continued Fraction*, Pacific J. Math., 104 (1983), 269–283.

MSC2010: 11B39, 11B83, 05A30.

Department of Mathematics Education, Başkent University, Baglica 06810 Ankara, Turkey
E-mail address: mcetin@baskent.edu.tr

Department of Mathematics, Gazi University, 06500 Ankara, Turkey
E-mail address: naimtuglu@gazi.edu.tr

A CAYLEY-HAMILTON AND CIRCULANT APPROACH TO JUMP SUMS

RUSSELL JAY HENDEL

Abstract. Define the Pascal Triangle jump sum by $\begin{bmatrix} n \\ k \end{bmatrix}_j = \sum_{m \equiv k \ (j)} \binom{n}{m}$, with $m \equiv k \ (j)$ meaning, as usual, $m \equiv k \pmod{j}$, and with with $\binom{n}{m} = 0$, if either $m < 0$ or $m > n$. The jump sum function adds every j-th entry in the n-th row of Pascal's Triangle starting at column k. The jump sum has been studied by several authors over the last 2 decades. Both recursions and explicit formulae have been given as well as several interesting number-theoretic applications. Varied proof methods have been presented including inductive, combinatoric, generating-function, and Riordan-array proofs. The goal of this paper is to provide an extremely compact proof of the recursions satisfied by the jump-sum functions using (i) the theory of circulant matrices and (ii) an extension of the Cayley-Hamilton Theorem that studies the values of a polynomial - whose zeroes are some, but not all, eigenvalues of a matrix - evaluated at that matrix. This matrix approach allows us to derive closed functional forms for some coefficients in the recursions.

1. Introduction

Define the (Pascal Triangle) jump-sum by

$$\begin{bmatrix} n \\ k \end{bmatrix}_j = \sum_{m \equiv k \ (n)} \binom{n}{m}, \tag{1.1}$$

with $m \equiv k \ (n)$ meaning, as usual, $m \equiv k \pmod{n}$, and with $\binom{n}{m} = 0$, if either $m < 0$ or $m > n$. The jump sum function adds every j-th entry in the n-th row of Pascal's Triangle, the summation process beginning at column k. Note, that although if say $k < 0$ that $\binom{n}{k} = 0$, nevertheless, $\begin{bmatrix} n \\ k \end{bmatrix}_j \neq 0$, since the value of $\begin{bmatrix} n \\ k \end{bmatrix}_j$ depends on the congruence class of k modulo j.

The jump-sums satisfy recursions and in fact, they "can be expressed in terms of some linearly recurrent sequences with orders bounded by $\phi(j)/2$," [19]. See also [20, 3].

Varied applications of the jump-sums exist including, values of Bernoulli and Euler polynomials at rational points [6, 20], values of quadratic characters [19, 13], as well as derivation of interesting new congruences for primes and various number theoretic quotients [16, 19].

Explicit formulas for $\begin{bmatrix} n \\ k \end{bmatrix}_j$ for $j = 3, 4, 5, 8, 10, 12$ may be found in [3, 15, 14, 16, 19].

A variety of proof methods have been applied including proofs by combinatorics [1], Riordan-arrays [10], and generating functions [12], as well as Jensen [2] and WZ proofs[5]. In this paper, we present a very compact proof based on the theory of circulants and extensions of the Cayley-Hamilton Theorem to the case where the factors of a polynomial contain some, but not all, of the eigenvalues of a matrix, and that polynomial is evaluated at that matrix.

To motivate our approach, we first review in Table 1 some numerical values of $\begin{bmatrix} 3 \\ k \end{bmatrix}_3$. Table 2 presents numerical values of $3\begin{bmatrix} n \\ k \end{bmatrix}_3 - 2^n$. Values of Table 2 can easily be computed from corresponding values in Table 1. Rows 3,6, and 9 of Table 2, suggest that the value of $3\begin{bmatrix} 3n \\ k \end{bmatrix}_3 -$

The author gratefully acknowledges useful comments on references and suggested focus of this paper from several attendees of the 16th Fibonacci conference held in Rochester, N.Y., July 2014.

n	Pascal Triangle Row	$k=0$	$k=1$	$k=2$
3	1 3 3 1	2	3	3
4	1 4 6 4 1	5	5	6
5	1 5 10 10 5 1	11	10	11
6	1 6 15 20 15 6 1	22	21	21
7	1 7 21 35 35 21 7 1	43	43	42
8	1 8 28 56 70 56 28 8 1	85	86	85
9	1 9 36 84 126 126 84 36 9 1	170	171	171

TABLE 1. Values of $\begin{bmatrix} n \\ k \end{bmatrix}_3$ based on (1.1), for small n.

n	Pascal Row	$k=0$	$k=1$	$k=2$
3	1 3 3 1	-2	1	1
4	1 4 6 4 1	-1	-1	2
5	1 5 10 10 5 1	1	-2	1
6	1 6 15 20 15 6 1	2	-1	-1
7	1 7 21 35 35 21 7 1	1	1	-2
8	1 8 28 56 70 56 28 8 1	-1	2	-1
9	1 9 36 84 126 126 84 36 9 1	-2	1	1

TABLE 2. Values of $3\begin{bmatrix} n \\ k \end{bmatrix}_3 - 2^n$ for small n. The corresponding values of $\begin{bmatrix} n \\ k \end{bmatrix}_3$ may be found in Table 1.

$n \equiv r\ (3)$	$k=0$	$k=1$	$k=2$
$r=0$	$2(-1)^n$	$-1(-1)^n$	$-1(-1)^n$
$r=1$	$-1(-1)^n$	$-1(-1)^n$	$2(-1)^n$
$r=2$	$-1(-1)^n$	$2(-1)^n$	$-1(-1)^n$

TABLE 3. Values of $c(k,n) = 3\begin{bmatrix} 3n \\ k \end{bmatrix}_3 - 2^{3n}$ based on the congruence class of n and k modulo 3.

$2^{3n} = c_3(k,n)$ depends only on the congruence class modulo 3 of n and k. An elementary proof based on the Pascal Recursion is presented in [3]. Table 3 presents all values of $c_3(k,n)$. Closed functional forms for $j\begin{bmatrix} jn \\ k \end{bmatrix}_j - 2^{jn} = c_j(k,n)$, depending only on the congruence class of k and n modulo j, have been computed for $j = 4, 5, 8, 10, 12$ [20, 3].

A further study of either Table 2 or Table 3 shows that for each fixed k and $l \in \{0,1,2\}$ the sequence $\{3\begin{bmatrix} 3n+l \\ k \end{bmatrix}_3 - 2^{3n+l}\}_{n\geq 1}$ satisfies the recursion $G_n + G_{n-1} = 0$.

We can exploit this uniformity to obtain a new approach to the jump-sum recursions based on matrices and vectors. Fix j and l with $0 \leq l \leq j-1$. Define the vector

$$G_n^{(j,l)} = G_n = \langle j\begin{bmatrix} jn+l \\ 0 \end{bmatrix}_j - 2^{jn+l}, j\begin{bmatrix} jn+l \\ 1 \end{bmatrix}_j - 2^{jn+l}, \ldots, j\begin{bmatrix} jn+l \\ k-1 \end{bmatrix}_j - 2^{jn+l}\rangle. \tag{1.2}$$

n	$3n+0$	$G_n^{(3,0)}$	$3n+1$	$G_n^{(3,1)}$	$3n+2$	$G_n^{(3,2)}$
1	3	$\langle -2,1,1\rangle$	4	$\langle -1,-1,2\rangle$	5	$\langle 1,-2,1\rangle$
2	6	$\langle 2,-1,-1\rangle$	7	$\langle 1,1,-2\rangle$	8	$\langle -1,2,-1\rangle$
3	9	$\langle -2,1,1\rangle$	10	$\langle -1,-1,2\rangle$	11	$\langle 1,-2,1\rangle$
4	12	$\langle 2,-1,-1\rangle$	13	$\langle 1,1,-2\rangle$	14	$\langle -1,2,-1\rangle$

TABLE 4. Values of $G_n^{(3,l)}, 0 \leq l \leq 2$, based on (1.2), with the values of the vector components previously computed in Tables 1-3. We have for all n and l, $G_n + G_{n-1} = 0$.

In the rest of the paper, we may notationally indicate such vectors by combining set notation with angle brackets as follows.

$$G_n^{(j,l)} = G_n = \langle j \begin{bmatrix} jn+l \\ k \end{bmatrix}_j - 2^{jn+l} : 0 \leq k \leq j-1\rangle.$$

When using such a notation, the angle brackets indicate that we are regarding the elements of the underlying set as ordered (that is, they are a vector). As an example of our notation, $\langle t : 3 \geq t \geq 1\rangle = \langle 3,2,1\rangle$ while $\langle t : 1 \leq t \leq 3\rangle = \langle 1,2,3\rangle$.

Table 4 shows values of G_n for initial values of n for $j=3$ and for all congruence classes of l modulo j. As can be seen, the vectors $\{G_n\}_{n\geq 1}$ satisfy the vector recursion $G_n + G_{n-1} = 0$, uniformly for all l.

The relationship between G_n and G_{n-1} can be described using a matrix. To do this we first recall that $\mathbf{Circ}(a_0, a_1, a_2, \ldots, a_{m-1})$ is the $m \times m$ matrix, Q, whose first row is $a_0, a_1, a_2, \ldots, a_{m-1}$ with $Q_{i,l} = Q_{i',l'}$ if $l - i \equiv l' - i'$ (m) [4].

Throughout the paper, the matrix entry in the x-th row and y-th column of a matrix Q will be denoted by either $Q(x,y)$ or $Q_{x,y}$. Similarly $v(k)$ or v_k will indicate the k-th component of the vector v. The notation $Q_{*,y}$ or $Q_{x,*}$ will indicate the y-th column or x-th row respectively. The notation Q_j (with one subscript) indicates Q evaluated at parameter j. We abuse vector notation so that e.g. the vector v in Qv is perceived as a column vector even though originally defined as a row vector.

Using these notations, we define

$$M_j = \mathbf{Circ}(\binom{j}{0} + \binom{j}{j}, \binom{j}{1}, \binom{j}{2}, \ldots, \binom{j}{j-1}). \tag{1.3}$$

Matrix M_j is closely related to the circulant matrix underlying Wendt's determinant [7, 21]. In fact, $W_j = \mathbf{Det}(M_j - I_j)$, where I_j is the $j \times j$ identity matrix. However, this fact will not be further used in this paper.

Proposition 1.1. *For any fixed $l, 0 \leq l \leq j-1$, and for all $n \geq 1$,*

$$M_j G_n^{(j,l)} = G_{n+1}^{(j,l)}. \tag{1.4}$$

.

Throughout the rest of the paper, except for the tables and examples, j will be fixed and hence, when notationally convenient, we omit mention of it.

Prior to presenting the proof, we summarize well-known binomial identities used throughout the paper.

Proposition 1.2. *For any positive integer x,*

a) $\displaystyle\sum_{p=0}^{x}\binom{x}{p} = 2^x.$

b) $\displaystyle\sum_{p=0}^{x}(-1)^p\binom{x}{p} = 0.$

c) $\displaystyle\sum_{p=0}^{\frac{x-1}{2}}\binom{x}{p} = 2^{x-1}, \qquad \textit{if } x \textit{ is odd.}$

d) Furthermore, for any integer z, $1 \le z \le x-1$, and any integer $y, 0 \le y \le x$, we have

$$\binom{x}{y} = \sum_{p=0}^{x}\binom{z}{p}\binom{x-z}{y-p}. \tag{1.5}$$

Proof. Well known. For example, (d) follows by comparing coefficients in the expansions of both sides of the identity, $(1+V)^x = (1+V)^z(1+V)^{x-z}$. When $z=1$ we obtain the traditional Pascal Recursion. □

Proof. We now return to the proof of (1.4).

Equation (1.4) is equivalent to the j equations,

$$M_{k,*}G_n = j\begin{bmatrix} j(n+1)+l \\ k \end{bmatrix}_j - 2^{j(n+1)+l}, \qquad 0 \le k \le j-1. \tag{1.6}$$

Equation (1.6) implies that for each k, $M_{k.*}$ defines a linear homogeneous recursion with constant coefficients on the sequence $\{j\left[\begin{smallmatrix} j(n+1)+l \\ k \end{smallmatrix}\right]_j - 2^{j(n+1)+l}\}_{n\ge1}$. Since the order-$j$ linear recursive sequences with constant coefficients form a vector space, to prove (1.6), it suffices to show that the recursion defined by $M_{k,*}$ holds for each summand in $j\left[\begin{smallmatrix} j(n+1)+l \\ k \end{smallmatrix}\right]_j - 2^{j(n+1)+l}$. We deal separately with each summand. Since the second summand is straightforward, we deal with it first.

Second summand.
By (1.3), the rows of M are permutations of the binomial coefficients with $\binom{j}{0}$ and $\binom{j}{j}$ added together. Hence, by Proposition 1.2(a),

$$2^{j(n+1)+l} = \sum_{p=0}^{j}\binom{j}{p}2^{jn+l}.$$

First summand. It suffices to prove

$$\begin{bmatrix} j(n+1)+l \\ k \end{bmatrix}_j = M_{k,*}\langle \begin{bmatrix} jn \\ q \end{bmatrix} : 0 \le q \le j-1\rangle, \qquad 0 \le k \le j-1. \tag{1.7}$$

By (1.3) and the identity $\binom{j}{x} = \binom{j}{j-x}$, we have

$$M_{k,*}(q) = \begin{cases} \binom{j}{k-q}, & \text{for } 0 \le q \le k-1, \\ \binom{j}{0} + \binom{j}{j}, & \text{for } q = k \\ \binom{j}{j-(q-k)}, & \text{for } k+1 \le q \le j-1. \end{cases} \tag{1.8}$$

Equation (1.1) shows that jump-sums are sums of binomial coefficients and hence they inherit the recursions satisfied by these binomial coefficients. Consequently, by (1.5),

$$\begin{aligned}
\begin{bmatrix} j(n+1)+l \\ k \end{bmatrix}_j &= \sum_{\substack{0\le q\le j \\ p+q\equiv k\ (j)}} \binom{j}{p} \begin{bmatrix} jn+l \\ q \end{bmatrix}_j \\
&= \sum_{\substack{0\le q\le k-1 \\ p+q\equiv k\ (j)}} \binom{j}{p} \begin{bmatrix} jn+l \\ q \end{bmatrix}_j \\
&+ \sum_{\substack{q=k \\ p+q\equiv k\ (j)}} \binom{j}{p} \begin{bmatrix} jn+l \\ q \end{bmatrix}_j \\
&+ \sum_{\substack{k+1\le q\le j-1 \\ p+q\equiv k\ (j)}} \binom{j}{p} \begin{bmatrix} jn+l \\ q \end{bmatrix}_j .
\end{aligned} \tag{1.9}$$

Using (1.9), we can prove (1.7) by showing that for the 3 cases in (1.8) corresponding to the three summands on the right hand side of (1.9) we have that the sum of q with the bottom argument of the binomial coefficient is congruent to k modulo j. But for the top case we clearly have $k-q+q=k$, for the middle case we trivially have $k+0=k+j\equiv k\ (j)$, and for the bottom case we similarly have $j-(q-k)+q\equiv k\ (j)$. This completes the proof of (1.7) and hence of (1.6). □

A similar proof, exploiting the fact that by (1.1) the jump-sum function inherits the recursions satisfied by the Pascal Triangle, yields the following proposition.

Proposition 1.3. *For any positive integers j, n and any non-negative integers $l, k, 0 \le l, k \le j-1$,*

$$\begin{bmatrix} jn+l+1 \\ k \end{bmatrix}_j = \begin{bmatrix} jn+l \\ k \end{bmatrix}_j + \begin{bmatrix} jn+l \\ k-1 \end{bmatrix}_j , \tag{1.10}$$

from which we derive

$$j\begin{bmatrix} jn+l+1 \\ k \end{bmatrix}_j - 2^{jn+l+1} = \left(j\begin{bmatrix} jn+l \\ k \end{bmatrix}_j - 2^{jn+l} \right) + \left(j\begin{bmatrix} jn+l \\ k-1 \end{bmatrix}_j - 2^{jn+l} \right). \tag{1.11}$$

From (1.7)*, we directly have*

$$M\langle \begin{bmatrix} jn \\ k \end{bmatrix}_j : 0 \le k \le j-1 \rangle = \langle \begin{bmatrix} j(n+1) \\ k \end{bmatrix}_j : 0 \le k \le j-1 \rangle. \tag{1.12}$$

Tables 1 and 2 illustrate (1.10) and (1.11) respectively. Note that by (1.1), when applying the Pascal recursions of (1.10) and (1.11), k is interpreted modulo j so that -1 is interpreted as $j-1$.

Matrices are an established technique to derive recursions [8]. Equation (1.4) immediately gives us recursions satisfied by $\{G_n\}_{n\ge1}$, since by letting $p=p_j$ be the characteristic polynomial of M_j, we have $p(M)=0$, and therefore $p(M)G_n=0$, for all $n\ge1$. Consequently, $p(X)$ is the associated polynomial of a recursion satisfied by the vector sequence $\{G_n\}_{n\ge1}$.

However, the degree of $p(X)$ is j while G in fact satisfies a recursion of order $\lfloor \frac{j-1}{2} \rfloor$. One approach to lowering the degree of p is to modify the Cayley-Hamilton polynomial by writing

$p'(X) = \prod(X - r_i)$, where the r_i are the *distinct* eigenvalues of p. This modified Cayley-Hamilton polynomial, under appropriate conditions (such as diagonalizability), also satisfies $p'(M) = 0$ [9] and hence $p'(M)G_n = 0$, for all $n \geq 1$. However, this too is not sufficient, since the degree of p' is greater (by 1 for odd j and by 2 for even j) than $\lfloor \frac{j-1}{2} \rfloor$. We must therefore extend the Cayley-Hamilton theory by studying polynomials, whose zeroes are a partial set of eigenvalues, evaluated at the underlying matrix.

This motivates the following outline of the rest of the paper. In Section 2, we present prerequisites summarizing important facts about circulants and values of polynomials whose roots are eigenvalues evaluated on the underlying matrices (Cayley-Hamilton theory). We also construct a modified Cayley-Hamilton polynomial, q. In Section 3, we show that although $q(M) \neq 0$, nevertheless, $q(M)G_n^{(j,l)} = 0$, for all j, n, l. Consequently, q is the associated polynomial of a recursion of order $\lfloor \frac{j-1}{2} \rfloor$. In Section 4, we derive exact formulas for some of the coefficients of $q(X)$.

2. Prerequisites

We need prerequisites on circulants, Vandermonde determinants, and Cayley-Hamilton.

The following proposition and definitions summarize major aspects of circulants [4].

Proposition 2.1. *Let $\zeta_j = e^{\frac{2\pi i}{j}}$, be a $j-th$ root of unity. Then the eigenvalues of any $j \times j$ circulant matrix are given by the following.*

$$\lambda_k = \sum_{p=0}^{j} \binom{j}{k} \zeta^{pk}, \qquad 0 \leq k \leq j-1. \tag{2.1}$$

Define the Vandermonde matrix V_j by

$$\sqrt{j} V_{i,k} = \zeta^{ik}, \qquad 0 \leq i, k \leq j-1. \tag{2.2}$$

Then $V^{-1} = \overline{V}$ and

$$M_j = V_j D_j V_j^{-1}, \tag{2.3}$$

with $D = D_j$ the diagonal matrix of eigenvalues of M_j, with

$$D_j(i,i) = \lambda_i, \qquad 0 \leq i \leq j-1. \tag{2.4}$$

.

Corollary 2.2. *The eigenvalues of M are given by* (2.1).

Proof. Proposition 2.1 applies to any $j \times j$ circulant and hence by (1.3) applies to M. □

The following proposition summarizes some basic facts about the eigenvalues, λ_k.

Proposition 2.3.
a) $\lambda_k = \sum_{i=0}^{j} \binom{j}{i} \zeta_j^{ki} = (1 + \zeta_j^k)^j$.
b) $\lambda_0 = 2^j$.
c) $\lambda_{\frac{j}{2}} = 0$, *if j is even.*
d) $\lambda_k = \lambda_{j-k}, k \neq 0$.
e) *λ_0 has multiplicity 1; when j is even, $\lambda_{\frac{j}{2}}$ has multiplicity 1; all other roots have multiplicity 2.*

Proof. (a) follows from (2.1) and the binomial expansion as shown. (b)-(d) follow from (a) and Proposition 1.2. (e) follows from (a). For example, $(1+\zeta^k)^j = 2^j$ requires $\zeta_k = 1$ which requires $k = 0$; hence, λ_0 has multiplicity 1. $\square$

We now present propositions about polynomials evaluated at matrices.

Proposition 2.4. *Let $B = HEH^{-1}$ be a matrix equation about $m \times m$ matrices. Let $r(X)$ be any polynomial. Then $r(B) = Hr(E)H^{-1}$.*

Proof. The proposition is clearly true for polynomials of the form $r(x) = X^t$ and hence extends to arbitrary polynomials by scalar multiplication and addition. $\square$

Define the corner matrix $C = C(x)$, by

$$C_x(i,j) = \begin{cases} x, & \text{if } (i,j) = (0,0), \\ 0, & \text{if } (i,j) \neq (0,0). \end{cases} \tag{2.5}$$

The corner matrices are useful because of the following proposition.

Proposition 2.5. *With $D = D_j$ defined by (2.4) and λ_k defined by (2.1), define a polynomial $q = q_j(x)$ by*

$$q(X) = \begin{cases} \displaystyle\prod_{k=1}^{\frac{j-1}{2}} (X - \lambda_k), & \text{if } j \text{ is odd}, \\ \displaystyle\prod_{k=1}^{\frac{j}{2}-1} (X - \lambda_k), & \text{if } j \text{ is even}. \end{cases} \tag{2.6}$$

Then

$$q(D) = C(q(\lambda_0)). \tag{2.7}$$

Comment 2.6. *The zeroes of $q(X)$ are the eigenvalues of M, without multiplicity, except for $\lambda_0 = 2^j$ and except for $\lambda_{\frac{j}{2}} = 0$, when j is even. We prove in Section 3 that even though $q(M) \neq 0$, $q(M)G_n = 0, n \geq 1$. Consequently, q_j is the associated polynomial of a recursion of order $\lfloor \frac{j-1}{2} \rfloor$ satisfied by the vector sequence $\{G_n\}_{n \geq 1}$. We thus see that $q(X)$ is the desired modification of the Cayley-Hamilton polynomial. Therefore, prior to the proof of Proposition 2.4, it might be worthwhile to see some examples.*

Example 2.7. *Let $j = 3$. Then by (2.1), the three eigenvalues of $M_3 = \mathbf{Circ}(2,3,3)$ are $\lambda_0 = 2^3 = 8$, $\lambda_1 = \lambda_2 = 2 + 3\omega + 3\omega^2$, with ω a primitive cube root of unity. In this case $q_3(X) = (X - \lambda_1)$. But $1 + \omega + \omega^2 = 0$ implying that $\lambda_1 = -1$, and consequently $q_3(x) = X + 1$, which is the associated polynomial of the recursion $G_n + G_{n-1} = 0$, which as we saw in Section 1, is satisfied by the vector sequence $\{G_n^{(3,0)}\}_{n \geq 1}$.*

Example 2.8. *Let $j = 4$. Then by (2.1), the 4 eigenvalues of $M_4 = \mathbf{Circ}(2,4,6,4)$, are $\lambda_0 = 16$, $\lambda_1 = \lambda_3 = 1 + 4i + 6i^2 + 4i^3 + i^4 = -4$, and $\lambda_2 = 1 + 4i^2 + 6i^4 + 4i^6 + i^8 = 0$. In this case, $q_4(X) = (X - \lambda_1) = X + 4$. One can check that $G_n + 4G_{n-1} = 0$.*

One can write down the coefficients of $q_i, i = 3, 4, \ldots$, with one polynomial per row. This gives rise to the jump sum recursion triangle [11], displayed in Table 5. The closed functional forms $2^{j-1} - j$ for the second column and $(\frac{j}{2})^{\frac{j}{2}}$ for right-most diagonal on rows where j is even, will be proven in Section 4.

j	Coefficients of $q_j(X)$
3	1, 1
4	1, 4
5	1, 11, -1
6	1, 26, -27
7	1, 57, -289, -1
8	1, 120, -2160, -256
9	1, 247, -13359, -13604, 1
10	1, 502, -73749, -383750, 3125

TABLE 5. Coefficients of $q_j(X)$, (2.6), in descending exponent order. The last coefficient has degree 0. For example, $q_5(X) = X^2 + 11X - 1$. $q(X)$ is the associated polynomial of a recursion on the vectors $\{G_n\}_{n\geq 1}$, (1.2). For example, if $j = 5$, $G_n + 11G_{n-1} - G_{n-2} = 0$, $n \geq 1$.

Proof. We return to the proof of Proposition 2.4. We may think of D as arranged in blocks of λ_i. By Proposition 2.3(e), λ_0 has multiplicity 1 so the upper left block has dimensions 1×1. Consider the effect of the factor $X - \lambda_i$ on M. (i) The upper left cell has $\lambda_0 - \lambda_i$, (ii) the block with λ_i down the diagonal has all zeroes, and (iii) other blocks have $\lambda_k - \lambda_i$ down the diagonal. Upon multiplication, we have zeroes in all blocks except the leftmost cell which has $(\lambda_0 - \lambda_1)(\lambda_0 - \lambda_2)\ldots = q(\lambda_0)$ as was to be shown. □

We need one more concept. Besides $I = I_j$ which is the $j \times j$ identity matrix we need a matrix $J = J_j$ defined as follows.

$$J_{x,y} = 1, \qquad 0 \leq x, y \leq j - 1. \tag{2.8}$$

We have the following elementary results.

Proposition 2.9.
a)$J^2 = jJ$.
b) $JM = MJ = 2^j J$.
c) For any positive integer n, $JM^n = M^n J = 2^{jn} J$
d) With V,C *and* q *defined by* (2.2),(2.5) *and* (2.6) *and for any complex* z_0, *we have* $q(VC(z_0)V^{-1}) = \frac{1}{j}q(z_0)J$.

Proof. (a) and (b) are clear. For example, to prove (b), all the rows of J are ones and hence the entries of JM are dot products of a vector of ones with the binomial coefficients in some permutation and therefore equal to 2^j by Proposition 1.2(a). (c) Follows from (b) by a routine induction. (d) follows from the fact that V and $\overline{V} = V^{-1}$ have a left column and top row of all ones. The $\frac{1}{j}$ comes from the normalization factor in (2.2). □

.

Proposition 2.10. *With* $M = M_j$ *defined by* (1.3) *and* $q = q_j$ *defined by* (2.6), *we have*

$$q(M) = \frac{1}{j}q(2^j)J. \tag{2.9}$$

Proof.

$$\begin{aligned} q(M) =& q(VDV^{-1}), && \text{by (2.3)},\\ =& Vq(D)V^{-1}, && \text{by Proposition 2.4},\\ =& VC(q(\lambda_0))V^{-1}, && \text{by Proposition 2.5},\\ =& VC(q(2^j))V^{-1}, && \text{by Proposition 2.3(b)},\\ =& \frac{1}{j}q(2^j)J, && \text{by Proposition 2.9(d)}. \end{aligned}$$

□

3. The Main Theorem

Theorem 3.1. *For any integers $n \geq 1$, $j \geq 3$, and $0 \leq l \leq j-1$, and with q, M, and G_n defined by* (2.6), (1.3), *and* (1.2) *respectively, we have*

$$q_j(M_j)G_n^{(j,l)} = 0, \qquad n \geq 1. \tag{3.1}$$

Corollary 3.2. *For fixed j, l, $q_j(X)$ is the associated polynomial to a recursion satisfied by the $\{G_n^{(j,l)}\}_{n\geq 1}$.*

Proof. We first prove (3.1) assuming $l = 0$.

By (1.2), (3.1) is equivalent to

$$q(M)\langle j\begin{bmatrix} jn \\ k \end{bmatrix}_j : 0 \leq k \leq j-1\rangle = q(M)\langle 2^{jn} : 0 \leq k \leq j-1\rangle. \tag{3.2}$$

But, by (2.8),

$$q(M)\langle 2^{jn} : 0 \leq k \leq j-1\rangle = q(M)2^{jn}J_{*,0}, \tag{3.3}$$

and similarly by (1.2) and (1.8) evaluated at $k = 0$,

$$G_1 = \langle\binom{j}{0} + \binom{j}{j}, \binom{j}{1}, \binom{j}{2}, \ldots, \binom{j}{j-1}\rangle = M_{0,*}. \tag{3.4}$$

Hence, by (1.12),

$$q(M)\langle j\begin{bmatrix} jn \\ k \end{bmatrix}_j : 0 \leq k \leq j-1\rangle = q(M)jM^{n-1}M_{0,*}. \tag{3.5}$$

In proving (3.2), a crucial step is replacement of the vectors in (3.3) and (3.5) by matrices. In other words, by (3.2)-(3.5), to prove (3.1) it *suffices* to prove

$$q(M)jM^{n-1}M = q(M)2^{jn}J. \tag{3.6}$$

.

We prove (3.6) by showing the left and right sides equal. By (2.9) and Proposition 2.9(c), we have

$$q(M)jM^n = jq(M)M^n = j\frac{1}{j}q(2^j)JM^n = q(2^j)2^{jn}J.$$

Similarly, by (2.9) and Proposition 2.9(a), we have

$$q(M)2^{jn}J = 2^{jn}\frac{1}{j}q(2^j)JJ = 2^{jn}\frac{1}{j}q(2^j)jJ = q(2^j)2^{jn}J.$$

This completes the proof of (3.1) when $l = 0$.

To prove (3.1) for $l > 1$, we use an inductive argument. We assume (3.1) is proven for the case l and proceed to prove it for the case $l+1$. The base case occurs when $l = 0$. But by (1.11), if a recursion holds for l then it holds for $l+1$.

This completes the proof of the Main Theorem. $\square$

4. Coefficient Results

Certain patterns emerge for the second and last coefficient in the jump sum triangle displayed in Table 5. We formally state them as a corollary to the Main Theorem.

Corollary 4.1. *With q defined by (2.6), let $m = \lfloor \frac{j-1}{2} \rfloor$. Further, define c_i by*

$$q(X) = X^m + c_1 X^{m-1} + c_2 X^{m-2} + \ldots + c_m. \tag{4.1}$$

Then

$$c_1 = 2^{j-1} - j. \tag{4.2}$$

Furthermore, if j is even, then

$$|c_m| = \left(\frac{j}{2}\right)^{\frac{j}{2}}. \tag{4.3}$$

.

Proof. Technically, by (2.6), to prove (4.2), we have to consider j even and odd separately. To prove (4.2), we assume j odd, the proof for the even case being similar and hence omitted. To prove (4.3), we assume j even. By (2.6), (4.1), (2.1) and Proposition 2.3(a), we have

$$-c_1 = \sum_{k=1}^{\frac{j-1}{2}} \lambda_k = \sum_{k=1}^{\frac{j-1}{2}} \sum_{p=0}^{j} \binom{j}{p} \zeta^{pk}; \qquad c_m = \prod_{p=1}^{\frac{j}{2}-1} \lambda_p = \prod_{p=1}^{\frac{j}{2}-1} (1+\zeta_j^p)^j. \tag{4.4}$$

Proof of (4.2)**.** In (4.4) we may interchange the order of summation and carve out the 0 and j term separately.

$$\begin{aligned} -c_1 &= \sum_{p=0}^{j} \binom{j}{p} \sum_{k=1}^{\frac{j-1}{2}} \zeta^{pk} \\ &= \sum_{p=0} \binom{j}{p} \sum_{k=1}^{\frac{j-1}{2}} \zeta^{pk} + \sum_{p=j} \binom{j}{p} \sum_{k=1}^{\frac{j-1}{2}} \zeta^{pk} + \sum_{p=1}^{j-1} \binom{j}{p} \sum_{k=1}^{\frac{j-1}{2}} \zeta^{pk} \\ &= j - 1 + \sum_{p=1}^{j-1} \binom{j}{p} \sum_{k=1}^{\frac{j-1}{2}} \zeta^{pk}. \end{aligned} \tag{4.5}$$

For the last summand in (4.5), since $\binom{j}{k} = \binom{j}{j-k}$, we have

$$\sum_{p=1}^{j-1} \binom{j}{p} \sum_{k=1}^{\frac{j-1}{2}} \zeta^{pk} = \sum_{p=1}^{\frac{j-1}{2}} \binom{j}{p} \sum_{k=1}^{\frac{j-1}{2}} \left(\zeta^{pk} + \zeta^{-pk}\right). \tag{4.6}$$

p is fixed in the inner summand. Since we are looking at exponents of j-th roots of unity we can evaluate these exponents modulo j. Let g equal the greatest common divisor of p and j. Then, for each fixed p, and evaluating modulo j, we have

$$\{0\}\bigcup\{kp : 1 \leq k \leq \frac{j-1}{2}\}\bigcup\{-kp : 1 \leq k \leq \frac{j-1}{2}\} = \{0, g, 2g, 3g, \dots, (\frac{j}{g}-1)g\}.$$

Hence,

$$\zeta^0 + \sum_{k=1}^{\frac{j-1}{2}}\left(\zeta^{pk} + \zeta^{-pk}\right) = 0. \tag{4.7}$$

Applying (4.7) to (4.6) and using Proposition 1.2(c), we have

$$\sum_{p=1}^{j-1}\binom{j}{p}\sum_{k=1}^{\frac{j-1}{2}}\zeta^{pk} = \sum_{p=1}^{\frac{j-1}{2}}\binom{j}{p}(-1) = -(2^{j-1}-1). \tag{4.8}$$

Equation (4.2) now follows from (4.1), (4.5) and (4.8).

Proof of (4.3)**.** Since j is assumed even, let

$$j = 2n. \tag{4.9}$$

By (4.4), (4.9) and Proposition 2.3(d),

$$c_m = \prod_{p=1}^{n-1}\lambda_p = \prod_{p=1}^{n-1}\lambda_{2n-p} = \prod_{p=1}^{n-1}(1+\zeta_{2n}^{p+n})^j = \prod_{p=1}^{n-1}(1-\zeta_{2n}^{p})^j. \tag{4.10}$$

Combining (4.4),(4.10) with Proposition 2.3(a) and using the identity $(1-\zeta_{2n}^p)(1+\zeta_{2n}^p) = (1-\zeta_n^p)$, we have

$$c_m^2 = \prod_{p=1}^{n-1}\lambda_p\prod_{p=1}^{n-1}\lambda_{2n-p} = \prod_{p=1}^{n-1}(1-\zeta_{2n}^p)^j\prod_{p=1}^{n-1}(1+\zeta_{2n}^p)^j = \prod_{p=1}^{n-1}(1-\zeta_n^p)^j. \tag{4.11}$$

To evaluate the last product we use the formula for geometric series and the fundamental theorem of algebra, to obtain

$$1 + X + \dots X^{n-1} = \frac{X^n - 1}{X-1} = (X-\zeta_n)(X-\zeta_n^2)\dots(X-\zeta_n^{n-1}).$$

Letting $X = 1$ in the last equation, yields

$$n = \prod_{p=1}^{n-1}(1-\zeta_n^p). \tag{4.12}$$

By (4.11), (4.9) and (4.12) we have $c_m^2 = n^{2n} = \left(\frac{j}{2}\right)^j$, proving (4.1). □

References

[1] D. Callan, *A combinatorial proof of Sun's curious identity,* Integers: Electron. J. Combin. Number Theory **4**, (2004), A05, 6pp.

[2] W. Chu, L. V. D. Claudio, *Jensen proof of a curious binomial identity,* Integers: Electron. J. Combin. Number Theory **3** (2003), A20, 3pp.

[3] Charles Cook and Rebecca Hillman, *Some Jump Sum Patterns for the Rows of Pascal and Related Triangles*, Congressus Numerantium, **200**,(2000), pp 255-267.

[4] P. J. Davis, *Circulant Matrices, 2nd ed.*, New York: Chelsea, 1994

[5] S. B. Ekhad, M. Mohammed, *A WZ proof of a curious identity*, Integers: Electron. J. Combin. Number Theory **3** (2003), A6, 2pp.
[6] A. Granville and Z.-W. Sun, *Values of Bernoulli polynomials*, Pacific Journal of Mathematics, **172**, (1996), pp. 117-138.
[7] Charles Helou, *On Wendt's Determinant*, Mathematics of Computation, **66**, (1997), pp. 1341-1346.
[8] Thomas Koshy, *Fibonacci and Lucas Numbers with Applications*, John Wiley, (2001), pp 362-401.
[9] Serge Lang, *Algebra, 3rd Edition*, MA: Addison-Wesley, (1993), p. 568, exercise 13.
[10] D. Merlini, R. Sprugnoli, *A Riordan array proof of a curious identity,* Integers: Electron. J. Combin. Number Theory **2** (2002), A8, 3pp.
[11] OEIS Foundation Inc. (2011), The On-Line Encyclopedia of Integer Sequences, `http://oeis.org/A244608.`
[12] A. Panholzer, H. Prodinger, *A generating functions proof of a curious identity,* Integers: Electron. J. Combin. Number Theory, **2**, (2002), A6, 3pp.
[13] Z.-H. Sun, *Combinatorial* $\sum_{k\equiv r\ (m)}\binom{n}{k}$ *and its applications in number theory (I)*, Nanjing University Journal. Mathematical Biquarterly, **9**, (1992), pp. 227- 240.
[14] Z.-H. Sun and Z.-W. Sun, *Fibonacci numbers and Fermat's Last theorem*, Acta Arithmetica, **60**, (1992), pp. 371-388.
[15] Z.-H. Sun, *Combinatorial* $\sum_{k\equiv r\ (m)}\binom{n}{k}$ *and its applications in number theory (II)*, Nanjing University Journal. Mathematical Biquarterly, **10**, (1993), pp. 105- 118.
[16] Z.-W. Sun, *A congruence for primes*, Proceedings of the American Mathematical Society, **123**,(1995), pp. 1341-1346.
[17] Z.-H. Sun, *Combinatorial* $\sum_{k\equiv r\ (m)}\binom{n}{k}$ *and its applications in number theory (III)*, Nanjing University Journal. Mathematical Biquarterly, **12**, (1995), pp. 90-102.
[18] Z.-W. Sun, *Products of binomial coefficients modulo* p^2, Acta Arithmetica, **97** (2001), pp. 87-98.
[19] Z.-W. Sun, *On the sum* $\sum_{k\equiv r\ (m)}\binom{n}{k}$ *and related congruences*, Israel J. Math,.**128**, (2002), pp. 135-156.
[20] Zhi-Wei Sun, *On Sums of Binomial Coefficients and Their Applications*, Discrete Mathematics **308**, (2008), pp. 4231-4245.
[21] E. Wendt, *Arithmetische Studien uber den letzten Fermatschen Satz, welcher aussagt, dass die Gleichung* $a^n = b^n + c^n$, *fur* $n > 2$ *in ganzen Zahlen nicht auflosbar ist,* J. reine angew. Math., **113**, (1894), pp. 335-347.

MSC2010: 11B39, 15A18,

Department of Mathematics, Towson University, Towson, MD., 21252
E-mail address: `RHendel@Towson.Edu`

THE INFINITE FIBONACCI TREE AND OTHER TREES GENERATED BY RULES

CLARK KIMBERLING AND PETER J. C. MOSES

ABSTRACT. Suppose that I is a subset of a set U and that C is a collection of operations f defined in U. Create a set S by these rules: every element of I is in S, and if x is in S, then $f(x)$ is in S for all f in C for which $f(x)$ is defined. Then S "grows" in successive generations. If I consists of a single number r then S can be regarded as a tree with root r. We examine several examples, including these: (1) $1 \in S$, and if $x \in S$ then $x+1 \in S$ and $1/x \in S$; (2) $1 \in S$, and if $x \in S$ then $x+1 \in S$ and $2x \in S$; (3) $1 \in S$, and if $x \in S$ then $x+1 \in S$, and if $x \neq 0$ then $-1/x \in S$; (4) $1 \in S$, and if $x \in S$ then $x+1 \in S$ and $\sqrt{-1}x \in S$, and if $x \neq 0$ then $1/x \in S$. The first of these examples is the infinite Fibonacci tree, in which every positive rational number occurs as a node.

1. Introduction

As early as 1619, Johannes Kepler created a tree of fractions using these rules: begin with $1/1$, and thereafter, each node i/j has two descendants, $(i+j)/i$ and $i/(i+j)$. Kepler's tree [1, 3] can be recast by saying that 1 is present, and if x is present, then so are $x+1$ and $1/(x+1)$. The tree starts with a single node which spawns 2 nodes (2 and $1/2$), which spawn 4 nodes $(3, 1/3, 3/2, 2/2)$, and so on, so that the nth generation has 2^n nodes. Moreover, every positive rational number occurs exactly once.

Now consider the set S defined by these rules: $1 \in S$, and if $x \in S$, then $x+1 \in S$ and $1/x \in S$. Deleting duplicates as they occur leaves the infinite Fibonacci tree, represented in Figure 1 (in Section 7) and discussed in Example 3.4. Another tree with Fibonacci connections is given by the rules $1 \in S$, and if $x \in S$, then $x+1 \in S$ and $2x \in S$, where duplicates are deleted as they occur. This tree, which includes every positive integer, is represented by Figure 2 and Corollary 2.2. A third tree, containing all the rational numbers, is given by the rules $1 \in S$, and if $x \in S$, then $x+1 \in S$ and $-1/x \in S$; a fourth tree, containing all the Gaussian rational numbers is given by the rules $1 \in S$, and if $x \in S$, then $x+1 \in S$ and $ix \in S$ and if $x \in S$ and $x \neq 0$, then $1/x \in S$.

The purpose of this paper is to discuss those four trees and others. Certain notations will be helpful; e.g., $a, b, c, d, e, f, g, h, k, m, n, r, s, t, u, v$ will denote integers, although f and g will also be used for functions. In particular, suppose that $f_1(x) = (ax+b)/(cx+d)$ and $f_2(x) = (ex+f)/(gx+h)$. For any initial x_0, we have a set S defined by the rules $x_0 \in S$, and if $x \in S$, then $f_1(x) \in S$ and $f_2(x) \in S$, and we shall refer to S not only as a set, but also as a tree determined by the rules, with deletion of duplicates as they occur. The set (and tree) is partitioned into generations $g(n)$ defined inductively by $g(1) = \{x_0\}$ and

$$g(n) = \{f_1(x) : x \in g(n-1)\} \cup \{f_2(x) : x \in g(n-1)\} \setminus \bigcup_{i=1}^{n-1} g(i). \tag{1.1}$$

for $n \geq 2$. Note that the generations are, by definition, pairwise disjoint. We are interested in cases in which S includes every positive integer, or every positive rational number, etc. Also of interest are the sizes $|g(n)|$ of the generations, recurrence relations for $|g(n)|$ and related sequences, and limits.

2. Multinacci Trees

Let S be the set defined by these rules: $1 \in S$, and if $x \in S$, then $x+1 \in S$ and $mx \in S$. The generations are given by $g(1) = \{1\}$ and, following (1.1),

$$g(n) = \{x+1 : x \in g(n-1)\} \cup \{mx : x \in g(n-1)\} \setminus \bigcup_{i=1}^{n-1} g(i).$$

Theorem 2.1. If $n \geq m \geq 2$, then

$$|g(n)| = |g(n-1)| + \cdots + |g(n-m)|. \tag{2.1}$$

Proof: Each number i in $\{0, 1, \ldots, m-1\}$ is generated by x in $g(n-i-1)$. Summing on i gives

$$|g(n)| \leq |g(n-1)| + \cdots + |g(n-m)|. \tag{2.2}$$

In view of (2.2), to prove (2.1), we must show that the numbers $mx+i$ in $g(n)$ are distinct. As an induction hypothesis, suppose that the numbers in $\bigcup_{i=1}^{n-1} g(i)$ are distinct. Suppose further that two numbers u and v in $g(n)$ are equal. Clearly they cannot both be of the form mx for x in $g(n-1)$, nor both of the form $y+1$ for y in $g(n-1)$, so, write $u = mx = y+1$, where $\{x, y\} \subseteq g(n-1)$. Then $y = mx - 1 \in g(n-1)$, whence $y - 1 = mx - 2 \in g(n-2)$, and so on to $y - (m-1) = mx - m = m(x-1) \in g(n-m)$, so that $(y-m+1)/m = x - 1 \in g(n-m-1)$. Consequently, $(y+1)/m = x \in g(n-m)$, contrary to the induction hypothesis, since $x \in g(n-1)$ and $m > 1$. Therefore, (2.1) holds. $\square$

Corollary 2.2. If $m = 2$, then $|g(n)| = F(n)$, the nth Fibonacci number.

This corollary is simply a special case of Theorem 2.1, and the proof of the theorem shows more: that $g(n)$ consists of $F(n-1)$ even numbers and $F(n-2)$ odd numbers. See Figure 2 in Section 7.

Corollary 2.3. Let $x(n,i)$ be the subset of $g(n)$ consisting of numbers $\equiv i \bmod m$. Then for each i, $|x(n,i)|$ satisfies the recurrence $|x(n,i)| = |x(n-1,i)| + \cdots + |x(n-m,i)|$.

A proof of this corollary is essentially given by the proof of Theorem 2.1.

3. More Fibonacci-Related Trees

Corollary 2.2 describes a tree of integers satisfying $|g(n)| = F(n)$; in this section, we consider other trees of fractions whose generations have sizes that are Fibonacci numbers. We begin with a lemma.

Lemma 3.1. Suppose that $m \geq 1$. The greatest k for which $k^2 + 4km$ is a square is $(m-1)^2$.

Proof: If $k = (m-1)^2$, then $k^2 + 4km = (m-1)^4 + 4(m-1)^2 m = (m^2-1)^2$. Now suppose that $k > (m-1)^2$. Then

$$(k + 2m - 2)^2 < k^2 + 4km < (k + 2m)^2,$$

so that if $k^2 + 4km$ is a square, then $k^2 + 4km = (k + 2m - 1)^2$. However, this implies $2k = (2m-1)^2$, contrary to the fact that $2m-1$ is odd. $\square$

Theorem 3.2. Suppose that k is a positive integer. Let S be the set defined by these rules: $1 \in S$, and if $x \in S$, then $x+k \in S$ and $k/x \in S$. Partition S into generations $g(n)$ inductively: $g(1) = \{1\}$, and for $n \geq 2$,

$$g(n) = \{x + k : x \in g(n-1)\} \cup \{k/x : x \in g(n-1)\} \setminus \bigcup_{i=1}^{n-1} g(i).$$

If $k = 1$, then $|g(n)| = F(n)$ for $n \geq 1$, and if $k > 1$, then $|g(n)| = F(n+1)$ for $n \geq 1$.

Proof: First, suppose that $k = 1$. Clearly $|g(n)| = F(n)$ for $n \leq 2$. Assume for arbitrary $n \geq 2$ that $g(n)$ consists of $F(n-1)$ numbers $x > 1$ together with $F(n-2)$ numbers $x \leq 1$. Each of the former spawns $x+1$ and $1/x$ in $g(n+1)$, and each of the others spawns the single number $x+1$ in $g(n+1)$. These numbers are distinct because the equation $x+1 = 1/x$ has no integer solution. Therefore, $g(n+1)$ consists of $2F(n-1)+F(n-2) = F(n+1)$ numbers.

Next, suppose that $k > 1$. Clearly $g(n) = F(n)$ for $n \leq 2$. Assume for arbitrary $n \geq 2$ that $g(n)$ consists of $F(n)$ numbers $x > k$ together with $F(n-1)$ numbers $x \leq k$. Each of the former spawns $x+k$ and k/x in $g(n+1)$, and each of the others spawns the single number $x+k$ in $g(n+1)$. To confirm that these numbers are distinct, suppose that $x+k = k/x$ for some x. Then $x^2 + kx - k = 0$, so that $k^2 + 4k$ must be a square, contrary to Lemma 1. Therefore, $g(n+1)$ consists of $2F(n) + F(n-1) = F(n+2)$ numbers. $\square$

Consider the rule "if $x \in S$, then $x + k \in S$" in the statement of Theorem 3.2. If this rule is changed to "if $x \in S$, then $x+1 \in S$" and the other rule remains "if $x \in S$, then $k/x \in S$", then the resulting tree, for $k > 1$, has generation sizes $|g(n)|$ which form a sequence not closely related to the Fibonacci sequence; indeed, the sequence appears to be not linearly recurrent. Nevertheless, the tree S contains every positive rational number, in accord with the following theorem.

Theorem 3.3. Suppose that k is a positive integer. Let S be the set defined by these rules: $1 \in S$, and if $x \in S$, then $x+1 \in S$ and $k/x \in S$. Then S is the set of positive rational numbers.

Proof: Clearly, every positive rational $b/1 \in S$. For arbitrary $d \geq 1$, assume that if u/v is a reduced positive rational with $v \leq d$, then $u/v \in S$. Suppose that $b/(d+1)$ is a reduced positive rational. As a first case, suppose that $b \leq d$. By the induction hypothesis, $(d+1)/b \in S$ and, by the same hypothesis, the number $x = k(d+1)/b \in S$. Consequently, $k/x \in S$; i.e., $b/(d+1) \in S$. To cover all remaining cases, suppose that $b > d+1$, so that $b = (d+1)q + r$, where $0 \leq r < d+1$. Then $b/(d+1) = q + r/(d+1)$. As in the first case, $r/(d+1) \in S$. Now q applications of $x \to x+1$ show that $b/(d+1) \in S$. $\square$

Example 3.4 Taking $k = 1$ in Theorem 3.2 and Theorem 3.3 gives the infinite Fibonacci tree represented by Figure 1. In the following array, row n shows the numbers in generation $g(n)$ arranged in decreasing order:

1												
2												
3	1/2											
4	3/2	1/3										
5	5/2	4/3	2/3	1/4								
6	7/2	7/3	5/3	5/4	3/4	2/5	1/5					
7	9/2	10/3	8/3	9/4	7/4	7/5	6/5	4/5	3/5	3/7	2/7	1/6

Note that the $F(n)$ numbers in row $n \geq 3$, taken in order, consist of $F(n-1)$ numbers $x+1$ from x in row $n-1$, followed by $F(n-2)$ numbers $1/(x+1)$ from x in row $n-2$.

Not every tree having $|g(n)| = F(n)$ for $n \geq 1$ is given by Corollary 2.2 and Example 3.4, as indicated by the following example.

Example 3.5. Let S be the tree defined by these rules: $1 \in S$, and if $x \in S$, then $1/x \in S$ and $1/(x+1) \in S$. Inductively, for $n \geq 2$, $g(n)$ consists of $F(n-2)$ numbers ≥ 1, each of the form $x+1$ for x in $g(n-2)$, together with $F(n-1)$ numbers < 1, each of the form $1/(x+1)$ for x in $g(n-1)$. Hence, $|g(n)| = F(n)$. It is easy to prove by induction that every fraction u/v in S is reduced to lowest terms and that if $v = 1$, then u is the only integer in $g(2u-1)$. Next, assume for arbitrary $v \geq 1$ that every fraction a/b with $b \leq v$ is in S, and suppose that $u/(v+1)$ is a fraction. If $u < v+1$, then by the induction hypothesis, $(v+1)/u \in S$, so that the rule $x \to 1/x$ applies, and $u/(v+1) \in S$. On the other hand, if $u > v+1$, write $u = (v+1)q + r$ with $0 \leq r < v+1$, so that $(v+1)/r \in S$. Then $r/(v+1) \in S$. Let $g(n)$ be the generation containing $r/(v+1)$. Then $u/(v+1) = r/(v+1) + q \in g(n+2q)$. Therefore, S contains every positive rational number.

Example 3.6. We have already seen examples of trees in which all the positive rational numbers occur. Consider next the tree S_1 given by the rules $1 \in S_1$, and if $x \in S_1$, then $x+4 \in S_1$ and $12/x \in S_1$. It is easy to see that the numbers 2 and 3 are missing from S_1. Starting another tree, S_3, with 3 and the same iterative membership requirements leads to a tree that includes 1 (in $g(5)$) and hence contains S_1 as a subtree, as in Figure 3. Regarding S_3, we observe that all positive integers not congruent to 2 mod 4 occur, that $|g(n)| = F(n+1)$ for $n \geq 1$, and that all fractions, as generated, are in reduced form. Since 2 is missing, it is natural to examine the tree S_2 having 2 as root, where, again, if $x \in S_2$, then $x+4 \in S_2$ and $12/x \in S_2$, as in Figure 4. The method of proof for Example 3.5 can be used to prove that $S_2 \cup S_3$ includes every positive rational number.

Example 3.7. Let S be the tree generated by these rules: $1 \in S$, and if $x \in S$, then $2x \in S$ and $1-x \in S$. To see that every integer h is in S, note first that this holds for $|h| \leq 2$, and assume for arbitrary $h \geq 2$ that if $|m| < h$, then $m \in S$. Now suppose that m satisfies $|m| = h > 2$. If m is even, write $m = 2k$, so that $k = m/2$, whence $|k| < |m| = h$, so that $k \in S$, whence $m \in S$. On the other hand, if $m = 2k+1$, then $k = (m-1)/2$, whence $|k| < |m| = h$, so that $-k \in S$; therefore $-2k \in S$, so that $1-(-2k)$, which is m, is in S. Thus, S contains every integer. Moreover, $|g(n)| = F(n)$ for $n \geq 3$. Conjecture: every generation $g(n)$ contains $\pm F$ for some Fibonacci number F.

Example 3.8. Let S be the tree generated by these rules: $1 \in S$, and if $x \in S$ then $1 + 1/x \in S$ and $1/x \in S$. An easy induction argument shows that in arbitrary $g(n)$, for $n > 2$, each node x greater than 1 begets a new node in $(0,1)$ and a new node in $(1,\infty)$, and each node x less than 1 begets a single new node in $(1,\infty)$. Thus, $g(n)$ consists of $F(n-1)$ nodes in $(1,\infty)$ and $F(n-2)$ nodes in $(0,1)$, leading to $|g(n)| = F(n)$ for $n \geq 1$. To see that every positive rational number is in S, the following lemma is useful: if $x \in S$ and $x > 1$ then $x-1 \in S$; to prove this, write $x = 1 + 1/u$, $u \in S$; then $x - 1 = 1/u$, which is in S. Clearly every $b/1$ is in S; suppose that b/d is an arbitrary fraction in reduced terms, with $d > 1$. By the lemma, we may assume that $b < d$, so that by induction hypothesis, $d/b \in S$. Consequently, $b/d \in S$. A final observation is that $F(n+1)/F(n) \in g(n)$, and that the numerator and denominator of this fraction are maximal for fractions in $g(n)$.

Example 3.9. Let S be the tree generated by these rules: $1 \in S$, and if $x \in S$, then $x/(x+1) \in S$ and $1/x \in S$. For every n, the set $g(n)$ has $F(n-1)$ numbers < 1 and $F(n-1)$ numbers ≥ 1, so that $|g(n)| = F(n)$ for all n. An induction proof on the size of denominators establishes that S contains every positive rational number. Another way to obtain this tree is to apply the reciprocation mapping $x \to 1/x$ to each node in the tree at Example 3.4.

4. All the Rational Numbers

Previous examples include trees which contain every positive integer, or every integer, or every positive rational number. We turn now to trees which contain every rational number.

Example 4.1. Decree that $0 \in S$ and that if $x \in S$, then $x+1 \in S$ and if $x+1 \neq 0$ then $-1/(x+1) \in S$. Then $g(1) = \{0\}$ and for all other generations, $g(n+1)$ consists of $F(n)$ negative numbers and $F(n)$ positive numbers, so that $|g(n+1)| = 2F(n)$. A proof that S contains every rational number depends on the method for Example 3.8: first, clearly every positive integer is in S; then inductively, every $1/n$ and $-n-1$ are in S, because $1/n = f_1(f_2(1/n))$ and $-n-1 = f_2(f_2(1/n))$, where $f_1(x) = x+1$ and $f_2(x) = -1/(x+1)$. The rest of the proof follows by induction on the size of denominators, together with reciprocation and the fact that if $x \in S$, then $x - 1 = f_2(f_2(f_1(f_2(f_2(x))))) \in S$. Every negative integer is a terminal node in S. The $F(n)$ positive numbers in $g(n+1)$ consist of $F(n-1)$ numbers $x+1$ from x in $g(n)$, together with $F(n-2)$ numbers $x/(x+1)$ from x in $g(n-1)$; the $F(n)$ negative numbers in $g(n+1)$ are the negative reciprocals of the positive numbers in $g(n+1)$.

Example 4.2. Let S be the tree generated by these rules: $1 \in S$, and if $x \in S$ then $x+1 \in S$, and if $x \in S$ and $x \neq 0$, then $-1/x \in S$. A proof that S contains every rational number is similar to the proof for Example 4.1; here, the corresponding lemma, that if $x \in S$ then $x-1 \in S$, stems from the fact that if $f_1(x) = x+1$ and $f_2(x) = -1/x$, then $x - 1 = f_2(f_1(f_2(f_1(f_2(x)))))$. For $n \geq 1$, let $S(n,i)$ be the set of nodes in $g(n)$ that have i offspring in $\bigcup_{h=1}^{n-1} g(h)$; e.g., $S(n,0)$ counts terminal nodes, and $S(n,2)$ counts nodes that beget 2 new nodes. The sequence $(S(n,i))$ satisfies the recurrence $a(n) = a(n-1) + a(n-3)$ for $n \geq 7$, so that the sequence $(|g(n)|)$, starting with $(1, 2, 3, 3, 5, 7, 10, 15, 22, \ldots,$ satisfies $|g(n)| = |g(n-1)| + |g(n-3)|$ for $n \geq 7$.

Example 4.3. Here, we show another way to generate the set S of Example 4.2, but in a more general manner. Suppose that $m \geq 3$, and define $h_m(n) = \{n\}$ for $n = 1, 2, \ldots, m$ and

$$h_m(n) = \{x+1 : x \in h_m(n-1)\} \cup \{x/(x+1) : x \in h_m(n-m)\},\ S_m = \bigcup_{n=1}^{\infty} h_m(n)$$

for $n \geq 4$. The now familiar proof by "denominator induction" shows that S_m is the set of positive rational numbers, and clearly, $|h_m(n)| = |h_m(n-1)| + |h_m(n-3)|$ for $n \geq m+1$, with $|h_m(n)| = 1$ for $n \leq m$. To obtain the numbers in the set S of Example 4.2, let $g(1) = h_3(1) = \{1\}$, $g(2) = \{-1, 2\}$, $g(3) = \{-1/2, 0, 3\}$, and for $n \geq 4$, let $g(n)$ be the set of numbers in $h_3(n)$ together with $-1/x$ for each x in $h_3(n-1)$. The array having $g(n)$ as row n, consisting of all the rational numbers, has these first six rows:

$$\begin{array}{ccccccc}
1 \\
-1 & 2 \\
-1/2 & 0 & 3 \\
-1/3 & 1/2 & 4 \\
-2 & -1/4 & 2/3 & 3/2 & 5 \\
-3/2 & -2/3 & -1/5 & 3/4 & 5/3 & 5/2 & 6
\end{array}$$

Example 4.4. As a generalization of Example 4.2, suppose that $m \geq 2$ and that "$-1/x \in S$" is replaced by "$-m/x \in S$". Then S contains every rational number.

5. Limits

Suppose that S is given by these rules: $1 \in S$, and if $x \in S$, then

$$\frac{ax+b}{cx+d} \in S \quad \text{and} \quad \frac{ex+f}{gx+h} \in S.$$

All the previously mentioned trees are special cases of S; e.g., the infinite Fibonacci tree is given by $(a,b,c,d) = (1,1,0,1)$ and $(e,f,g,h) = (0,1,1,0)$. When S includes all the positive rationals, every convergent sequence of rationals can be identified with a sequence of nodes in S. If the nodes lie in a single path, their limit is of interest. In order to study such paths, call an edge of the form $x \to \dfrac{ax+b}{cx+d}$ an *up-edge,* denoted by U, and an edge of the form $x \to \dfrac{ex+f}{gx+h}$ a *down-edge,* denoted by D. An up-edge followed by a down-edge corresponds to

$$x \to \frac{ax+b}{cx+d} \to \frac{(ae+cf)x+be+df}{(ag+ch)x+bg+dh} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix}, \tag{5.1}$$

where

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} e & f \\ g & h \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and a down edge followed by an up-edge corresponds to

$$x \to \frac{ex+f}{gx+h} \to \frac{(ae+bg)x+af+bh}{(ce+dg)x+cf+dh} = \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix}, \tag{5.2}$$

where

$$\begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix}.$$

In (5.1) and (5.2), the matrix product notation has the usual meaning but also serves as a useful way to represent the indicated fraction. An infinite path of the form $UDUDUD\ldots$ is a *zigzag path.* Call nodes of the form (5.1) *upper nodes* and those of the form (5.2) *lower nodes.* We shall see that under suitable conditions, the upper nodes converge and the lower nodes converge. In order to state the conditions, let

$$\Delta = (ae - dh + cf - bg)^2 + 4(be + df)(ag + ch),$$

and deem S *regular* if $\Delta \neq 0$, $ag + ch \neq 0$ and $ce + dg \neq 0$, where a, b, c, d, e, f, g are all nonnegative. A first theorem about convergence along paths in S follows.

Theorem 5.1: Suppose that path p is a zigzag graph in a regular tree S and that the limits of the upper nodes and lower nodes on p exist. The limits are, respectively,

$$\frac{ae - dh + cf - bg + \sqrt{\Delta}}{2(ag + ch)} \quad \text{and} \quad \frac{ae - dh - cf + bg + \sqrt{\Delta}}{2(ce + dg)}. \tag{5.3}$$

Proof: We begin with upper nodes, for which the limit, if it exists, is given by iterating the mapping (5.1). Let p be a zigzag path $UDUDUD\ldots$. Then the node given by $x(UD)^n$ has the form

$$\begin{pmatrix} \alpha_n & \beta_n \\ \gamma_n & \delta_n \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^n \begin{pmatrix} x \\ 1 \end{pmatrix}, \tag{5.4}$$

so that

$$\begin{pmatrix} \alpha_{n+1} & \beta_{n+1} \\ \gamma_{n+1} & \delta_{n+1} \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^n \begin{pmatrix} x \\ 1 \end{pmatrix},$$

or equivalently,

$$\frac{\alpha_{n+1}x + \beta_{n+1}}{\gamma_{n+1}x + \delta_{n+1}} = \frac{(\alpha\alpha_n x + \beta\gamma_n)x + \alpha\beta_n + \beta\delta_n}{(\gamma\alpha_n x + \delta\gamma_n)x + \gamma\beta_n + \delta\delta_n}. \tag{5.5}$$

Let $u = \lim\limits_{n\to\infty} \alpha_n/\gamma_n$, $v = \lim\limits_{n\to\infty} \beta_n/\gamma_n$, $w = \lim\limits_{n\to\infty} \delta_n/\gamma_n$. Taking limits in (5.5) gives

$$\frac{ux + v}{x + w} = \frac{(\alpha u + \beta)x + \alpha v + \beta w}{(\gamma u + \delta)x + \gamma v + \delta w}.$$

Cross-multiplying, collecting coefficients of $x^2, x, 1$, and regarding x as an indeterminate, we find $u^2\gamma + u(\delta - \alpha) - \beta = 0$ and $v^2\gamma + vw(\delta - \alpha) - \beta w^2 = 0$, so that

$$u = v/w = (\alpha - \delta \pm \sqrt{\Delta})/2\gamma. \tag{5.6}$$

The hypothesis that a, b, c, d, e, f, g are all nonnegative forces u to be the greater of the two possibilities, that is,

$$u = \frac{v}{w} = \frac{\alpha - \delta + \sqrt{(\alpha - \delta)^2 + 4\beta\gamma}}{2\gamma}. \tag{5.7}$$

The limit is then simply u, since, from $uw = v$, we have $ux + uw = ux + v$, so that $(ux+v)/(x+w) = u$. Now substituting $\alpha = ea + fc$, $\beta = eb + fd$, $\gamma = ga + hc$, $\delta = gh + hd$ into (5.7) gives (5.3). A proof for lower nodes following the same steps finds a discriminant $(\alpha' - \delta')^2 + 4\beta'\gamma' = (\alpha - \delta)^2 + 4\beta\gamma$. Of course, by (5.2), the second limit in (5.3) is $(eu + f)/(gu + h)$. □

Limits for selected choices of a, b, c, d, e, f, g, h are shown below. The first two rows match the infinite Fibonacci tree (Example 3.4) and the Kepler tree of fractions (Section 1).

a, b, c, d	e, f, g, h	$(UD)^\infty$	$(DU)^\infty$
$1, 1, 0, 1$	$0, 1, 1, 0$	$(-1 + \sqrt{5})/2$	$(1 + \sqrt{5})/2$
$1, 1, 0, 1$	$0, 1, 1, 1$	$-1 + \sqrt{2}$	$\sqrt{2}$
$1, -1, 0, 1$	$1, 0, 1, 5$	$(-3 + \sqrt{5})/2$	$(-5 + \sqrt{5})/2$
$1, -1, 0, 1$	$1, 0, 1, 6$	$-2 + \sqrt{3}$	$-3 + \sqrt{3}$
$2, 1, 0, 1$	$0, 1, 1, 0$	$1/2$	2
$3, 1, 0, 1$	$0, 1, 1, 0$	$(-1 + \sqrt{13})/6$	$(1 + \sqrt{13})/2$
$1, 2, 1, 1$	$0, 1, 1, 0$	$(-1 + \sqrt{5})/2$	$(1 + \sqrt{5})/2$
$1, 3, 1, 1$	$0, 1, 1, 0$	$-1 + \sqrt{2}$	$1 + \sqrt{2}$
$1, 3, 2, 1$	$0, 1, 1, 0$	$(-1 + \sqrt{5})/2$	$(1 + \sqrt{5})/2$
$2, 5, 3, 1$	$0, 1, 1, 0$	$(-1 + \sqrt{3})/2$	$1 + \sqrt{3}$

Limits other than those indicated by $(UD)^\infty$ and $(DU)^\infty$ are also of interest. Consider an infinite path p of the form $U^{k_1}DU^{k_2}D \cdots U^{k_m}D \cdots$ in the infinite Fibonacci tree. Clearly, the nodes of p converge if and only if the sequence (k_i) is bounded. Assuming (k_i) bounded, we now study limits along periodic paths—where a period is a finite branch of the form $B = U^{k_1}DU^{k_2}D \cdots U^{k_m}D$, and the periodic path is the infinite concatenation $BBB \cdots$, denoted by B^∞.

Theorem 5.2. Suppose that $f(x) = (\alpha x + \beta)/(\gamma x + \delta)$, where $\alpha \neq 0$, $\gamma \neq 0$, and $(\alpha - \delta)^2 + 4\beta\gamma > 0$. Define $f_1(x) = f(x)$ and $(f_n(x) = f(f_{n-1}(x))$ for $n \geq 2$. Then $\lim_{n\to\infty} f_n(x) = (v - \delta)/\gamma$, where v is an eigenvalue of $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$.

Proof: Define $(\alpha_n, \beta_n, \gamma_n, \delta_n)$ as in (5.4), so that (7) and (5.6) hold, which is to say that the number $u = \lim_{n\to\infty} f_n(x)$ is one of the two numbers

$$u_1 = \frac{\alpha - \delta + \sqrt{(\alpha-\delta)^2 + 4\beta\gamma}}{2\gamma}, \quad u_2 = \frac{\alpha - \delta - \sqrt{(\alpha-\delta)^2 + 4\beta\gamma}}{2\gamma}.$$

The eigenvalues of $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ are

$$v_1 = \frac{\alpha + \delta + \sqrt{(\alpha-\delta)^2 - 4(\alpha\delta - \beta\gamma)}}{2\gamma}, \quad v_2 = \frac{\alpha + \delta - \sqrt{(\alpha-\delta)^2 - 4(\alpha\delta - \beta\gamma)}}{2\gamma}.$$

Thus, if $u = u_1$, then $u = (v_1 - \delta)/\gamma$, and if $u = u_2$, then $u = (v_2 - \delta)/\gamma$. $\square$

Returning to any suitable choice of a, b, c, d, e, f, g, h, let $U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $D = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$; that is, $f_1(x) = (ax + b)/(cx + d)$ and $f_2(x) = (ex + f)/(gx + h)$. Then

$$f_1^{k_1} f_2 f_1^{k_2} f_2 \cdots f_1^{k_m} f_2(x) = (\alpha x + \beta)/(\gamma x + \delta), \tag{5.8}$$

where

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = B = U^{k_1} D U^{k_2} D \cdots U^{k_m} D, \tag{5.9}$$

Therefore, Theorem 5.2 applies, so that the limit along B^∞, starting at any node x in S, is $(v - \delta)/\gamma$, where v is an eigenvalue of B. Next, we show the connection between such a limit and its continued fraction as determined by B.

Corollary 5.3. Let S be the infinite Fibonacci tree given by $(a, b, c, d, e, f, g, h) = (1, 1, 0, 1, 0, 1, 1, 0)$. Let B^∞ be the infinite path formed by concatenating the finite path $B = U^{k_1} D U^{k_2} D \cdots U^{k_m} D$, represented by the matrix in (5.9), and let $u = \lim_{n\to\infty} f_n(x)$. Then $u = [0, \overline{k_m, k_{m-1}, \ldots, k_1}]$.

Proof: The assertion follows from the fact that left multiplication $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^n$ matches attaching $[k_m, k_{m-1}, \ldots, k_1]$ at the end of the continued fraction consisting of n copies of $[k_m, k_{m-1}, \ldots, k_1]$.

Example 5.4. In the infinite Fibonacci tree of Corollary 5.3, let $B = UUDUUUDUD$, so that $(k_1, k_2, k_3) = (2, 3, 1)$. By Corollary 5.3, $\lim_{n\to\infty} f_n(x) = [0, \overline{1, 3, 2}] = (-3 + \sqrt{37})/4$.

6. Gaussian Fractions

In this section, the set (or tree) S is given by these rules: $1 \in S$, and if $x \in S$, then $x + 1 \in S$ and $ix \in S$, and if $x \neq 0$, then $1/x \in S$. We shall prove that S contains every Gaussian rational number; that is, every number $(a + bi)/(c + di)$, where $c^2 + d^2 > 0$.

Lemma 6.1. Suppose that b, c, d are integers. If any one of the numbers $(b + ci)/d$, $(bi - c)/d$, $(-b - ci)/d$, $(bi + c)/d$ is in S, then the other three are also in S.

Proof: Iterating the rule that if $x \in S$, then $ix \in S$ shows that ix, $-x$, and $-ix$ are in S. $\square$

Lemma 6.2. If $x \in S$, then $x - 1 \in S$.

Proof: If $x \in S$, then by Lemma 6.1, $-x \in S$. Consequently, $-x + 1 \in S$, so that $x - 1 \in S$ by Lemma 6.1. $\square$

Lemma 6.3. If $x \in S$ and $a + bi$ is a Gaussian integer, then $x + a + bi \in S$.

Proof: Suppose that $x \in S$ and that a is a real integer. If $a > 0$, then $x \to x + 1 \to x + 2 \to \cdots \to x + a$ are in S; if $a < 0$, then $x \to x - 1 \to x - 2 \to \cdots \to x - a$ are in S, by Lemma 6.2. So, we have $x + a$ in S for every integer a. Suppose now that b is an integer. Then $-i(x + a) \in S$, by Lemma 6.1, whence $-ix - ia + b \in S$, by Lemma 6.3. Then $i(-ix - ia + b) \in S$, which is to say that $x + a + bi \in S$. $\square$

Theorem 6.4. Every Gaussian rational number $(a + bi)/(c + di)$ is in S.

Proof: $1 \in S$, so that $-1 \in S$ by Lemma 6.1. Then $0 \in S$, by rule 1, whence $a + bi \in S$ for every Gaussian integer $a + bi$. Now suppose that w/z is an arbitrary Gaussian rational, where w and z are Gaussian integers and $|z|$ is least possible. If $|z| = 1$, then w is a Gaussian integer, so that $w/z \in S$. Assume then, that $w/z \notin S$. Then there is a least integer $\delta > 1$ for which there is a Gaussian rational w'/z' such that $|z'| = \delta$ and $w'/z' \notin S$. We may and do assume that $w' = w$ and $z' = z$. By the division algorithm [2], there exist Gaussian integers q and r such that $w = qz + r$, where $|r| < |z|$. Then $w/z = q + r/z$. If $r/z \in S$, then $w/z \in S$, by Lemma 6.3, a contradiction. On the other hand, if $r/z \notin S$, then $z/r \in S$ since $|r| < |z|$. But then $r/z \in S$, another contradiction. Therefore, $w/z \in S$. $\square$

We conclude this section with a tree of Gaussian integers.

Example 6.5. Let S be the tree generated by these rules: $0 \in S$, and if $x \in S$ then $x + 1 \in S$ and $ix \in S$. Iterating the mapping $x \to x + 1$ shows that every positive integer n is in S. Then in, $-n$, and $-in$ are in S, so that $0 = -1 + 1 \in S$, and $b'i + 1, b'i + 2, b'i + 3, ...$ are in S for every integer b'. For each of these numbers $b'i + c$, the number $-b'i - c$ is in S, so that, in conclusion, S includes every Gaussian integer $a + bi$. For $n \geq 1$, let $S(n, i)$ be the set of nodes in $g(n)$ that have i offspring in $\bigcup_{h=1}^{n-1} g(h)$; e.g., $S(n, 0)$ counts terminal nodes, and $S(n, 2)$ counts nodes that beget 2 new nodes. We conjecture that $S(n, 0) = n - 5$ for $n \geq 5$, that $S(n, 1) = 2n - 7$ for $n \geq 4$, and that $S(n, 2) = n - 1$ for $n \geq 1$, so that, if the conjectures are true, then the sequence $(g(n))$, starting with $1, 1, 2, 4, 7, 11, 15, 19, 23, \ldots$, satisfies $|g(n)| = 4n - 13$ for $n \geq 5$.

7. Figures

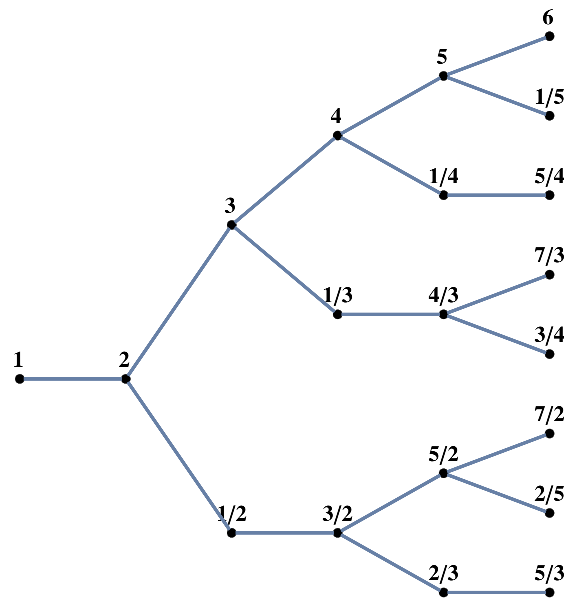


FIGURE 1. $x \to x + 1, x \to 1/x$; Example 3.4

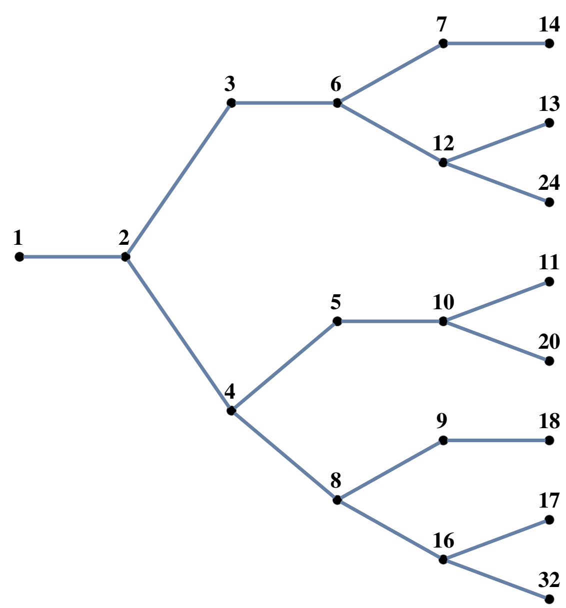


FIGURE 2. $x \to x + 1, x \to 2x$; Theorem 2.1

8. Concluding Remarks

In many of the foregoing trees of rational numbers, the numbers in $g(n)$ occur as "already reduced" fractions This observation leads to the question of conditions on a, b, c, d under which

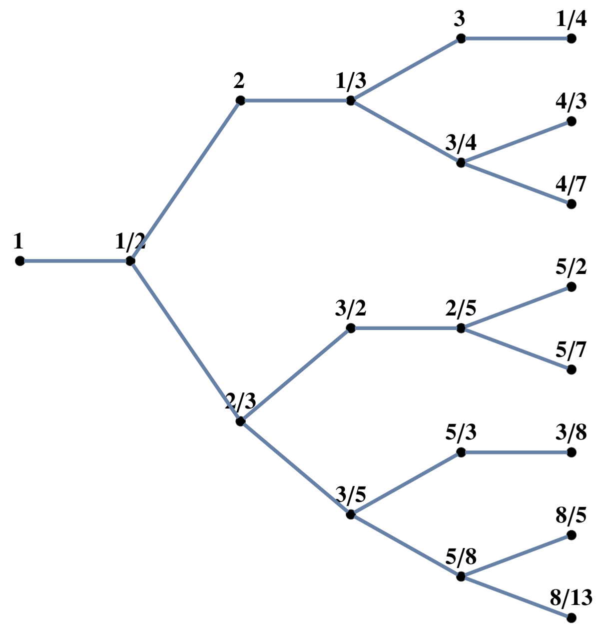


FIGURE 3. $x \to 1/x, x \to 1/(x+1)$; Example 3.5

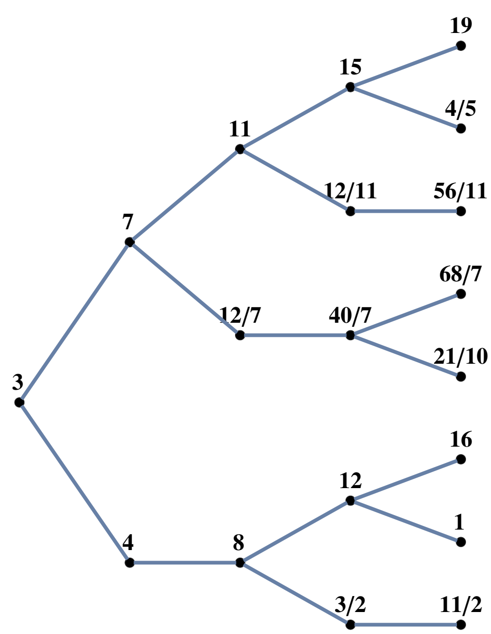


FIGURE 4. $x \to x+4, x \to 12/x$, from 3; Example 3.6

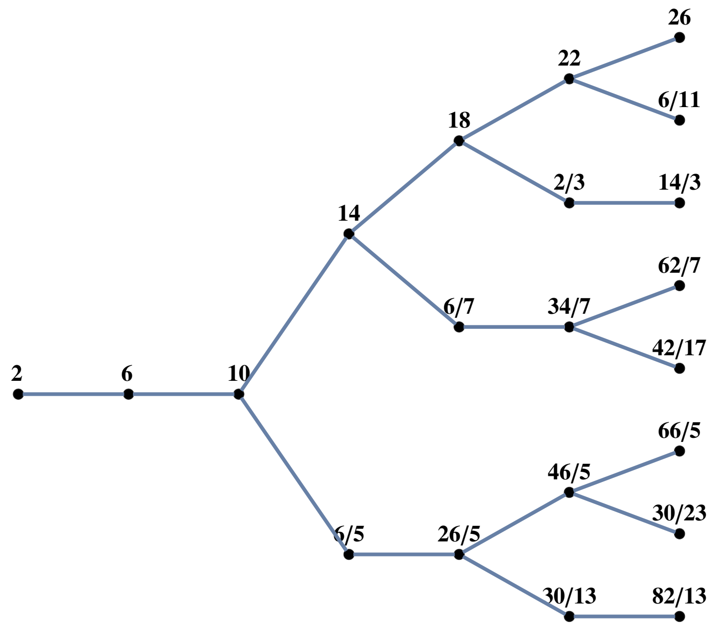


FIGURE 5. $x \to x + 4, x \to 12/x$; from 2; Example 3.6

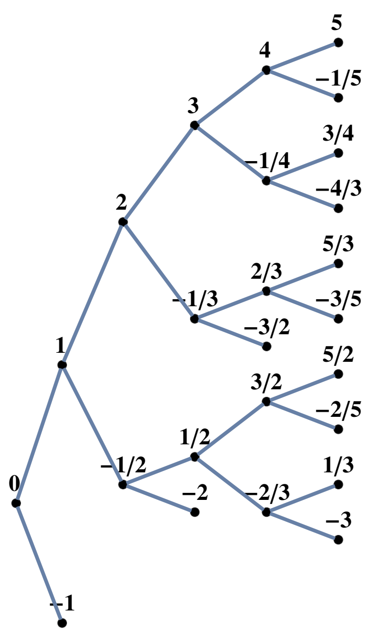


FIGURE 6. $x \to x + 1, x \to -1/(x + 1)$; Example 4.1

the numbers in $g(n)$ given by (1.1) are reduced fractions; i.e., for each $x = u/v$ in $g(n-1)$, the integers $au + bv$ and $cu + dv$ in the fraction $(au + bv)/(cu + dv)$ are relatively prime. It is easy to prove that one such condition (which holds for many of the trees considered in this paper) is that

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \pm 1.$$

Examples in which the fractions are not automatically reduced are given by Theorem 3.3 with $k > 1$. A number of the trees are represented in the Online Encyclopedia of Integer Sequences [3]; for Kepler's tree, see A020651, and for a list of others, see the Comments section at A226080. We conclude with three representative Mathematica (version ≥ 7) programs which may be useful for further research.

Program 1. All the positive rational numbers, generated as in Figure 1, Example 3.4, and A226080

```
z=10;g[1]={1};g[2]={2};g[3]={3,1/2};
d[s_,t_]:=Part[s,Sort[Flatten[Map[Position[s,#]&,Complement[s,t]]]]];
n=3;While[n<=z,n++;g[n]=d[Riffle[g[n-1]+1,1/g[n-1]],g[n-2]]];
Table[g[n],{n,z}]
```

Program 2. All the rational numbers, generated as in Example 4.1, with a ListPlot of the 20th generation

```
g[1]= {0};f1[x_]:=x+1;f2[x_]:=-1/(x+1);h[1]=g[1];
b[n_]:=b[n]=Union[f1[g[n-1]],f2[g[n-1]]];
h[n_]:=h[n]=Union[h[n-1],g[n-1]];
g[n_]:=g[n]=Complement[b[n],Intersection[b[n],h[n]]]
Table[g[n], {n,12}]
ListPlot[g[20]]
```

Program 3. All the Gaussian rationals, generated as in Theorem 6.4, with positions of real integers

```
Off[Power::infy];x= {0};
Do[x=DeleteDuplicates[
    Flatten[Transpose[{x,x+1,1/x,I*x}/.ComplexInfinity-> 0]]
                    ], {6}];x
On[Power::infy];
t1=Flatten[Position[x, _?(IntegerQ[#] && NonNegative[#]&)]] (*A233694*)
t2=Flatten[Position[x, _?(IntegerQ[#] && Negative[#]&)]] (*A233695*)
Union[t1,t2] (*A233696*)
```

References

[1] Johannes Kepler, [excerpt from Book III of *Harmonices Mundi*, 1619, translated with an Introduction and Notes by E. J. Aiton, A. M. Duncan, and J. V. Field, *Memoirs of the American Philosophical Society*, 209 (1997), p. 163 (in Book III, Chapter II, "On the Harmonic Division of the String").

[2] William Judson LeVeque, *Topics in Number Theory,* v. 1, Addison Wesley, 1956, 130-131.

[3] Online Encyclopedia of Integer Sequences, **https://oeis.org/**

Department of Mathematics, University of Evansville, 1800 Lincoln Avenue, Evansville, Indiana 47722, USA.
E-mail address: ck6@evansville.edu

Engineering Division, Moparmatic Co., 1154 Evesham Road, Astwood Bank, Nr. Redditch, Worcestershire, B96 6DT. UK.
E-mail address: mows@mopar.freeserve.co.uk

HIGHER-ORDER IDENTITIES FOR FIBONACCI NUMBERS

TAKAO KOMATSU, ZUZANA MASÁKOVÁ, AND EDITA PELANTOVÁ

Abstract. Let F_n be the n-th Fibonacci number. In this paper, we give some explicit expressions of $\sum_{l=0}^{2r-3}\binom{2r-3}{l}\sum_{\substack{j_1+\cdots+j_r=n-2l\\ j_1,\dots,j_r\ge 1}} F_{j_1}\cdots F_{j_r}$ as well as $\sum_{\substack{j_1+\cdots+j_r=n\\ j_1,\dots,j_r\ge 1}} F_{j_1}\cdots F_{j_r}$.

1. Introduction

It is known that the generating function $f(x)$ of Fibonacci numbers F_n is given by

$$f(x) := \frac{x}{1-x-x^2} = \sum_{n=0}^\infty F_n x^n\,.$$

Then $f(x)$ satisfies the relation

$$f(x)^2 = \frac{x^2}{1+x^2} f'(x) \tag{1.1}$$

or

$$(1+x^2)f(x)^2 = x^2 f'(x)\,. \tag{1.2}$$

The left-hand side of (1.2) is

$$\begin{aligned}
&(1+x^2)\left(\sum_{n=0}^\infty F_n x^n\right)\left(\sum_{m=0}^\infty F_m x^m\right)\\
&= (1+x^2)\sum_{n=0}^\infty\sum_{j=0}^n F_j F_{n-j} x^n\\
&= \sum_{n=0}^\infty\sum_{j=0}^n F_j F_{n-j} x^n + \sum_{n=2}^\infty\sum_{j=0}^{n-2} F_j F_{n-j-2} x^n\,.
\end{aligned}$$

The right-hand side of (1.2) is

$$x^2\left(\sum_{n=1}^\infty nF_n x^{n-1}\right) = \sum_{n=1}^\infty (n-1)F_{n-1}x^n\,.$$

Research supported in part by the grant of Wuhan University and Hubei Provincial 100 Talents Program.

Comparing the coefficients of both sides, we get

$$\begin{aligned}(n-1)F_{n-1} &= \sum_{j=0}^{n} F_j F_{n-j} + \sum_{j=0}^{n-2} F_j F_{n-j-2} \\ &= \sum_{j=1}^{n-1} F_j F_{n-j} + \sum_{j=0}^{n-2} F_j F_{n-j-2} \\ &= \sum_{j=1}^{n-1} (F_j F_{n-j} + F_{j-1}F_{n-j-1}) \,.\end{aligned}$$

Hence, we get the identity which can be identical with $F_{m+n} = F_{m+1}F_n + F_m F_{n-1}$ (see e.g. [1, Lemma 5].

Theorem 1.1. *For $n \ge 1$, we have*

$$nF_n = \sum_{j=1}^{n} (F_j F_{n-j+1} + F_{j-1}F_{n-j}) \,.$$

Differentiating both sides of (1.1) by x and dividing them by 2, we obtain

$$f(x)f'(x) = \frac{x}{(1+x^2)^2} f'(x) + \frac{x^2}{2(1+x^2)} f''(x) \,. \tag{1.3}$$

By (1.1) and (1.3), we get

$$\begin{aligned} f(x)^3 &= \frac{x^2}{1+x^2} f(x)f'(x) \\ &= \frac{x^3}{(1+x^2)^3} f'(x) + \frac{x^4}{2(1+x^2)^2} f''(x) \end{aligned} \tag{1.4}$$

or

$$(1+x^2)^3 f(x)^3 = x^3 f'(x) + \frac{1}{2} x^4 (1+x^2) f''(x) \,. \tag{1.5}$$

The left-hand side of (1.5) is

$$\begin{aligned} &(1+3x^2+3x^4+x^6) \sum_{n=0}^{\infty} \sum_{\substack{j_1+j_2+j_3=n \\ j_1,j_2,j_3 \ge 0}} F_{j_1}F_{j_2}F_{j_3} x^n \\ &= \sum_{l=0}^{3} \sum_{n=2l}^{\infty} \binom{l}{3} \sum_{\substack{j_1+j_2+j_3=n-2l \\ j_1,j_2,j_3 \ge 1}} F_{j_1}F_{j_2}F_{j_3} x^n \,. \end{aligned}$$

The right-hand side of (1.5) is

$$\begin{aligned} &x^3 \sum_{n=1}^{\infty} nF_n x^{n-1} + \frac{x^4}{2} \sum_{n=2}^{\infty} n(n-1)F_n x^{n-2} + \frac{x^6}{2} \sum_{n=2}^{\infty} n(n-1)F_n x^{n-2} \\ &= \sum_{n=2}^{\infty} \frac{(n-1)(n-2)}{2} F_{n-2} x^n + \sum_{n=4}^{\infty} \frac{(n-4)(n-5)}{2} F_{n-4} x^n \,. \end{aligned}$$

Comparing the coefficients of both sides, we get the following.

Theorem 1.2. *For $n \geq 6$, we have*

$$\sum_{l=0}^{3} \binom{3}{l} \sum_{\substack{j_1+j_2+j_3=n-2l \\ j_1,j_2,j_3 \geq 1}} F_{j_1}F_{j_2}F_{j_3} = \binom{n-1}{2} F_{n-2} + \binom{n-4}{2} F_{n-4} \,.$$

In this paper, we give some explicit expressions of $\sum_{l=0}^{2r-3} \binom{2r-3}{l} \sum_{\substack{j_1+\cdots+j_r=n-2l \\ j_1,\dots,j_r \geq 1}} F_{j_1} \cdots F_{j_r}$ as well as $\sum_{\substack{j_1+\cdots+j_r=n \\ j_1,\dots,j_r \geq 1}} F_{j_1} \cdots F_{j_r}$.

2. Main result

In general, we can state the following.

Theorem 2.1. *Let $r \geq 2$. Then for $n \geq 3r-5$, we have*

$$\sum_{l=0}^{2r-3} \binom{2r-3}{l} \sum_{\substack{j_1+\cdots+j_r=n-2l \\ j_1,\dots,j_r \geq 1}} F_{j_1} \cdots F_{j_r}$$
$$= \sum_{k=1}^{r-1} \frac{n-2k-r+3}{r-1} \binom{n-2k+1}{r-k-1} \binom{n-k-2r+3}{k-1} F_{n-2k-r+3} \,.$$

Lemma 2.2. *For $r \geq 2$, we have*

$$f(x)^r = \frac{x^{2r-2} f^{(r-1)}(x)}{(r-1)!(1+x^2)^{r-1}} + \sum_{k=1}^{r-2} \frac{\sum_{j=0}^{k-1} (-1)^j \binom{k}{j} \binom{r-2}{k-j-1} x^{2r-k-2+2j}}{k(r-k-2)!(1+x^2)^{r+k-1}} f^{(r-k-1)}(x) \,. \qquad (2.1)$$

Proof. The proof is done by induction. It is trivial to see that the identity holds for $r = 2$. Suppose that the identity holds for some r. Differentiating both sides by x, we obtain

$$
\begin{aligned}
&rf(x)^{r-1}f'(x)\\
&= \frac{x^{2r-2}f^{(r)}(x)}{(r-1)!(1+x^2)^{r-1}} + \frac{(2r-2)x^{2r-3}f^{(r-1)}(x)}{(r-1)!(1+x^2)^r}\\
&\quad + \sum_{k=1}^{r-2} \frac{\sum_{j=0}^{k-1}(-1)^j\binom{k}{j}\binom{r-2}{k-j-1}x^{2r-k-2+2j}}{k(r-k-2)!(1+x^2)^{r+k-1}} f^{(r-k)}(x)\\
&\quad + \sum_{k=1}^{r-2} \frac{\sum_{j=0}^{k-1}(-1)^j(2r-k-2+2j)\binom{k}{j}\binom{r-2}{k-j-1}x^{2r-k-3+2j}}{k(r-k-2)!(1+x^2)^{r+k}} f^{(r-k-1)}(x)\\
&\quad - \sum_{k=1}^{r-2} \frac{\sum_{j=0}^{k-1}(-1)^j(3k-2j)\binom{k}{j}\binom{r-2}{k-j-1}x^{2r-k-1+2j}}{k(r-k-2)!(1+x^2)^{r+k}} f^{(r-k-1)}(x)\\
&= \frac{x^{2r-2}f^{(r)}(x)}{(r-1)!(1+x^2)^{r-1}} + \frac{2x^{2r-3}f^{(r-1)}(x)}{(r-2)!(1+x^2)^r}\\
&\quad + \sum_{k=1}^{r-2} \frac{\sum_{j=0}^{k-1}(-1)^j\binom{k}{j}\binom{r-2}{k-j-1}x^{2r-k-2+2j}}{k(r-k-2)!(1+x^2)^{r+k-1}} f^{(r-k)}(x)\\
&\quad + \sum_{k=2}^{r-1} \frac{\sum_{j=0}^{k-2}(-1)^j(2r-k-1+2j)\binom{k-1}{j}\binom{r-2}{k-j-2}x^{2r-k-2+2j}}{(k-1)(r-k-1)!(1+x^2)^{r+k-1}} f^{(r-k)}(x)\\
&\quad + \sum_{k=2}^{r-1} \frac{\sum_{j=1}^{k-1}(-1)^j(3k-2j-1)\binom{k-1}{j-1}\binom{r-2}{k-j-1}x^{2r-k-2+2j}}{(k-1)(r-k-1)!(1+x^2)^{r+k-1}} f^{(r-k)}(x)\\
&= \frac{x^{2r-2}f^{(r)}(x)}{(r-1)!(1+x^2)^{r-1}} + r\sum_{k=1}^{r-1} \frac{\sum_{j=0}^{k-1}(-1)^j\binom{k}{j}\binom{r-1}{k-j-1}x^{2r-k-2+2j}}{k(r-k-1)!(1+x^2)^{r+k-1}} f^{(r-k)}(x)\,.
\end{aligned}
$$

Here, we used the relations

$$\frac{2}{(r-2)!} + \frac{1}{(r-3)!} = \frac{r}{(r-2)!} \quad (k=1)$$

and

$$
\begin{aligned}
&\frac{r-k-1}{k}\binom{k}{j}\binom{r-2}{k-j-1} + \frac{2r-k-1+2j}{k-1}\binom{k-1}{j}\binom{r-2}{k-j-2}\\
&\qquad + \frac{3k-2j-1}{k-1}\binom{k-1}{j-1}\binom{r-2}{k-j-1}\\
&= \frac{r}{k}\binom{k}{j}\binom{r-1}{k-j-1} \quad (k \geq 2)\,.
\end{aligned}
$$

Together with (1.1), we get

$$\begin{aligned}f(x)^{r+1} &= \frac{x^2}{1+x^2} f(x)^{r-1} f'(x)\\ &= \frac{x^2}{1+x^2}\left(\frac{x^{2r-2}f^{(r)}(x)}{r!(1+x^2)^{r-1}} + \sum_{k=1}^{r-1}\frac{\sum_{j=0}^{k-1}(-1)^j\binom{k}{j}\binom{r-1}{k-j-1}x^{2r-k-2+2j}}{k(r-k-1)!(1+x^2)^{r+k-1}} f^{(r-k)}(x)\right)\\ &= \frac{x^{2r}f^{(r)}(x)}{r!(1+x^2)^r} + \sum_{k=1}^{r-1}\frac{\sum_{j=0}^{k-1}(-1)^j\binom{k}{j}\binom{r-1}{k-j-1}x^{2r-k+2j}}{k(r-k-1)!(1+x^2)^{r+k}} f^{(r-k)}(x)\,.\end{aligned}$$

□

Proof of Theorem 2.1. By Lemma 2.2 we get

$$\begin{aligned}(1+x^2)^{2r-3}f(x)^r &= (1+x^2)^{r-2}\frac{x^{2r-2}f^{(r-1)}(x)}{(r-1)!}\\ &\quad+\sum_{k=1}^{r-2}(1+x^2)^{r-k-2}\frac{\sum_{j=0}^{k-1}(-1)^j\binom{k}{j}\binom{r-2}{k-j-1}x^{2r-k-2+2j}}{k(r-k-2)!} f^{(r-k-1)}(x)\,. \qquad (2.2)\end{aligned}$$

Since $F_0 = 0$, the left-hand side of (2.2) is equal to

$$\begin{aligned}&(1+x^2)^{2r-3}\sum_{n=0}^{\infty}\sum_{\substack{j_1+\cdots+j_r=n\\ j_1,\dots,j_r\ge 0}} F_{j_1}\cdots F_{j_r}x^n\\ &=\sum_{l=0}^{2r-3}\sum_{n=2l}^{\infty}\binom{2r-3}{l}\sum_{\substack{j_1+\cdots+j_r=n-2l\\ j_1,\dots,j_r\ge 1}} F_{j_1}\cdots F_{j_r}x^n\,.\end{aligned}$$

On the other hand,

$$\begin{aligned}&(1+x^2)^{r-2}\frac{x^{2r-2}f^{(r-1)}(x)}{(r-1)!}\\ &=\sum_{i=0}^{r-2}\binom{r-2}{i}x^{2i}\frac{x^{2r-2}}{(r-1)!}\sum_{n=r-1}^{\infty}\frac{n!}{(n-r+1)!}F_n x^{n-r+1}\\ &=\frac{1}{(r-1)!}\sum_{i=0}^{r-2}\binom{r-2}{i}\sum_{n=2r+2i-2}^{\infty}\frac{(n-r-2i+1)!}{(n-2r-2i+2)!}F_{n-r-2i+1}x^n\,.\end{aligned}$$

For $i = r-2$, we have

$$\begin{aligned}&\frac{1}{(r-1)!}\sum_{n=4r-6}\frac{(n-3r+5)!}{(n-4r+6)!}F_{n-3r+5}x^n\\ &=\sum_{n=3r-5}\frac{n-3r+5}{r-1}\binom{n-3r+4}{r-2}F_{n-3r+5}x^n\,,\end{aligned}$$

which yields the term for $k = r-1$ on the right-hand side of the identity in Theorem 2.1. Notice that

$$\binom{\gamma'}{\gamma} = 0 \quad (\gamma' < \gamma)\,.$$

The second term of the right-hand side of (2.2) is

$$
\begin{aligned}
&\sum_{k=1}^{r-2}\sum_{i=0}^{r-k-2}\binom{r-k-2}{i}x^{2i}\frac{1}{k(r-k-2)!}\sum_{j=0}^{k-1}(-1)^j\binom{k}{j}\binom{r-2}{k-j-1}x^{2r-k-2+2j}\\
&\qquad\times\sum_{n=r-k-1}^{\infty}\frac{n!}{(n-r+k+1)!}F_n x^{n-r+k+1}\\
&=\sum_{i=0}^{r-3}\sum_{j=0}^{r-i-3}\sum_{k=j}^{r-i-3}\frac{1}{(k+1)(r-k-3)!(n-2r+k-2i-2j+3)!}\binom{r-k-3}{i}\\
&\qquad\times\binom{k+1}{j}\binom{r-2}{k-j}\sum_{n=2r+2i+2j-k-3}(-1)^j(n-r-2i-2j+1)!F_{n-r-2i-2j+1}x^n\\
&=\sum_{i=0}^{r-3}\sum_{\kappa=i+1}^{r-2}\sum_{k=\kappa-i-1}^{r-i-3}\frac{1}{(k+1)(r-k-3)!(n-2r+k-2\kappa+5)!}\binom{r-k-3}{i}\\
&\qquad\times\binom{k+1}{\kappa-i-1}\binom{r-2}{k-\kappa+i+1}\sum_{n=2r+2\kappa-k-5}(-1)^{\kappa-i-1}(n-r-2\kappa+3)!F_{n-r-2\kappa+3}x^n\,.
\end{aligned}
$$

Together with the first term of the right-hand side of (2.2) we can prove that

$$
\begin{aligned}
&\frac{1}{(r-1)!}\binom{r-2}{k-1}\frac{(n-r-2k+3)!}{(n-2r-2k+4)!}\\
&\quad+\sum_{i=0}^{k-1}\sum_{l=k-i-1}^{r-i-3}\frac{1}{(l+1)(r-l-3)!(n-2r+l-2k+5)!}\\
&\quad\times\binom{r-l-3}{i}\binom{l+1}{k-i-1}\binom{r-2}{l-k+i+1}(-1)^{k-i-1}(n-r-2k+3)!\\
&=\frac{n-2k-r+3}{r-1}\binom{n-2k+1}{r-k-1}\binom{n-k-2r+3}{k-1}\,. \qquad (2.3)
\end{aligned}
$$

Then the proof is done. ☐

3. Examples 1

When $r=2$ and $r=3$, Theorem 2.1 is reduced to Theorem 1.1 and Theorem 1.2, respectively. When $r=3,4,5$ in Theorem 2.1, we get the following Corollaries as examples.

Theorem 3.1. *For $n \ge 7$, we have*

$$
\begin{aligned}
&\sum_{l=0}^{5}\binom{5}{l}\sum_{\substack{j_1+j_2+j_3+j_4=n-2l\\ j_1,j_2,j_3,j_4\ge 1}}F_{j_1}F_{j_2}F_{j_3}F_{j_4}\\
&\qquad=\binom{n-1}{3}F_{n-3}+\frac{(n-3)(n-5)(n-7)}{3}F_{n-5}+\binom{n-7}{3}F_{n-7}\,.
\end{aligned}
$$

Theorem 3.2. *For $n \ge 10$, we have*

$$\sum_{l=0}^{7}\binom{7}{l}\sum_{\substack{j_1+j_2+j_3+j_4+j_5=n-2l\\ j_1,j_2,j_3,j_4,j_5\ge 1}} F_{j_1}F_{j_2}F_{j_3}F_{j_4}F_{j_5}$$
$$= \binom{n-1}{4}F_{n-4} + \frac{(n-3)(n-4)(n-6)(n-9)}{8}F_{n-6}$$
$$+ \frac{(n-5)(n-8)(n-10)(n-11)}{8}F_{n-6} + \binom{n-10}{4}F_{n-10}\,.$$

Theorem 3.3. *For $n \ge 13$, we have*

$$\sum_{l=0}^{9}\binom{9}{l}\sum_{\substack{j_1+\cdots+j_6=n-2l\\ j_1,\dots,j_6\ge 1}} F_{j_1}\cdots F_{j_6}$$
$$= \binom{n-1}{5}F_{n-5} + \frac{(n-3)(n-4)(n-5)(n-7)(n-11)}{30}F_{n-7}$$
$$+ \frac{(n-5)(n-6)(n-9)(n-12)(n-13)}{20}F_{n-9}$$
$$+ \frac{(n-7)(n-11)(n-13)(n-14)(n-15)}{30}F_{n-11} + \binom{n-13}{5}F_{n-13}\,.$$

4. Another result

In this section, we shall give an expression of $\sum_{\substack{j_1+\cdots+j_r\\ j_1,\dots,j_r\ge 1}} F_{j_1}\cdots F_{j_r}$.

The left-hand side of (1.1) is

$$\left(\sum_{n=0}^{\infty}F_n x^n\right)\left(\sum_{m=0}^{\infty}F_m x^m\right) = \sum_{n=0}^{\infty}\sum_{j=0}^{n}F_j F_{n-j}x^n\,.$$

The right-hand side of (1.1) is

$$x^2\left(\sum_{j=0}^{\infty}(-1)^j x^{2j}\right)\left(\sum_{m=1}^{\infty}mF_m x^{m-1}\right) = x^2\left(\sum_{j=0}^{\infty}\alpha_j x^j\right)\left(\sum_{m=0}^{\infty}(m+1)F_{m+1}x^m\right)$$
$$= x^2\sum_{n=0}^{\infty}\sum_{m=0}^{n}\alpha_{n-m}(m+1)F_{m+1}x^n$$
$$= \sum_{n=0}^{\infty}\sum_{m=0}^{n-2}\alpha_{n-m-2}(m+1)F_{m+1}x^n\,,$$

where $\alpha_j = \cos\frac{j\pi}{2}$ $(j \ge 0)$, satisfying $\{\alpha_j\}_{j\ge 0} = 1,0,-1,0,1,0,-1,0,1,0,-1,0,\dots$. Comparing the coefficients of both sides, we have the following.

Theorem 4.1. *For $n \ge 2$, we have*

$$\sum_{j=0}^{n} F_j F_{n-j} = \sum_{m=1}^{n-1} m F_m \cos\frac{(n-m-1)\pi}{2}\,. \tag{4.1}$$

In general, we have the following.

Theorem 4.2. *For $n \ge r \ge 2$, we have*

$$\sum_{\substack{j_1+\cdots+j_r=n\\ j_1,\dots,j_r\ge 1}} F_{j_1}\cdots F_{j_r}$$
$$= \frac{C_{r-2}}{(2r-4)!2^{2r-4}} \sum_{m=1}^{n-r+1} \frac{(n+m+r-3)!!(n-m+r-3)!!}{(n+m-r+1)!!(n-m-r+1)!!} m F_m \cos\frac{(n-m-r+1)\pi}{2}\,,$$

where C_n is the n-th Catalan number ([4, A000108]*) given by*

$$C_n = \frac{1}{n+1}\binom{2n}{n} \quad (n \ge 0)$$

and $n!! = n(n-2)(n-4)\cdots 1$ if n is odd; $n!! = n(n-2)(n-4)\cdots 2$ if n is even.

Proof. The left-hand side of (2.1) in Lemma 2.2 is equal to

$$\sum_{n=0}^{\infty} \sum_{\substack{j_1+\cdots+j_r=n\\ j_1,\dots,j_r\ge 1}} F_{j_1}\cdots F_{j_r} x^n\,.$$

The first term on the right-hand side of (2.1) in Lemma 2.2 is equal to

$$\begin{aligned}
&\frac{x^{2r-2}f^{(r-1)}(x)}{(r-1)!(1+x^2)^{r-1}}\\
&=\frac{x^{2r-2}}{(r-1)!}\sum_{i=0}^{\infty}\binom{i+r-2}{r-2}x^{2i}\sum_{m=0}^{\infty}\frac{(m+r-1)!}{m!}F_{m+r-1}x^m\\
&=\frac{x^{2r-2}}{(r-1)!}\sum_{k=0}^{\infty}\frac{1}{(r-2)!2^{r-2}}\frac{(k+2r-4)!!}{k!!}\cos\frac{k\pi}{2}x^k\sum_{m=0}^{\infty}\frac{(m+r-1)!}{m!}F_{m+r-1}x^m\\
&=\frac{x^{2r-2}}{(r-1)!}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{1}{(r-2)!2^{r-2}}\frac{(n-m+2r-4)!!}{(n-m)!!}\\
&\qquad\times\cos\frac{(n-m)\pi}{2}\frac{(m+r-1)!}{m!}F_{m+r-1}x^n\\
&=\frac{1}{(r-1)!(r-2)!2^{r-2}}\sum_{n=2r-2}^{\infty}\sum_{m=0}^{n-2r+2}\frac{(n-m-2)!!}{(n-m-2r+2)!!}\\
&\qquad\times\cos\frac{(n-m-2r+2)\pi}{2}\frac{(m+r-1)!}{m!}F_{m+r-1}x^n\\
&=\frac{1}{(r-1)!(r-2)!2^{r-2}}\sum_{n=2r-2}^{\infty}\sum_{m=r-1}^{n-r+1}\frac{(n-m+r-3)!!}{(n-m-r+1)!!}\\
&\qquad\times\cos\frac{(n-m-r+1)\pi}{2}\frac{m!}{(m-r+1)!}F_mx^n\,.
\end{aligned}$$

Concerning the second term, we have

$$\begin{aligned}
&\frac{\sum_{j=0}^{k-1}(-1)^j\binom{k}{j}\binom{r-2}{k-j-1}x^{2r-k-2+2j}}{(1+x^2)^{r+k-1}}f^{(r-k-1)}(x)\\
&=\sum_{j=0}^{k-1}(-1)^j\binom{k}{j}\binom{r-2}{k-j-1}x^{2r-k-2+2j}\sum_{i=0}^{\infty}(-1)^i\binom{i+r+k-2}{r+k-2}x^{2i}\\
&\qquad\times\sum_{m=0}^{\infty}\frac{(m+r-k-1)!}{m!}F_{m+r-k-1}x^m\\
&=\sum_{j=0}^{k-1}(-1)^j\binom{k}{j}\binom{r-2}{k-j-1}x^{2r-k-2+2j}\sum_{l=0}^{\infty}\frac{1}{(r+k-2)!2^{r+k-2}}\frac{(l+2r+2k-4)!!}{l!!}\\
&\qquad\times\cos\frac{l\pi}{2}x^k\sum_{m=0}^{\infty}\frac{(m+r-k-1)!}{m!}F_{m+r-k-1}x^m
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=0}^{k-1}(-1)^j\binom{k}{j}\binom{r-2}{k-j-1}x^{2r-k-2+2j}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{1}{(r+k-2)!2^{r+k-2}}\frac{(n-m+2r+2k-4)!!}{(n-m)!!}\\
&\qquad\times\cos\frac{(n-m)\pi}{2}\frac{(m+r-k-1)!}{m!}F_{m+r-k-1}x^n\\
&= \frac{1}{(r+k-2)!2^{r+k-2}}\sum_{j=0}^{k-1}(-1)^j\binom{k}{j}\binom{r-2}{k-j-1}\sum_{n=2r-k-2+2j}^{\infty}\sum_{m=0}^{n-2r+k+2-2j}\\
&\qquad\frac{(n-m+3k-2-2j)!!}{(n-m-2r+k+2-2j)!!}\cos\frac{(n-m-2r+k+2-2j)\pi}{2}\frac{(m+r-k-1)!}{m!}F_{m+r-k-1}x^n\,.
\end{aligned}
$$

Since

$$
\frac{(n-m+r+2k-3-2j)!!}{(n-m-r+k+3-2j)!!}=0\quad\text{if}\quad m=n-2r+k+2-2j\quad(j=1,2,\dots,k-2)
$$

and

$$
\cos\frac{(n-m-r+1-2j)\pi}{2}=0\quad\text{if}\quad m=n-2r+k+1-2j\quad(j=0,1,\dots,k-1)\,,
$$

this is equal to

$$
\begin{aligned}
&\frac{1}{(r+k-2)!2^{r+k-2}}\sum_{j=0}^{k-1}(-1)^j\binom{k}{j}\binom{r-2}{k-j-1}\sum_{n=2r-k-2}^{\infty}\sum_{m=0}^{n-2r+k+2}\\
&\times\frac{(n-m+3k-2-2j)!!}{(n-m-2r+k+2-2j)!!}\cos\frac{(n-m-2r+k+2-2j)\pi}{2}\frac{(m+r-k-1)!}{m!}F_{m+r-k-1}x^n\\
&= \frac{1}{(r+k-2)!2^{r+k-2}}\sum_{n=2r-k-2}^{\infty}\sum_{m=r-k-1}^{n-r+1}\sum_{j=0}^{k-1}(-1)^j\binom{k}{j}\binom{r-2}{k-j-1}\\
&\quad\times\frac{(n-m+r+2k-3-2j)!!}{(n-m-r+k+3-2j)!!}\cos\frac{(n-m-r+1-2j)\pi}{2}\frac{m!}{(m-r+k+1)!}F_m x^n\,.
\end{aligned}
$$

Since $\cos(\alpha+\pi)=-\cos\alpha$, this is also equal to

$$
\begin{aligned}
&\frac{1}{(r+k-2)!2^{r+k-2}}\sum_{n=2r-k-2}^{\infty}\sum_{m=r-k-1}^{n-r+1}\sum_{j=0}^{k-1}\binom{k}{j}\binom{r-2}{k-j-1}\frac{(n-m+r+2k-3-2j)!!}{(n-m-r+k+3-2j)!!}\\
&\quad\times\cos\frac{(n-m-r+1)\pi}{2}\frac{m!}{(m-r+k+1)!}F_m x^n\\
&= \frac{1}{(r+k-2)!2^{r+k-2}}\sum_{n=2r-k-2}^{\infty}\sum_{m=r-k-1}^{n-r+1}\frac{(n-m+r-1)!!}{(n-m-r+1)!!}\binom{r+k-2}{k-1}\frac{(n-m+r-3)!!}{(n-m+r-2k-1)!!}\\
&\quad\times\cos\frac{(n-m-r+1)\pi}{2}\frac{m!}{(m-r+k+1)!}F_m x^n\,.
\end{aligned}
$$

Therefore, the right-hand side of the relation in Theorem 4.2 is

$$\frac{1}{(r-1)!(r-2)!2^{r-2}}\sum_{n=2r-2}^{\infty}\sum_{m=r-1}^{n-r+1}\frac{(n-m+r-3)!!}{(n-m-r+1)!!}\cos\frac{(n-m-r+1)\pi}{2}\frac{m!}{(m-r+1)!}F_m x^n$$
$$+\sum_{k=1}^{r-1}\frac{1}{k(r-k-2)!}\frac{1}{(r+k-2)!2^{r+k-2}}\sum_{n=2r-k-2}^{\infty}$$
$$\sum_{m=r-k-1}^{n-r+1}\frac{(n-m+r-1)!!}{(n-m-r+1)!!}\binom{r+k-2}{k-1}\frac{(n-m+r-3)!!}{(n-m+r-2k-1)!!}$$
$$\times\cos\frac{(n-m-r+1)\pi}{2}\frac{m!}{(m-r+k+1)!}F_m x^n$$
$$=\frac{1}{(r-1)!(r-2)!2^{r-2}}\sum_{n=2r-2}^{\infty}\sum_{m=r-1}^{n-r+1}\frac{(n-m+r-3)!!}{(n-m-r+1)!!}\cos\frac{(n-m-r+1)\pi}{2}\frac{m!}{(m-r+1)!}F_m x^n$$
$$+\sum_{n=r-1}^{\infty}\frac{1}{(r-1)!2^{r-2}}\sum_{m=1}^{r-2}\sum_{k=r-m-1}^{r-2}\frac{1}{k!(r-k-2)!2^k}\frac{(n-m+r-1)!!}{(n-m-r+1)!!}$$
$$\times\frac{(n-m+r-3)!!}{(n-m+r-2k-1)!!}\cos\frac{(n-m-r+1)\pi}{2}\frac{m!}{(m-r+k+1)!}F_m x^n$$
$$+\sum_{n=r-1}^{\infty}\frac{1}{(r-1)!2^{r-2}}\sum_{m=r-1}^{n-r+1}\sum_{k=1}^{r-2}\frac{1}{k!(r-k-2)!2^k}\frac{(n-m+r-1)!!}{(n-m-r+1)!!}$$
$$\times\frac{(n-m+r-3)!!}{(n-m+r-2k-1)!!}\cos\frac{(n-m-r+1)\pi}{2}\frac{m!}{(m-r+k+1)!}F_m x^n\,.$$

Since for $1\le m\le r-2$ we have

$$\frac{1}{(r-1)!2^{r-2}}\sum_{k=r-m-1}^{r-2}\frac{1}{k!(r-k-2)!2^k}\frac{(n-m+r-1)!!}{(n-m-r+1)!!}$$
$$\times\frac{(n-m+r-3)!!}{(n-m+r-2k-1)!!}\frac{m!}{(m-r+k+1)!}$$
$$=\frac{1}{(r-1)!(r-2)!2^{2r-4}}\frac{(n+m+r-3)!!}{(n+m-r+1)!!}\frac{(n-m+r-3)!!}{(n-m-r+1)!!}m$$

and for $r-1\le m\le n-r+1$ we have

$$\frac{1}{(r-1)!(r-2)!2^{r-2}}\frac{(n-m+r-3)!!}{(n-m-r+1)!!}\frac{m!}{(m-r+1)!}$$
$$+\frac{1}{(r-1)!2^{r-2}}\sum_{k=r-m-1}^{r-2}\frac{1}{k!(r-k-2)!2^k}\frac{(n-m+r-1)!!}{(n-m-r+1)!!}$$
$$\times\frac{(n-m+r-3)!!}{(n-m+r-2k-1)!!}\frac{m!}{(m-r+k+1)!}$$
$$=\frac{1}{(r-1)!(r-2)!2^{2r-4}}\frac{(n+m+r-3)!!}{(n+m-r+1)!!}\frac{(n-m+r-3)!!}{(n-m-r+1)!!}m\,,$$

the proof is done. □

5. Examples 2

When $r = 2$, Theorem 4.2 is reduced to Theorem 4.1. When $r = 3, 4, 5$, we have the following results as examples.

Theorem 5.1. *For $n \ge 3$, we have*

$$\sum_{\substack{j_1+j_2+j_3=n\\ j_1,j_2,j_3\ge 1}} F_{j_1}F_{j_2}F_{j_3} = \sum_{m=1}^{n-2} \frac{(n+m)(n-m)mF_m}{8} \cos\frac{(n-m-2)\pi}{2}\,,$$

Theorem 5.2. *For $n \ge 4$, we have*

$$\sum_{\substack{j_1+j_2+j_3+j_4=n\\ j_1,j_2,j_3,j_4\ge 1}} F_{j_1}F_{j_2}F_{j_3}F_{j_4}$$
$$= \sum_{m=1}^{n-3} \frac{(n+m+1)(n+m-1)(n-m+1)(n-m-1)mF_m}{4!2^3} \cos\frac{(n-m-3)\pi}{2}\,.$$

Theorem 5.3. *For $n \ge 5$, we have*

$$\sum_{\substack{j_1+\cdots+j_5=n\\ j_1,\dots,j_5\ge 1}} F_{j_1}\cdots F_{j_5}$$
$$= \sum_{m=1}^{n-4} \frac{5(n+m+2)(n+m)(n+m-2)(n-m+2)(n-m)(n-m-2)mF_m}{6!2^6}$$
$$\times \cos\frac{(n-m)\pi}{2}\,.$$

6. Remarks

In [3, Theorem 32.4], it is shown that $\sum_{j=0}^{n} F_jF_{n-j} = h_{2,n}$, where $h_{i,j} = h_{i,j-2} + h_{i,j-1} + h_{i-1,j-1}$ ($i \ge 1$, $j \ge 2$) with $h_{0,j} = 0$ ($j \ge 2$), $h_{j,j} = 1$ ($j \ge 1$) and $h_{i,j} = 0$ ($i > j$). In addition, an explicit form is given by $h_{2,n} = \big((n-1)F_n + 2nF_{n-1}\big)/5$ ([3, (32.13)]). We can show that Theorem 4.1 matches this fact.

In addition, in [3, Theorem 32.4 and (32.14)], it is shown that the left-hand side of (5.1) is equal to $\big((5n^2-3n-2)F_n - 6nF_{n-1}\big)/50$.

Proposition 6.1. *For $n \ge 2$*

$$\sum_{m=1}^{n-1} mF_m \cos\frac{(n-m-1)\pi}{2} = \frac{(n-1)F_n + 2nF_{n-1}}{5}\,.$$

Lemma 6.2. *For $n \ge 0$ and $k \ge 1$ with $k > j$, we have*

$$\sum_{i=0}^{n} F_{ki+j} = \frac{F_{(n+1)k+j} - (-1)^k F_{nk+j} - F_j - (-1)^j F_{k-j}}{L_k - (-1)^k - 1}\,, \tag{6.1}$$

$$\begin{aligned}\sum_{i=0}^{n} iF_{ki-j} = & \frac{1}{\bigl(L_k - (-1)^k - 1\bigr)^2}\Bigl(nF_{(n+2)k-j} - \bigl(2(-1)^k n + n + 1\bigr)F_{(n+1)k-j} \\ & + \bigl(2(-1)^k(n+1) + n\bigr)F_{nk-j} - (n+1)F_{(n-1)k-j} \\ & \quad -(-1)^{k+j}F_{k+j} + F_{k-j} + 2(-1)^{k+j}F_j\Bigr)\,,\end{aligned} \tag{6.2}$$

Proof. (6.1) is Theorem 5.11 in [3]. We shall prove (6.2). Since

$$z + 2z^2 + 3z^3 + \cdots + nz^n = z\frac{d}{dz}(1 + z + z^2 + \cdots + z^n) = \frac{nz^{n+2} - (n+1)z^{n+1} + z}{(z-1)^2}\,,$$

by $\alpha = (1+\sqrt{5})/2$ and $\beta = (1+\sqrt{5})/2$ with $\alpha\beta = -1$, we have

$$\begin{aligned}\sum_{i=1}^{n} iF_{ki-j} &= \sum_{i=1}^{n} i\frac{\alpha^{ki-j} - \beta^{ki-j}}{\sqrt{5}} \\ &= \frac{1}{\sqrt{5}}\left(\frac{1}{\alpha^j}\frac{n(\alpha^k)^{n+2} - (n+1)(\alpha^k)^{n+1} + \alpha}{(\alpha^k - 1)^2} - \frac{1}{\beta^j}\frac{n(\beta^k)^{n+2} - (n+1)(\beta^k)^{n+1} + \beta}{(\beta^k - 1)^2}\right) \\ &= \frac{1}{\sqrt{5}\bigl((\alpha\beta)^k - (\alpha^k + \beta^k) + 1\bigr)^2}\bigl(n(\alpha^{nk-j} - \beta^{nk-j}) - (n+1)(\alpha^{(n-1)k-j} - \beta^{(n-1)k-j}) \\ &\quad - (-1)^k(\alpha^k\beta^{-j} - \beta^k\alpha^{-j}) - 2n(-1)^k(\alpha^{(n+1)k-j} \\ &\quad - \beta^{(n+1)k-j}) + 2(-1)^k(n+1)(\alpha^{nk-j} - \beta^{nk-j}) - 2(-1)^k(\alpha^{-j} - \beta^{-j}) \\ &\quad + n(\alpha^{(n+2)k-j} - \beta^{(n+2)k-j}) - (n+1)(\alpha^{(n+1)k-j} - \beta^{(n+1)k-j}) + (\alpha^{k-j} - \beta^{k-j})\bigr) \\ &= \frac{1}{\bigl(L_k - (-1)^k - 1\bigr)^2}\Bigl(nF_{(n+2)k-j} - \bigl(2(-1)^k n + n + 1\bigr)F_{(n+1)k-j} \\ &\quad + \bigl(2(-1)^k(n+1) + n\bigr)F_{nk-j} - (n+1)F_{(n-1)k-j} \\ &\quad\quad - (-1)^{k+j}F_{k+j} + F_{k-j} + 2(-1)^{k+j}F_j\Bigr)\,.\end{aligned}$$

Here, we used the fact $F_{-j} = (-1)^{j-1}F_j$ $(j \ge 1)$. □

Proof of Proposition 6.1. Let $n = 4k$. Other cases $n \not\equiv 0 \pmod 4$ can be proven similarly. By Lemma 6.2

$$\begin{aligned}
&\sum_{m=1}^{n-1} mF_m \cos\frac{(n-m-1)\pi}{2} \\
&= -\sum_{l=1}^{k}(4l-3)F_{4l-3} + \sum_{l=1}^{k}(4l-1)F_{4l-1} \\
&= 4\sum_{l=1}^{k} lF_{4l-2} + \sum_{l=1}^{k} F_{4l-5} \\
&= \frac{4}{25}\big(kF_{4k+6} - (3k+1)F_{4k+2} + (3k+2)F_{4k-2} - (k+1)F_{4k-6} - 5\big) \\
&\quad + \frac{1}{5}(F_{4k-1} - F_{4k-5} + 4) \\
&= \frac{(4k-1)F_{4k} + 2F_{4k-1}}{5} = \frac{(n-1)F_n + 2F_{n-1}}{5}\,.
\end{aligned}$$

□

7. Acknowledgement

This work has been partly done when the first author visited the Czech Technical University in Prague in June 2014. He would like to thank the Department of Mathematics FNSPE for the kind hospitality.

References

[1] A. T. Benjamin and J. Quinn, *Proofs that really count: The art of combinatorial proof*, Dolciani Mathematical Expositions No.27, MAA, 2003.
[2] T. Komatsu, *Convolution identities for Cauchy numbers*, Acta Math. Hungary. **144** (2014), 76–91.
[3] T. Koshy, *Fibonacci and Lucas numbers with applications*, Wiley, New York, 2001.
[4] OEIS Foundation Inc. (2014), The On-Line Encyclopedia of Integer Sequences, http://oeis.org.

MSC2010: 05A15, 05A19, 11B39

School of Mathematics and Statistics, Wuhan University, Wuhan, 430072, China
E-mail address: komatsu@whu.edu.cn

Department of Mathematics FNSPE, Czech Technical University in Prague, Trojanova 13, 120 00 Prague 2, Czech Republic
E-mail address: zuzana.masakova@fjfi.cvut.cz

Department of Mathematics FNSPE, Czech Technical University in Prague, Trojanova 13, 120 00 Prague 2, Czech Republic
E-mail address: edita.pelantova@fjfi.cvut.cz

ON FIBONACCI NUMBERS WHICH ARE ELLIPTIC KORSELT NUMBERS

FLORIAN LUCA AND PANTELIMON STĂNICĂ

Abstract. Here, we show that if E is a CM elliptic curve with CM field $\mathbb{Q}(\sqrt{-d})$, then the set of n for which the nth Fibonacci number F_n satisfies an elliptic Korselt criterion for $\mathbb{Q}(\sqrt{-d})$ (defined in the paper) is of asymptotic density zero.

1. Introduction

Let $b \geq 2$ be an integer. A composite integer n is a pseudoprime to base b if the congruence $b^n \equiv b \pmod{n}$ holds. There are infinitely many pseudoprimes with respect to any base b, but they are less numerous than the primes. That is, putting $\pi_b(x)$ for the number of base b pseudoprimes $n \leq x$, a result of Pomerance [9] shows that the inequality

$$\pi_b(x) \leq x/L(x)^{1/2} \qquad \text{where} \qquad L(X) = \exp\left(\log x \log\log\log x / \log\log x\right)$$

holds for all sufficiently large x. It is conjectured that $\pi_b(x) = x/L(x)^{1+o(1)}$ as $x \to \infty$.

Let $\{F_n\}_{n\geq 0}$ be the sequence of Fibonacci numbers $F_{n+1} = F_n + F_{n-1}$ for $n \geq 1$ with $F_0 = 0, F_1 = 1$, and $\{L_n\}_{n\geq 0}$ be its companion Lucas sequence satisfying the same recurrence with initial conditions, $L_0 = 2, L_1 = 1$. For the Fibonacci sequence $\{F_n\}_{n\geq 1}$ is was shown in [7] that the set of $n \leq x$ such that F_n is a prime or a base b pseudoprime is of asymptotic density zero. More precisely, it was shown that the number of such $n \leq x$ is at most $5x/\log x$ if x is sufficiently large.

Since elliptic curves have become very important in factoring and primality testing, several authors have defined and proved many results on elliptic pseudoprimes. To define an elliptic pseudoprime, let E be an elliptic curve over $\mathbb{Q}$ with complex multiplication by $\mathbb{Q}(\sqrt{-d})$. Here, $d \in \{1, 2, 3, 7, 11, 19, 43, 67, 163\}$. If p is a prime not dividing $6\Delta_E$, where Δ_E is the discriminant of E, and additionally $(-d|p) = -1$, where $(a|p)$ denotes the Legendre symbol of a with respect to p, then the order of the group of points on E modulo p denoted $\#E(\mathbb{F}_p)$, equals $p + 1$. In case $p \nmid \Delta_E$ and $(-d|p) = 1$, we have $\#E(\mathbb{F}_p) = p + 1 - a_p$ for some nonzero integer a_p with $|a_p| < 2\sqrt{p}$. Gordon [3], used the simple formula for $\#\mathbb{E}(\mathbb{F}_p)$ in the case $(-d|p) = -1$ to define the following test of compositeness: Let Q be a point in $E(\mathbb{Q})$ of infinite order. Let $N > 163$ be a number coprime to 6 to be tested. We compute $(-d|N)$. If it is 1 we do not test and if it is 0, then N is composite. If it is -1, then we compute $[N + 1]Q \pmod{N}$. If it is not O (the identity element of $E(\mathbb{Q})$), then N is composite while if it is O, then we declare N to be a probable prime for $Q \in E$. So, we can define N to be a pseudoprime for $Q \in E$ if it is composite and probable prime for $Q \in E$. The counting function of elliptic pseudoprimes for $Q \in E$ has also been investigated by several authors. The record belongs to Gordon and Pomerance [4], who showed that this function is at most $\exp(\log x - \frac{1}{3}\log L(x))$ for x sufficiently large depending on Q and E. We are not aware of research done on the set of indices n for which F_n can be an elliptic pseudoprime for $Q \in E$.

There are composite integers n which are pseudoprimes for all bases b. They are called Carmichael numbers and there exist infinitely many of them as shown by Alford, Granville and Pomerance in 1994 in [1]. They are also characterized by the property that n is composite, squarefree and $p-1 \mid n-1$ for all prime factors p of n. This characterization is referred to as the *Korselt criterion.*

Analogously, given a fixed curve E having CM by $\mathbb{Q}(\sqrt{-d})$, a composite integer n which is an elliptic pseudoprime for all points Q of infinite order on E is called an elliptic Carmichael number for E. Fix $d \in D$. The authors of [2] defined the following elliptic Korselt criterion which ensures that n is an elliptic Carmichael number for any E with CM by $\mathbb{Q}(\sqrt{-d})$ provided that $(N, \Delta_E) = 1$.

Theorem 1.1. *(EPT) Let N be squarefree, coprime to 6, composite, with an odd number of prime factors p all satisfying $(-d|p) = -1$ and $p+1 \mid N+1$. Then N is an elliptic Carmichael number for any E with CM by $\mathbb{Q}(\sqrt{-d})$ provided that $(N, \Delta_E) = 1$.*

We call positive integers N satisfying the first condition of Theorem 1.1 *elliptic Korselt for* $\mathbb{Q}(\sqrt{-d})$. In [2], it is shown that there are infinitely many elliptic Korselt numbers for $\mathbb{Q}(\sqrt{-d})$ for all $d \in D$ under some believed conjectures from the distribution of prime numbers. It was recently shown by Wright [10] that the number of elliptic Carmichael numbers up to x is

$$\geq \exp\left(\frac{K \log x}{(\log\log\log x)^2}\right) \quad \text{with some positive constant} \quad K$$

for all $x > 100$.

Here, we fix $d \in D := \{1, 2, 3, 7, 11, 19, 43, 67, 163\}$ and look at the set of numbers

$$\mathcal{N}^{(d)} = \{n : F_n \text{ is elliptic Korselt for } \mathbb{Q}(\sqrt{-d})\}.$$

It is easy to prove that $\mathcal{N}^{(1)} = \emptyset$. Namely, since $F_{2n+1} = F_n^2 + F_{n+1}^2$, it follows that if $r \geq 5$ is an odd prime, then all prime factors of F_r are congruent to 1 modulo 4. In particular, $(-1|p) = 1$ for all prime factors p of F_r. Since $F_r \mid F_n$ for all $r \mid n$, then the primes $p|F_r$ (recall that they all satisfy $(-1|p) = 1$) would divide F_n but that is impossible since F_n is Korselt and its prime factors must satisfy $(-1|p) = -1$. This shows that if $n \in \mathcal{N}^{(1)}$, then n cannot have prime factors $r \geq 5$, therefore $n = 2^a \cdot 3^b$, which is impossible since F_n must be coprime to 6. It is likely that $\mathcal{N}^{(d)}$ is finite for all $d \in D\backslash\{1\}$ (or even empty) but we do not know how to prove such a strong result. Instead, we settle for a more modest goal and prove that $\mathcal{N}^{(d)}$ is of asymptotic density 0. For a subset $\mathcal{A}$ of the positive integers and a positive real number x put $\mathcal{A}(x) = \mathcal{A} \cap [1, x]$.

2. The result

We prove the following result.

Theorem 2.1. *For $d \in D\backslash\{1\}$, we have*

$$\mathcal{N}^{(d)}(x) \ll \frac{x(\log\log x)^{1/2}}{(\log x)^{1/2}}.$$

Proof. Let $\mathcal{Q}$ be the set of primes $q \equiv 2, 3 \pmod 5$. Let x be a large positive real number and y be some parameter depending on x to be made more precise later. Consider $n \in \mathcal{N}(x)$, where we omit the dependence on d for simplicity. Put $N = F_n$. Our proof uses the fact that N is coprime to 6 but it does not use the fact that $(-d|p) = -1$ for all prime factors p of N. We distinguish several cases.

Case 1. $n \in \mathcal{N}_1(x) = \{n \le x : q \nmid n \text{ for any } q \in \mathcal{Q} \cap (y, x)\}$.

By Brun's sieve (see, for example, Theorem 2.3 on Page 70 in [5]), we have

$$\#\mathcal{N}_1(x) \ll x \prod_{\substack{p \in \mathcal{Q} \\ y \le p \le x}} \left(1 - \frac{1}{p}\right) \ll x \left(\frac{\log y}{\log x}\right)^{1/2}. \tag{2.1}$$

From now on, we work with $n \in \mathcal{N}(x) \backslash \mathcal{N}_1(x)$, so there exists $q \in \mathcal{Q}$ with $q \ge y$ such that $q \mid n$. Since such $q \equiv 2, 3 \pmod 5$, it follows that $F_q \equiv -1 \pmod q$. Furthermore, let p be any prime factor of F_q. Then $p \equiv \pm 1 \pmod q$. Since $F_q \equiv -1 \pmod q$, at least one of the prime factors p of F_q has the property that $p \equiv -1 \pmod q$. Thus, $q \mid p + 1$. Since $p + 1 \mid F_n + 1$, we get that $q \mid F_n + 1$. Note that $4 \nmid n$ because otherwise F_n is a multiple of $F_4 = 3$, which is not possible. We now use the fact that

$$F_n + 1 = F_{(n+\delta)/2} L_{(n-\delta)/2},$$

for some $\delta \in \{\pm 1, \pm 2\}$ such that $n \equiv \delta \pmod 4$. Thus,

$$q \mid F_{(n+\delta)/2} L_{(n-\delta)/2} \mid F_{n-\delta} F_{n+\delta}.$$

Hence, either $q \mid F_{n-\delta}$ or $q \mid F_{n+\delta}$. This shows that if we put $z(q)$ for the index of appearance of q in the Fibonacci sequence, then $n \equiv \pm\delta \pmod{z(q)}$.

Put $\mathcal{R} = \{q : z(q) \le q^{1/3}\}$. By a classical argument due to Hooley [6], we have

$$\#\mathcal{R}(t) \ll t^{2/3}. \tag{2.2}$$

Case 2. $\mathcal{N}_2(x) = \{n \in \mathcal{N}_1(x) \backslash \mathcal{N}(x) : q \in \mathcal{R}\}$.

If $n \in \mathcal{N}_2(x)$, then $q \mid n$ for some $q > y$ in $\mathcal{R}$. For a fixed q, the number of such $n \le x$ is $\lfloor x/q \rfloor \le x/q$. Hence,

$$\#\mathcal{N}_2(x) \le \sum_{\substack{y \le q \le x \\ q \in \mathcal{R}}} \frac{x}{q} \le x \sum_{\substack{q \ge y \\ q \in \mathcal{R}}} \frac{1}{q} \ll \frac{x}{y^{1/3}}, \tag{2.3}$$

where the last estimate follows from estimate (2.2) by the Abel summation formula.

Case 3. $\mathcal{N}_3(x) = \mathcal{N}(x) \backslash \left(\mathcal{N}_1(x) \cup \mathcal{N}_2(x)\right)$.

If $n \in \mathcal{N}_3(x)$, then we saw that there exists $q \ge y$ in $\mathcal{Q} \backslash \mathcal{R}$ dividing n such that $n \equiv \delta \pmod{z(q)}$ for some $\delta \in \{\pm 1, \pm 2\}$. Since $q \equiv 2, 3 \pmod 5$, $z(q)$ divides $q + 1$, therefore q and $z(q)$ are coprime. Fixing q and writing $n = qm$, the congruences $mq \equiv \delta \pmod{z(q)}$ put $m \le x/q$ into one of four possible arithmetic progressions modulo $z(q)$. The number of such integers for a fixed q is therefore at most $4\lfloor x/qz(q) \rfloor + 4 \le 4x/qz(q) + 4$. Summing up the above bound over all $q \le x$ in $\mathcal{Q} \backslash \mathcal{R}$, we get that

$$\#\mathcal{N}_3(x) \le 4 \sum_{\substack{y \le q \le x \\ q \notin \mathcal{R}}} \frac{x}{qz(q)} + 4\pi(x) \le 4x \sum_{q \ge y} \frac{1}{q^{4/3}} + 4\pi(x) \ll \frac{x}{y^{1/3}} + \frac{x}{\log x}. \tag{2.4}$$

Comparing estimates (2.1), (2.3), (2.4), it follows that we should choose y such that

$$y^{1/3} = (\log x / \log y)^{1/2}, \qquad \text{giving} \qquad y = (2/3 + o(1)) \frac{(\log x)^{3/2}}{(\log \log x)^{3/2}}$$

as $x \to \infty$. With this choice for y, we get the desired result from (2.1), (2.3) and (2.4), because

$$\#\mathcal{N}(x) \le \#\mathcal{N}_1(x) + \#\mathcal{N}_2(x) + \#\mathcal{N}_3(x).$$

□

3. Comments and Remarks

If $d \neq 1$, we used neither the condition that $(-d|p) = -1$ for all prime factors p of F_n, nor the condition that F_n is squarefree and has an odd number of prime factors. It is likely that if one can find a way to make use of these conditions, then one can give sharper (smaller) upper bound on $\#\mathcal{N}^{(d)}(x)$ than that of Theorem 2.1. Finally, there are other definitions of elliptic Carmichael numbers N which apply to elliptic curves without CM (see for example [7]). It was shown in [7] that the set of N which are Carmichael for E in that sense is of asymptotic density zero. It would be interesting to show that the set of n such that F_n is elliptic Carmichael in that sense is also a set of asymptotic density zero. The methods of this paper do not seem to shed much light on this modified problem.

4. Acknowledgements

We thank the referee for pointing out a logical mistake in a previous version of this paper. This paper was written during a visit of P. S. to the School of Mathematics of the University of the Witwatersrand. This author thanks this institution for hospitality.

References

[1] W. R. Alford, A. Granville and C. Pomerance, *There are infinitely many Carmichael numbers*, Ann. of Math. (2) **139** (1994), 703–722.

[2] A. Ekstrom, C. Pomerance and D. S. Thakur, *Infinitude of elliptic Carmichael numbers*, J. Aust. Math. Soc. **92** (2012), 45–60.

[3] D. M. Gordon, *Pseudoprimes on elliptic curves*, in J. M. DeKoninck and C. Levesque, eds. Number Theory, Proc. Internat. Number Theory Conf., Laval 1987, 291–305, de Gruyter, New York, 1989.

[4] D. M. Gordon and C. Pomerance, *The distribution of Lucas and elliptic pseudoprimes*, Math. Comp. **57** (1991), 825–838.

[5] H. Halberstam and H.-E. Richert, *Sieve methods*, Academic Press, London, UK, 1974.

[6] C. Hooley, *On Artin's conjecture*, J. Reine Angew. Math. **226** (1967), 209–220.

[7] F. Luca and I. E. Shparlinski, *Pseudoprime values of the Fibonacci sequence, polynomials and the Euler function*, Indag. Math. (N.S.) **17** (2006), 611–625.

[8] F. Luca and I. E. Shparlinski, *On the counting function of elliptic Carmichael numbers*, Canad. Math. Bull. **57** (2014), 105–112.

[9] C. Pomerance, *On the distribution of pseudoprimes*, Math. Comp. **37** (1981), 587–593.

[10] T. Wright, *There are infinitely many elliptic Carmichael numbers*, preprint, 2014; available at http://webs.wofford.edu/wrighttj/job/carmichael_elliptic_2.pdf.

MSC2010: Primary 11G20,11Y16 ; Secondary 11T23

School of Mathematics, University of the Witwatersrand, P. O. Box Wits 2050, South Africa
E-mail address: `florian.luca@wits.ac.za`

Naval Postgraduate School, Applied Mathematics Department, Monterey, CA 93943, USA
E-mail address: `pstanica@nps.edu`

THREE ANALOGUES OF STERN'S DIATOMIC SEQUENCE

SAM NORTHSHIELD

ABSTRACT. We present three analogues of Sterns diatomic sequence by altering various definitions of that sequence: the first involves replacing addition by another binary operation, the second by replacing a pair of complementary sequences by another, the third by replacing the binary representation of an integer by its Zeckendorf representation.

1. INTRODUCTION

Stern's diatomic sequence $a_1 = 1, a_{2n} = a_n, a_{2n+1} = a_n + a_{n+1}$ is a particularly well studied sequence (see, e.g., [1], [8], [9] and references therein, as well as [13]). The first section is devoted to showing that this sequence is *interesting*. In particular, we shall look at the following properties.

- $n \mapsto a_{n+1}/a_n$ is a bijection between the positive natural numbers and the positive rational numbers,
- $n/2^k \mapsto a_n/a_{n+2^k}$ extends to a continuous strictly increasing function on $[0, 1]$ known as "Conway's box function" (it's inverse is $?(x)$, Minkowski's question-mark function),
- It shares a number of similarities to the Fibonacci sequence; in particular, it has a Binet type formula.

The remaining three sections are devoted to three analogues of Stern's sequence:

- We replace addition by another binary operation; in particular, we define $b_1 = 0, b_{2n} = b_n, b_{2n+1} = b_n \oplus b_{n+1}$ where $x \oplus y = x + y + \sqrt{4xy + 1}$. This sequence is related to Stern's sequence and arises from certain sphere packings. It has apparently not appeared before in the literature.
- We replace the complementary indexing sequences $\{2n\}$ and $\{2n + 1\}$ by another pair of complementary sequences; in particular, let $R_1 = 1, R_{\alpha(n)} = R_n, R_{\beta(n)} = R_n + R_{n+1}$ where $\alpha(n) = \lfloor n\phi - 1/\phi^2 \rfloor$, $\beta(n) = \lfloor n\phi^2 + \phi \rfloor$ form a specific pair of complementary Beatty sequences. This sequence has been extensively studied as R_n is the number of ways n can be represented as a sum of distinct Fibonacci numbers.
- The known Binet type formula for Stern's sequence [9] is written in terms of the sequence $s_2(n)$ (:= the number of terms in the binary expansion of n). We replace $s_2(n)$by $s_F(n)$(:= the number of terms in the Zeckendorf representation of n). This new sequence, apparently not studied before, is an integer sequence with several interesting (and several conjectural) properties.

2. Stern's Diatomic Sequence

Consider the following "diatomic array" [1] formed as a variant of Pascal's triangle; each entry is either the value directly above or else the sum of the two above it.

$$
\begin{array}{ccccccccccccccccc}
1 & & & & & & & & & & & & & & & & 1 \\
1 & & & & & & & & 2 & & & & & & & & 1 \\
1 & & & & 3 & & & & 2 & & & & 3 & & & & 1 \\
1 & & 4 & & 3 & & 5 & & 2 & & 5 & & 3 & & 4 & & 1 \\
1 & 5 & 4 & 7 & 3 & 8 & 5 & 7 & 2 & 7 & 5 & 8 & 3 & 7 & 4 & 5 & 1 \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & .
\end{array}
$$

The word "diatomic" is used here since every entry of the diatomic array gets its value from either one or two entries above and gives that value to three entries below, hence has "valence" 4 or 5 (hence the diatomic array models a kind of crystalline alloy of two elements).

Ignoring the right most column and reading the numbers as in a book, we get Stern's diatomic sequence:

$$1, 1, 2, 1, 3, 2, 3, 1, 4, 3, 5, 2, 5, 3, 4, 1, 5, ...$$

The sequence is thus defined by the recurrence

$$a_1 = 1, a_{2n} = a_n, a_{2n+1} = a_n + a_{n+1}. \tag{1}$$

We define a_0 to be 0 (the value consistent with $a_{2\cdot 0+1} = a_0 + a_1$).

Perhaps the most celebrated property of this sequence is that every positive rational number is represented exactly once as a_{n+1}/a_n. See, for example, [4] or [9]. We rephrase this fact as a theorem.

Theorem 2.1. *Every ordered pair of relatively prime positive integers appears exactly once in the sequence* $\{(a_n, a_{n+1})\}$.

Proof. For an ordered pair, consider the process of subtracting the smallest from largest (stop if equal). For example, $(4, 5) \mapsto (4, 1)$ and $(7, 3) \mapsto (4, 3)$. By the definition of Stern's sequence,

$$(a_{2n}, a_{2n+1}), (a_{2n+1}, a_{2n+2}) \longmapsto (a_n, a_{n+1}).$$

Every relatively prime pair appears (if not, then there is an ordered pair not on the list with lowest sum. Apply the process; the result has lower sum and so is (a_n, a_{n+1}) for some n and so the original pair is either (a_{2n}, a_{2n+1}) or (a_{2n+1}, a_{2n+2})). Every relatively prime pair appears *exactly* once since, if not, then there exist $m < n$ with $(a_m, a_{m+1}) = (a_n, a_{n+1})$ and such that m is as small as possible. Applying the process to both implies $\lfloor m/2 \rfloor = \lfloor n/2 \rfloor$ and thus $a_m = a_{m+1} = a_{m+2}$ which is impossible. □

One can then rewrite any sum over relatively prime pairs in terms of Stern's sequence. As an example, we rephrase the Riemann hypothesis. First note that $n \longmapsto a_{2n}/a_{2n+1}$ is an explicit bijection from $\mathbb{Z}^+$ to $\mathbb{Q} \cap (0, 1)$. Then the Riemann hypothesis is equivalent to

$$\sum_{a_{2n+1}<x} e^{2\pi i a_{2n}/a_{2n+1}} = O(x^{1/2+\epsilon}).$$

Briefly why this is so: the Möbius function can be written as $\mu(n) := \sum_{1\leq k\leq n, \gcd(n,k)=1} e^{2\pi i k/n}$ and so the left hand side is really just Merten's function $M(x) := \sum_{n<x} \mu(n)$. The connection between Merten's function and the Riemann hypothesis is well-known; see for example [5].

Minkowski's question mark function was introduced in 1904 as an example of a "singular function" (it is strictly increasing yet its derivative exists and equals 0 almost everywhere). It is defined in terms of continued fractions:

$$?(x) = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2^{a_1+a_2+\ldots+a_n}}$$

where $x = 1/(a_1+1/(a_2+1/(a_3+\ldots)))$. By Lagrange's theorem that states that the continued fraction representation of a quadratic surd must eventually repeat, it is clear that $?(x)$ takes quadratic surds to rational numbers.

The function

$$f : \frac{k}{2^n} \longmapsto \frac{a_k}{a_{2^n+k}}$$

extends to a continuous strictly increasing function on $[0,1]$. This function is known as "Conway's box function" and its inverse is Minkowski's question mark function $?(x)$. See [9] for a

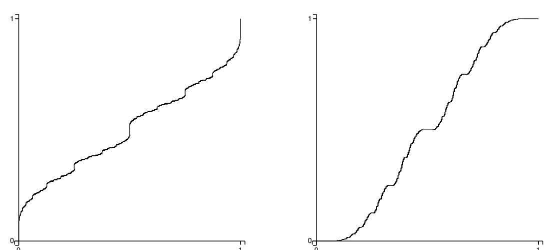


FIGURE 1. The graphs of $y = f(x)$ and its inverse $y =?(x)$.

proof.

The functions $f(x)$ and $?(x)$ extends to homeomorphisms (or, equivalently, are restrictions of homeomorphisms) between two fractals.

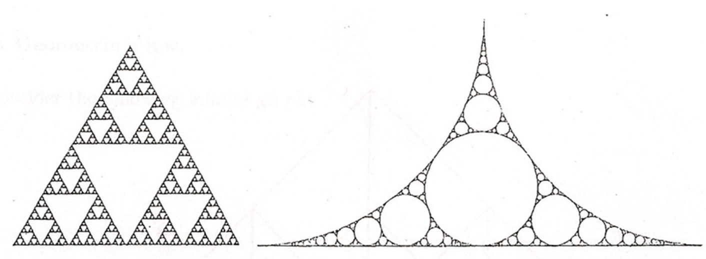


FIGURE 2. Sierpinski gasket and an Apollonian circle packing

Stern's sequence is related to the Fibonacci sequence in a number of ways. For example, the Fibonacci sequence is a subsequence:

$$
\begin{array}{ccccccccccccccccc}
\boxed{1} & & & & & & & & & & & & & & & & 1 \\
1 & & & & & & & & \boxed{2} & & & & & & & & 1 \\
1 & & & & \boxed{3} & & & & 2 & & & & 3 & & & & 1 \\
1 & & 4 & & 3 & & \boxed{5} & & 2 & & 5 & & 3 & & 4 & & 1 \\
1 & 5 & 4 & 7 & 3 & \boxed{8} & 5 & 7 & 2 & 7 & 5 & 8 & 3 & 7 & 4 & 5 & 1 \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & .
\end{array}
$$

It is easy to see that

$$a_{J(n)} = F_n \tag{2}$$

where $J(n) := (2^n - (-1)^n)/3$ is the Jacobsthal sequence [12, A001405].

A new result by Coons and Tyler [3] identifies and proves the asymptotic upper bound:

$$\limsup_{n\to\infty} \frac{a_n}{(3n)^{\log_2 \phi}} = \frac{1}{\sqrt{5}}.$$

The constants involved in this formula are perhaps not so surprising since, by formula (2), it is clear that

$$\lim_{n\to\infty} a_{J(n)}/((3J(n))^{\log_2 \phi}) = 1/\sqrt{5}.$$

Stern's sequence has a few remarkable similarities to the Fibonacci sequence (see [9] and [10]). For example, Stern's sequence satisfies a modified Fibonacci recurrence:

$$a_{n+1} = a_n + a_{n-1} - 2(a_{n-1} \mod a_n).$$

Next, certain diagonal sums across Pascal's triangle yield the Fibonacci sequence while the corresponding sums across Pascal's triangle modulo 2 yield Stern's sequence:

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & \boxed{1} & & \\
 & 1 & & \boxed{3} & & 3 & & 1 & \\
\boxed{1} & & 4 & & 6 & & 4 & & 1 \\
 & . & . & . & . & . & . & &
\end{array}
,\quad
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 0 & & \boxed{1} & & \\
 & 1 & & \boxed{1} & & 1 & & 1 & \\
\boxed{1} & & 0 & & 0 & & 0 & & 1 \\
 & . & . & . & . & . & . & &
\end{array}
$$

$$F_{n+1} = \sum_{2i+j=n} \binom{i+j}{i}, \qquad a_{n+1} = \sum_{2i+j=n} \left[\binom{i+j}{i} \bmod 2 \right]$$

Recall Binet's formula

$$F_{n+1} = \frac{\phi^{n+1} - \overline{\phi}^{n+1}}{\phi - \overline{\phi}} = \sum_{k=0}^{n} \phi^k \overline{\phi}^{n-k}. \tag{3}$$

Stern's sequence satisfies a similar formula:

$$a_{n+1} = \sum_{k=0}^{n} \sigma^{s_2(k)} \overline{\sigma}^{s_2(n-k)}$$

where $\sigma := (1 + \sqrt{-3})/2$ is a sixth root of unity and $s_2(n)$ is the number of ones in the binary expansion of n [12, A000120]:

$$
\begin{array}{cccccccccc}
n & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8... \\
s_2(n) & 0 & 1 & 1 & 2 & 1 & 2 & 2 & 3 & 1...
\end{array}
$$

Why this is true: If $G(x) := \sum \sigma^{s_2(n)} x^n$ and $F(x)$ is the generating function for $\{a_{n+1}\}$, then $G(x) = (1+\sigma x)G(x^2)$ and $F(x) = (1+x+x^2)F(x^2)$. For real x, since $(1+\sigma x)(1+\overline{\sigma}x) = 1+x+x^2$, $|G(x)|^2 = F(x)$ and the result follows by equating coefficients.

3. Replacing addition by another operation

Definition 3.1. *For non-negative real numbers a, b, let*

$$\begin{aligned} a \oplus b &= a+b+\sqrt{4ab+1} \\ a \ominus b &= a+b-\sqrt{4ab+1} \end{aligned}.$$

Proposition 3.2. *If $a, b, c, d > 0$ and $|ad - bc| = 1$ then $(ac) \oplus (bd) = (a+b)(c+d)$.*

Proof. If $(ad-bc)^2 = 1$, then

$$(ad+bc)^2 = 1 + 4abcd$$

and thus

$$(ac) \oplus (bd) = ac + bd + \sqrt{4abcd+1} = ac + bd + ad + bc.$$

□

Remark 3.3. By the Fibonacci identity

$$F_{n-1}F_{n+1} = F_n^2 + (-1)^n,$$

it follows that

$$(F_{n-1}F_n) \oplus (F_nF_{n+1}) = (F_{n-1}+F_n)(F_n+F_{n+1}) = F_{n+1}F_{n+2}.$$

and so the sequence $x_n := F_nF_{n+1}$ satisfies the modified Fibonacci recurrence

$$x_{n+1} = x_n \oplus x_{n-1}.$$

Here we define the first new sequence.

Definition 3.4. *Let $b_1 = 0$, and for $n \geq 1$,*

$$\begin{aligned} b_{2n} &= b_n \\ b_{2n+1} &= b_n \oplus b_{n+1}. \end{aligned}$$

The sequence begins

$$0, 0, 1, 0, 2, 1, 2, 0, 3, 2, 6, 1, 6, 2, 3, 0, 4, 3, 10, 2, 15, 6, 12, 1, 12, 6, 15, ...$$

It is not immediately clear that this sequence must always be integral. One way to show this is to express each b_k as a product of elements of Stern's sequence (Theorem 3.6, below). First we must prove a lemma.

Lemma 3.5. *For $m, n \geq 0$, if $m+n = 2^j - 1$ then $a_{m+1}a_{n+1} - a_ma_n = 1$.*

Proof. We prove this by induction on j. If $m+n = 1$, then $a_{m+1}a_{n+1} - a_ma_n = a_1a_2 - a_0a_1 = 1$ and the result holds for $j = 1$. Suppose now that the result holds for a fixed j and that $m+n = 2^{j+1}-1$. Without loss of generality, $m = 2k+1$ and $n = 2l$ for some $k, l \geq 0$ (and so $k + l = 2^j - 1$). Then

$$\begin{aligned} a_{m+1}a_{n+1} - a_ma_n &= a_{2k+2}a_{2l+1} - a_{2k+1}a_{2l} \\ &= a_{k+1}(a_l + a_{2l+1}) - (a_k + a_{k+1})a_l = a_{k+1}a_{l+1} - a_ka_l = 1 \end{aligned}$$

and the result follows. ∎

Theorem 3.6. *If $2^j \le k \le 2^{j+1}$, then*

$$b_k = a_{2^{j+1}-k}a_{k-2^j}.$$

Proof. If $k = 2^j$ then, because $a_0 = 0$, $b_k = 0 = a_{2^{j+1}-k}a_{k-2^j} = a_{2^j-k}a_{k-2^{j-1}}$.

Let $x_k := a_{2^{j+1}-k}a_{k-2^j}$ where $k \in (2^j, 2^{j+1})$. Then $2k, 2k+1 \in (2^{j+1}, 2^{j+2})$ and thus

$$x_{2k} = a_{2^{j+1}-2k}a_{2k-2^j} = a_{2^j-k}a_{k-2^{j-1}} = x_k$$

and, by lemma 3.5 and proposition 3.2,

$$\begin{aligned} x_{2k+1} &= a_{2^{j+1}-(2k+1)}a_{2k+1-2^j} \\ &= a_{2(2^j-k-1)+1}a_{2(k-2^{j-1})+1} \\ &= (a_{2^j-k-1} + a_{2^j-k}) \cdot (a_{k-2^{j-1}} + a_{k+1-2^{j-1}}) \\ &= (a_{2^j-k-1}a_{k+1-2^{j-1}}) \oplus (a_{2^j-k}a_{k-2^{j-1}}) = x_{k+1} \oplus x_k \end{aligned}$$

Hence $b_k = x_k$ for all k, and the result follows. ∎

Corollary 3.7. $b_n \in \mathbb{N}$.

As seen in section 2, Stern's diatomic sequence leads to a construction of Conway's box function $f(x)$, the inverse of Minkowski's question-mark function. The sequence $\{b_k\}$ gives rise to a similar function that turns out to be closely related to $f(x)$.

Definition 3.8. *For $k, n \in \mathbb{N}$, $k \le 2^n$, let*

$$g\left(\frac{k}{2^n}\right) := \frac{b_k}{b_{2^n+k}}.$$

Theorem 3.9. *The function $g(x)$ extends to a continuous function on $[0,1]$ that satisfies, for $x \in (2^{-j-1}, 2^{-j})$,*

$$g(x) = f(2^{j+1}x - 1)[1 - jf(2x)]$$

where $f(x)$ is Conway's box function.

Proof. Let $x = k/2^n$. Then $2^{n-j-1} \le k \le 2^{n-j}$ for some $j \ge 0$. Since $2^n \le 2^n + k \le 2^{n+1}$, it follows from Theorem 3.6 that

$$b_k = a_{2^{n-j}-k}a_{k-2^{n-j-1}} \text{ and } b_{2^n+k} = a_{2^n-k}a_k.$$

By [9, formulas (2) and (3)],

$$a_{2^n-k} = ja_k + a_{2^{n-j}-k}$$

and thus

$$\begin{aligned} g(x) = g\left(\frac{k}{2^n}\right) &= \frac{b_k}{b_{2^n+k}} = \frac{a_{2^{n-j}-k}a_{k-2^{n-j-1}}}{a_{2^n-k}a_k} \\ &= \frac{(a_{2^n-k} - ja_k)a_{k-2^{n-j-1}}}{a_{2^n-k}a_k} = \frac{a_{k-2^{n-j-1}}}{a_k}\left(1 - \frac{ja_k}{a_{2^n-k}}\right) \\ &= f\left(\frac{k}{2^{n-j-1}} - 1\right)\left[1 - jf\left(\frac{k}{2^{n-1}}\right)\right] = f(2^{j+1}x - 1)[1 - jf(2x)]. \end{aligned}$$

The extension of $g(x)$ to a continuous function on $[0,1]$ follows from the facts that f extends to a continuous function on $[0,1]$ and $f(2^{-j}) = 1/(j+1)$. ∎

The restriction of $g(x)$ to $[1/2, 1]$ is just a scaled version of $f(x)$:

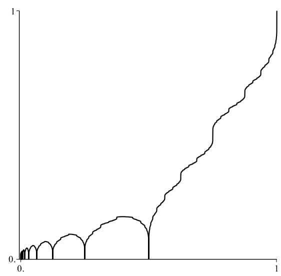


FIGURE 3. Singular function associated with $\{b_n\}$

Corollary 3.10. *$g(x) = f(2x-1)$ for $x \in [1/2, 1]$.*

Recall that every positive rational number appears exactly once in the set $\{a_{k+1}/a_k : k \in \mathbb{N}\}$. We prove an analogue for the sequence $\{b_k\}$. We use the expression "$A = \square$" to mean that $A = n^2$ for some integer n.

Theorem 3.11. *Every element of $\{(a,b) \in \mathbb{N}^2 : 4ab+1 = \square\}$ appears exactly once in the sequence $\{(b_k, b_{k+1}) : k \in \mathbb{N}\}$.*

Proof. Consider the following analogue of the (slow) Euclidean algorithm.

$$M_{\oplus} : (a,b) \longmapsto \begin{cases} (a, a \ominus b) & \text{if } a < b, \\ (a \ominus b, b) & \text{if } b < a, \\ \text{stop} & \text{if } a = b. \end{cases}$$

Suppose $(a,b) \in \mathbb{N}^2$, with $4ab+1 = \square$. If $a \ominus b < 0$ then it is easy to see that $(a-b)^2 < 1$ and thus $a = b$. In this case, since $4a^2+1 \neq \square$ unless $a = 0$, the only possibility is $a = b = 0$. Hence, $M_{\oplus}((a,b)) \in \mathbb{N}^2$ and, if this algorithm terminates at all, it must terminate at $(0,0)$.

With $(a,b) \in \mathbb{N}^2$, with $4ab+1 = \square$, let $k := \sqrt{4ab+1}$. If $0 < a < b$, then $a^2 < ak$ and thus

$$a(a \ominus b) = a(a+b-k) = a^2 + ab - ak < ab.$$

In general, the product of numbers in $M_{\oplus}((a,b))$ is strictly less than the product ab and thus the algorithm will eventually reach, without loss of generality, $(0,b)$. If $b = 0$ then the algorithm stops. On the other hand, if $b > 0$, it is easy to see that $M_{\oplus}((0,b)) = (0, b-1)$, and thus the algorithm will terminate at $(0,0)$.

Let $B_n := (b_n, b_{n+1})$. By the definition of the sequence $\{b_k\}$, it's easy to see that for $n > 1$,

$$M_{\oplus} : B_{2n}, B_{2n+1} \longmapsto B_n$$

and, moreover, if $M_{\oplus} : (a,b) \mapsto B_n$, then either $(a,b) = B_{2n}$ or $(a,b) = B_{2n+1}$.

If $(a,b) \in \mathbb{N}^2$, with $4ab+1 = \square$ is not of the form B_n for some n, then all of its successors under $M_{\oplus}$, including $(0,0)$, are not either – a contradiction. Hence every $(a,b) \in \mathbb{N}^2$, with $4ab+1 = \square$ is of the form B_n for some n.

The pair $(0,0)$ appears only once and, in general, no pair appears more than once in $\{B_n\}$ for, otherwise, there exists a smallest $n > 1$ such that $B_n = B_m$ for some $m > n$. Applying $M_{\oplus}$ to both B_m and B_n forces $\lfloor n/2 \rfloor = \lfloor m/2 \rfloor$ and therefore $m = n+1$. Thus $b_n = b_{n+1} = b_{n+2}$, a contradiction. $\square$

A generalization of $\oplus$ is as follows: For a given number N, define

$$x \underset{N}{\oplus} y := x + y + \sqrt{4xy + N}.$$

Remark 3.12. *$a, b, a \underset{N}{\oplus} b$ solve*

$$2(x^2 + y^2 + z^2) - (x + y + z)^2 = N.$$

Defining $\underset{N}{\ominus}$ in the obvious manner,

$$(a \underset{N}{\oplus} b) \underset{N}{\ominus} b = a.$$

Every non-zero complex number z can be represented uniquely as $re^{i\theta}$ for some positive r and some $\theta \in [0, 2\pi)$ and so we define $\sqrt{z} := \sqrt{r}e^{i\theta/2}$. Hence $\underset{N}{\oplus}$ and $\underset{N}{\ominus}$ are well defined for complex N.

We may then generalize $\{b_k\}$.

Definition 3.13. *Given a (complex) number A, let $c_1 = c_2 = A$ and, for $n \geq 1$,*

$$\begin{aligned} c_{2n} &= c_n \\ c_{2n+1} &= c_n \underset{N}{\oplus} c_{n+1}. \end{aligned}$$

It turns out that such a sequence can be expressed as a linear combination of the sequences $\{a_k^2\}$ and $\{b_k\}$. We first need a lemma.

Lemma 3.14. *For $k \geq 1$,*

$$a_k^2 b_{k+1} + a_{k+1}^2 b_k + 1 = a_k a_{k+1} \sqrt{4 b_k b_{k+1} + 1}.$$

Proof. Let $s_k := \sqrt{4b_k b_{k+1} + 1}$. Note that

$$b_{2k+1} = b_k + b_{k+1} + s_k.$$

Then

$$\begin{aligned} s_{2k}^2 &= 4b_{2k}b_{2k+1} + 1 = 4b_k(b_k + b_{k+1} + s_k) + 1 \\ &= 4b_k^2 + s_k^2 + 4b_k s_k = (2b_k + s_k)^2 \end{aligned}$$

and so

$$s_{2k} = 2b_k + s_k.$$

Similarly,

$$\begin{aligned} s_{2k+1}^2 &= 4b_{2k+1}b_{2k+2} + 1 = 4b_{k+1}(b_k + b_{k+1} + s_k) + 1 \\ &= 4b_{k+1}^2 + s_k^2 + 4b_{k+1} s_k = (2b_{k+1} + s_k)^2 \end{aligned}$$

and so

$$s_{2k+1} = 2b_{k+1} + s_k.$$

Note that

$$a_1^2 b_2 + a_2^2 b_1 + 1 = 1 = a_1 a_2 \sqrt{4b_1 b_2 + 1}$$

and so the lemma holds for $k = 1$.

Suppose the lemma holds for a particular k. We show it works for $2k$ and $2k+1$ and thus, by induction, the lemma will be shown.

$$\begin{aligned}a_{2k}^2 b_{2k+1} a_{2k+1}^2 b_{2k} + 1 &= a_k^2(b_k + b_{k+1} + s_k) + (a_k + a_{k+1})^2 b_k + 1\\ &= a_k^2 b_k + a_k^2 b_{k+1} + a_k^2 s_k + a_k^2 b_k + 2a_k a_{k+1} b_k + a_{k+1}^2 b_k + 1\\ &= a_k^2(2b_k + s_k) + 2a_k a_{k+1} b_k + a_k^2 b_{k+1} + a_{k+1}^2 b_k + 1\\ &= a_k^2(2b_k + s_k) + 2a_k a_{k+1} b_k + a_k a + k + 1 s_k\\ &= a_k(a_k + a_{k+1})(2b_k + s_k) = a_{2k} a_{2k+1} s_{2k}\end{aligned}$$

and thus the lemma works for $2k$.

$$\begin{aligned}a_{2k+1}^2 b_{2k+2} a_{2k+2}^2 b_{2k+1} + 1 &= (a_k + a_{k+1})^2 b_{k+1} + a_{k+1}^2(b_k + b_{k+1} + s_k) + 1\\ &= a_k^2 b_{k+1} + 2a_k a_{k+1} b_{k+1} + a_{k+1}^2 b_{k+1} + a_{k+1}^2 b_k + a_{k+1}^2 b_{k+1} + a_{k+1}^2 s_k + 1\\ &= a_k^2(2b_{k+1} + s_k) + a_k^2 b_{k+1} + a_{k+1}^2 b_k + 1 + 2a_k a_{k+1} b_{k+1}\\ &= a_k^2(2b_{k+1} + s_k) + a_k a_{k+1} s_k + 2a_k a_{k+1} b_{k+1}\\ &= a_{k+1}(a_k + a_{k+1})(2b_{k+1} + s_k) = a_{2k+2} a_{2k+1} s_{2k+1}\end{aligned}$$

and thus the lemma works for $2k+1$. □

Theorem 3.15. *Given A, B, let $c_k := Aa_k^2 + Bb_k$. Then $\{c_k\}$ has $c_1 = c_2 = A$ and, for $N = 4AB + B^2$,*

$$\begin{aligned}c_{2n} &= c_n\\ c_{2n+1} &= c_n \underset{N}{\oplus} c_{n+1}.\end{aligned}$$

Proof.

$$\begin{aligned}c_k c_{k+1} + AB &= (Aa_k^2 + Bb_k)(Aa_{k+1}^2 + Bb_{k+1}) + AB\\ &= A^2 a_k^2 a_{k+1}^2 + B^2 b_k b_{k+1} + AB(a_{k+1}^2 b_k + a_k^2 b_{k+1} + 1)\\ &= A^2 a_k^2 a_{k+1}^2 + B^2 b_k b_{k+1} + ABa_k a_{k+1}\sqrt{4b_k b_{k+1} + 1}\end{aligned}$$

and so

$$\begin{aligned}4c_k c_{k+1} + N &= 4A^2 a_k^2 a_{k+1}^2 + 4B^2 b_k b_{k+1} + B^2 + 4ABa_k a_{k+1}\sqrt{4b_k b_{k+1} + 1}\\ &= (2Aa_k a_{k+1} + B\sqrt{4b_k b_{k+1} + 1})^2\end{aligned}$$

and thus

$$\begin{aligned}c_k \underset{N}{\oplus} c_{k+1} &= (Aa_k^2 + Bb_k) + (Aa_{k+1}^2 + Bb_{k+1}) + \sqrt{4c_k c_{k+1} + N}\\ &= Aa_k^2 + Bb_k + Aa_{k+1}^2 + Bb_{k+1} + 2Aa_k a_{k+1} + B\sqrt{4b_k b_{k+1} + 1}\\ &= A(a_k + a_{k+1})^2 + B(b_k + b_{k+1} + \sqrt{4b_k b_{k+1} + 1}\\ &= Aa_{2k+1}^2 + Bb_{2k+1} = c_{2k+1}.\end{aligned}$$

Since

$$c_{2k} = Aa_{2k}^2 + Bb_{2k} = Aa_k^2 + Bb_k = c_k,$$

the theorem is shown. □

Example 3.16. *Let $N = -3$, $c_1 = c_2 = 1$, we see that $A = 1$, $B = -1$, and thus $c_k = a_k^2 - b_k$.*

If to every local cut point P in the fractal CP appearing in figure 2 one attaches a sphere above but tangent to the plane at that point with curvature (1/radius) equal to the sum of the curvatures of the two circles meeting there, then one gets a 3-dimensional generalization of Ford circles. The curvatures (similarly, the product of local cut points and corresponding curvatures) along any circular arc are from a sequence $\{c_n\}$ for appropriately chosen N (see [11] and references therein for a discussion of various types of "Ford spheres").

Consider the sequence $\{b_k\}$ written in tabular form:

$$\begin{array}{lllllllll} 0 \\ 0 & 1 \\ 0 & 2 & 1 & 2 \\ 0 & 3 & 2 & 6 & 1 & 6 & 2 & 3 \\ 0 & 4 & 3 & 10 & 2 & 15 & 6 & 12 & \ldots \\ . & . & . & . & . & . & . & . & . \end{array}$$

It is apparent that every column is an arithmetic sequence and, moreover, the defining differences are respectively

$$0, 1, 1, 4, 1, 9, 4, 9, \ldots,$$

the squares of Stern's diatomic sequence $\{a_k^2\}$. This is, in fact, true. We shall express this result as a formula.

Theorem 3.17. *For $0 \leq k < 2^j$,*

$$b_{2^{j+1}+k} = a_k^2 + b_{2^j+k}.$$

Proof. Assume $0 \leq k < 2^j$. Since $2^j \leq 2^j + k < 2^{j+1}$, Theorem 3.6 implies

$$b_{2^{j+1}+k} = b_{2^j+2^j+k} = a_{2^{j+1}-(2^j+k)} a_{2^j+k-2^j} = a_{2^j-k} a_k.$$

By [9, formulas (2) and (3)] ,

$$a_{2^{j+1}-k} = a_k + a_{2^j-k}$$

and thus

$$b_{2^{j+2}+k} = a_{2^{j+1}-k} a_k = (a_k + a^{2^j-k}) a_k = a_k^2 + a_{2^j-k} a_k = a_k^2 + b_{2^{j+1}+k}.$$

The result follows by induction. ☐

Remark 3.18. $\{b_{2k-1}\}$ *appears as* [12, A119272], *the product of numerators and denominators in the Stern-Brocot tree.*

Remark 3.19. *For a fixed (x, y), $z = x \oplus y$ and $z = x \ominus y$ are the two solutions of*

$$2(x^2 + y^2 + z^2) - (x + y + z)^2 = 1.$$

4. Fibonacci representations

A *Fibonacci representation* of a number n is a way of writing that number as a sum of distinct Fibonacci numbers. One such representation is, of course, the Zeckendorf representation which is gotten by the greedy algorithm and which is characterized by having no two consecutive Fibonacci numbers. In general, a given n has several Fibonacci representations, the number of such we call R_n. The sequence $\{R_n\}$ is extremely well studied; see papers by Klarner [7], Bicknell-Johnson [1, 2], and Stockmeyer [14], for example.

A string of 0s and 1s is a finite word with alphabet $\{0, 1\}$ (equivalently, an element of $\{0, 1\}^*$). Often we denote such a word by ω. We shall think of such strings as Fibonacci representations: we shall assign a numerical value $[\omega]$ to a string ω by the formula

$$[i_1 i_2 ... i_k] = \sum i_j F_{k+2-j}.$$

For example, $[0100] = [0011] = 3$ and $[01010011] = 21 + 8 + 2 + 1 = 32$.

The generating function for $\{R_n\}$ has an obvious product formulation.

Proposition 4.1. *The sequence* (R_n) *satisfies*

$$\sum_{n=0}^{\infty} R_n x^n = \prod_{i=2}^{\infty} \left(1 + x^{F_i}\right)$$

where F_n *denotes the* n*th Fibonacci number.*

Next, we define the *Fibonacci shift*:

$$\rho(n) := \lfloor n\phi + 1/\phi \rfloor$$

that satisfies $\rho([\omega]) = [\omega 0]$ for every string ω. This shift has been studied before; for example, it appears in [6, graffiti, p. 301].

Theorem 4.2. *For* $c_i \in \{0, 1\}$, $i = 2, ..., N$,

$$\rho\left(\sum_{i=2}^{N} c_i F_i\right) = \sum_{i=2}^{N} c_i F_{i+1}.$$

Proof. By Binet's formula (3),

$$\phi F_n = F_{n+1} - \overline{\phi}^n.$$

For any choice $c_i \in \{0, 1\}$ for $i = 2, ..., N$, note that

$$-1/\phi^2 = \sum_{n=1}^{\infty} \overline{\phi}^{2n+1} < \sum_{i=2}^{N} c_i \overline{\phi}^i < \sum_{n=1}^{\infty} \overline{\phi}^{2n} = 1/\phi$$

and therefore

$$0 < -\sum_{i=2}^{N} c_i \overline{\phi}^i - \overline{\phi} < 1.$$

Hence,

$$\begin{aligned} \rho\left(\sum_{i=2}^{N} c_i F_i\right) &= \left\lfloor \phi \sum_{i=2}^{N} c_i F_i - \overline{\phi} \right\rfloor \\ &= \sum_{i=2}^{N} c_i F_{i+1} + \left\lfloor -\sum_{i=2}^{N} c_i \overline{\phi}^i - \overline{\phi} \right\rfloor = \sum_{i=2}^{N} c_i F_{i+1}. \end{aligned}$$

□

In terms of $\rho(n)$, we may define $\{R_n\}$ recursively. Clearly, $R_0 = R_1 = 1$. A representation of n either ends in 0 in which case $n = [\omega 0]$ where $\rho([\omega]) = n$ or else it ends in 1 in which case $n = [\omega 1]$ and so $n - 1 = [\omega 0] = \rho([\omega])$. Hence, for all $n \geq 1$,

$$R_n := \sum_{\rho(i) \in \{n, n-1\}} R_i.$$

Note that the function $\rho_2(n) := \rho(\rho(n)) = \lfloor n\phi^2 + 1/\phi \rfloor$ is an example of a Beatty sequence (i.e., of the form $\lfloor an + b \rfloor$) and so has a complementary Beatty sequence, namely $T(n) := \lfloor n\phi + 2/\phi \rfloor$. For example,

$$\rho_2(n) = 0, 3, 5, 8, 11, 13, 16, 18, 21, 24, ...$$

and

$$T(n) = 1, 2, 4, 6, 7, 9, 10, 12, 14, 15, ...$$

The following characterization could be used as a new definition of $\{R_n\}$.

Theorem 4.3. *For $n \geq 1$, and $T(n) := \lfloor n\phi + 2/\phi \rfloor$,*

$$R_{\rho_2(n)} = R_n + R_{n-1}$$

and

$$R_{T(n)} = R_n.$$

Proof. Since $\phi \in (1, 2)$, $\rho(n) \in \{\rho(n + 1) - 1, \rho(n + 1) - 2\}$. Since $2\phi > 3$, $(n - 1)\phi + 1/\phi \leq (n + 1)\phi + 1/\phi - 3$ and so $\rho(n - 1) < \rho(n + 1) - 2$. Note that

$$T(n) = \lfloor n\phi + 2/\phi \rfloor = \lfloor (n + 1)\phi + 1/\phi \rfloor - 1 = \rho(n + 1) - 1$$

and therefore

$$R_{T(n)} = \sum_{\rho(i) \in \{\rho(n+1)-1, \rho(n+1)-2\}} R_i = R_n.$$

We show the first equation in the theorem by a counting argument. By the definition of $\rho(n)$, $\rho_2(n) = \rho(n) + n$ and so

$$\rho_2(n + 1) - \rho_2(n) = \rho(n + 1) - \rho(n) + 1 \in \{2, 3\}.$$

For a given n, if $n = [\omega]$ then $\rho(\rho(n)) = [\omega 00]$ and $\rho(\rho(n + 1))$ equals either $[\omega 10]$ or $[\omega 11]$.

Suppose $\rho_2(n + 1) - \rho_2(n) = 2$. The map $\omega \mapsto \omega 00$ is a bijection from representations of n to the representations of $\rho_2(n)$ ending in 00 while the map $\omega \mapsto \omega 10$ is a bijection from representations of $n - 1$ to the representations of $\rho_2(n)$ not ending in 00. Hence the first equation holds.

A similar argument holds when $\rho_2(n + 1) - \rho_2(n) = 3$. □

Remark 4.4. *The sequence $\{R_n\}$ is thus analogous to the alternative form of Stern's sequence:*

$$a_{2n} = a_n, a_{2n-1} = a_n + a_{n-1}.$$

For every word $\omega := \omega_0\omega_1...\omega_n \in \{0, 1\}^*$, we let $|\omega| := n + 1$ denote the length of ω and define a point in the complex plane

$$P(\omega) := \sum_{k=0}^{n} \phi^{-k}(2\omega_k - 1 - i).$$

We form a graph **G** by putting an edge between $P(\omega)$ and $P(\omega j)$ for $j = 0, 1$, $\omega \in \{0, 1\}^*$. This graph is illustrated in Figure 4 below. Note further that $P(\omega) = P(\omega')$ iff $|\omega| = |\omega'|$ and $[\omega] = [\omega']$. Hence, we may consistently assign the integer $[\omega]$ to each vertex $P(\omega)$ of the graph. This shows that $R_{[\omega]}$ is the number of downward paths from $P(*)$ to $P(\omega)$ and the graph can be thought of as a kind of hyperbolic Pascal's triangle. In fact, the portion between 0,01,010,0101,... and 1,10,101,1010,... is really just the "Fibonacci diatomic array" appearing in [2].

For v a vertex of the Fibonacci representation graph, let $[v]$ be the number of downward paths from the top vertex to v.

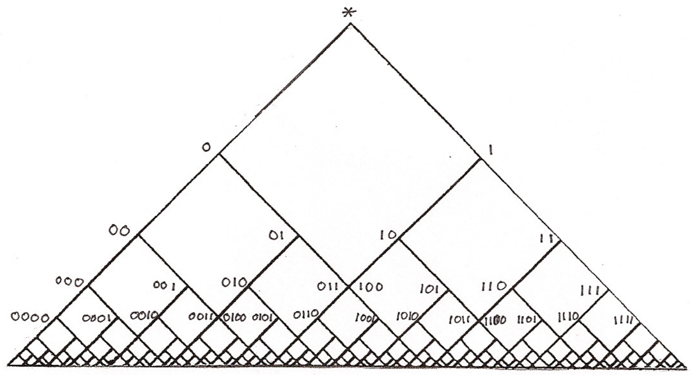


FIGURE 4. Fibonacci Representation Graph with words in $\{0,1\}^*$.

Lemma 4.5. *Along the nth row of the graph* $\mathbf{G}$*, the function* $[v]$ *forms an increasing sequence of consecutive integers* $0, \ldots, F_{n+2} - 2$.

Proof. Iterates of $\rho(n)+1$ starting at 0 yields the sequence $0, 1, 3, 6, 11, ..., F_{n+2}-2, ...$ (provable by induction). Hence the last value of $[v]$ in the nth row is $F_{n+2} - 2$. Since $\rho(n+1) - \rho(n) \in \{1, 2\}$, the lemma follows. $\square$

A consequence is the following surprising formula:

$$\rho(\rho(\rho(n) + 1))) = \rho(\rho(\rho(n)) + 1) + 1.$$

This graph has numerous interesting properties:

- Every quadrilateral in the closure of the graph is either a square or a golden rectangle.
- All the squares (actually hexagons) are congruent in hyperbolic space with area $\ln \phi$ (and, as hexagons, each edge has length $\ln \phi$). The figure is thus an aperiodic tiling of part of the upper half-plane $\mathbf{H}$ (and can be extended to all of $\mathbf{H}$) where all the tiles are congruent!
- The points along any row, when embedded in $\mathbb{R}$ form part of a one-dimensional quasicrystal. The lengths of the segments, appropriately scaled, form a word: $\phi, 1, \phi, \phi, 1, \phi, ...$, the "Fibonacci word".
- The vertices form a quasicrystal in $\mathbf{H}$.
- The graph is the Cayley graph of the "Fibonacci monoid" $\langle a, b | abb = baa \rangle$.
- The graph can be constructed by the following recursive procedure starting with a single vertex; from each of the latest generation of vertices, draw two edges going southeast and southwest respectively, connect if a hexagon can be formed. Repeat.

Something new with respect to the study of $\{R_n\}$ is the development of an analog of Conway's box function. For $k < F_{n-1}$, define

$$q(k, F_n) := R_k / R_{F_n + k}.$$

Lemma 4.6. *For* $k = 0, ..., F_{n-1} - 1$,

$$q(T(k), F_{n+1}) = q(k, F_n)$$

and

$$q(\rho_2(k), F_{n+2}) = q(k, F_n) * q(k-1, F_n)$$

where $*$ *denotes "mediant addition".*

Proof. Note that

$$T(n) = \rho(n+1) - 1$$

and so, if $k \le F_{n-1} - 1$,

$$T(F_n + k) = F_{n+1} + T(k).$$

Then

$$\begin{aligned} q(k, F_n) &= \frac{R_k}{R_{F_n+k}} = \frac{R_{T(k)}}{R_{T(F_n+k)}} \\ &= \frac{R_{T(k)}}{R_{F_{n+1}+T(k)}} = q(T(k), F_{n+1}) \end{aligned}$$

and the first equation follows. Similarly,

$$\begin{aligned} q(k, F_n) * q(k-1, F_n) &= \frac{R_k}{R_{F_n+k}} * \frac{R_{k-1}}{R_{F_n+k-1}} = \frac{R_k + R_{k-1}}{R_{F_n+k} + R_{F_n+k-1}} \\ &= \frac{R_{\rho_2(k)}}{R_{\rho_2(F_n+k)}} = \frac{R_{\rho_2(k)}}{R_{F_{n+2}+\rho_2(k)}} \\ &= q(\rho_2(k), F_{n+2}) \end{aligned}$$

and the second equation follows. □

As a consequence, if, as $n \to \infty$, k/F_n converges to $x \in [0, 1/\phi]$, then $q(k, F_n)$ converges to some value, say $Q(x)$. The function $Q : [0, 1/\phi] \to [0, 1]$ is increasing and continuous.

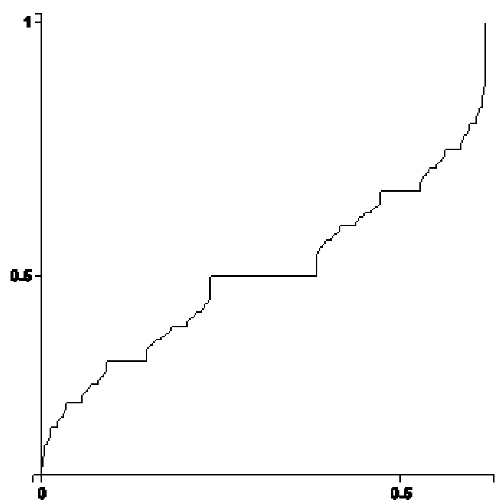


Figure 5. Analogue of Conway's box function

Note, however, it is not strictly increasing.

Lemma 4.7. *For* $j = 0, ..., F_{n-1} - 1$,

$$R_{F_{n+2}+j} = R_{F_n+j} + R_j.$$

Theorem 4.8. *The inverse of Q satisfies, on its irrational points of continuity,*

$$Q^{-1}(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\phi^{2(c_1+c_2+\ldots+c_k)-1}}$$

where x has continued fraction decomposition $x = 1/(c_1 + 1/(c_2 + 1/(c_3 + \ldots)))$.

Proof. Recall that $R_{F_m+k} = R_{F_{m+1}-k}$ and so

$$\frac{1}{n+q(k,F_m)} = \frac{R_{F_{m+1}-k}}{R_k + nR_{F_{m+1}-k}} = q(F_{m+1}-k, F_{m+2n}).$$

Letting $k/F_m \to x$ where x is a point of continuity of B, we see that

$$\frac{1}{n+Q(x)} = Q\left(\frac{\phi - x}{\phi^{2n}}\right).$$

We may then rewrite:

$$\frac{\phi - Q^{-1}(x)}{\phi^{2n}} = Q^{-1}\left(\frac{1}{n+x}\right)$$

and the theorem follows. □

The function $Q(x)$ extends past $1/\phi$ but is no longer monotonic.

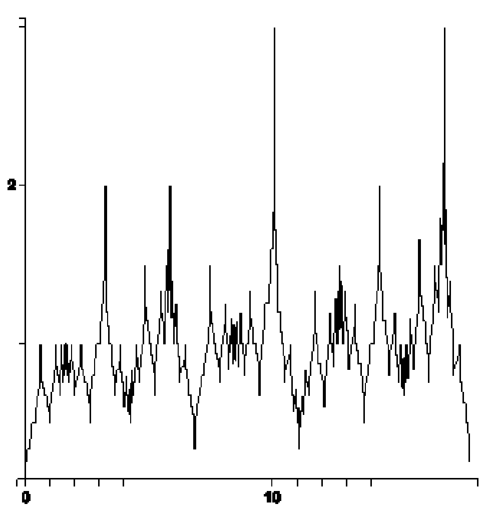


FIGURE 6. Analogue of Conway's box function; larger domain

Patterns can be found by looking at the "crushed array" which is found by stacking rows of terms $R_{F_n-1}, ..., R_{F_{n+1}-2}$ sliding terms to the left on rows:

$$
\begin{array}{ccccccccccccccccc}
1 \\
1 & 2 \\
1 & 2 & 2 \\
1 & 3 & 2 & 2 & 3 \\
1 & 3 & 3 & 2 & 4 & 2 & 3 & 3 \\
1 & 4 & 3 & 3 & 5 & 2 & 4 & 4 & 2 & 5 & 3 & 3 & 4 \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & .
\end{array}
$$

The kth column satisfies: $x_{n+2} = x_n + c$ with common difference $c = R_k$ $(R_0 = 0)$.

Alternatively, $x_{n+1} = x_n + x_{n-1} - x_{n-2}$ (a "dying rabbit" sequence).

$$x_{n+1} = x_n + x_{n-1} - x_{n-2}$$

Characteristic polynomial factors $x^3 - x^2 - x + 1 = (x-1)^2(x+1)$ so every example is of the form $x_n = a + bn + c(-1)^n$. Hence, $\{x_{2n}\}$ and $\{x_{2n+1}\}$ are arithmetic sequences.

$$x_{n+1} = x_n + x_{n-1} - x_{n-3}$$

e.g., [12, A023434] $x^4 - x^3 - x^2 + 1 = (x-1)(x^3 - x - 1)$, so every example is of the form $a + br_1^n + cr_2^n + dr_3^n$ where r_1 is the "plastic constant", 1.32471795..., the smallest Pisot number, and r_2, r_3 are its algebraic conjugates. Such examples are always a constant plus a Padovan sequence $y_{n+1} = y_{n-1} + y_{n-2}$. E.g., [12, A000931]

$$x_{n+1} = x_n + x_{n-1} - x_{n-1},$$

is always a constant sequence.

5. Extending Binet's formula

Let $s_F(n)$ be the number of terms in the Zeckendorf representation of n (e.g., $s_F(27) = 3$). Equivalently, $s_F(n)$ is the least number of Fibonacci numbers that sum to n. This sequence, for $n = 0, 1, ...$, is [A007895] and starts

$$0, 1, 1, 1, 2, 1, 2, 2, 1, 2, 2, 2, 3, 1, 2, 2, 2, 3, 2, 3, 3, ...$$

Using notation of the previous section, we see that $s_F(n)$ satisfies the recursion:

$$s_F([\omega 0]) = s_F([\omega]), s_F([\omega 01]) = s_F([\omega]) + 1$$

which translates to

$$s_F(\rho(n)) = s_F(n), s_F(\rho_2(n) + 1) = s_F(n) + 1$$

where $\rho(n)$ is the "Fibonacci shift" defined in Section 4 (just after Proposition 4.1). The crushed array for this sequence is

$$\begin{array}{ccccccccccccccccc}
1 \\
1 \\
1 & 2 \\
1 & 2 & 2 \\
1 & 2 & 2 & 2 & 3 \\
1 & 2 & 2 & 2 & 3 & 2 & 3 & 3 \\
1 & 2 & 2 & 2 & 3 & 2 & 3 & 3 & 2 & 3 & 3 & 3 & 4 \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & .
\end{array}$$

Note that columns are constant and that the limiting row is $s_F(n)+1$.

Replacing $s_2(n)$ by $s_F(n)$ in Binet's formula for Stern's sequence yields our third variant of Stern's sequence:

$$c_{n+1} = \sum_{k=0}^{n} \sigma^{s_F(k)} \overline{\sigma}^{s_F(n-k)}.$$

The sequence starts, for $n = 1, 2, \ldots,$

$$1, 1, 2, 3, 2, 4, 3, 3, 6, 4, 6, 6, 4, 8, 6, 7, 10, 6, 9, 7, 5, 11, 8, \ldots.$$

It is always integral since c_{n+1} is an algebraic integer in $\mathbb{Z}[\sigma]$ invariant under complex conjugation.

A crushed array for this sequence is:

$$\begin{array}{ccccccccccccc}
1 \\
1 \\
2 & 3 \\
2 & 4 & 3 \\
3 & 6 & 4 & 6 & 6 \\
4 & 8 & 6 & 7 & 10 & 6 & 9 & 7 \\
5 & 11 & 8 & 11 & 13 & 8 & 14 & 10 & 9 & 15 & 9 & 13 & 11 \\
7 & 15 & 11 & 15 & 19 & 12 & 19 & 14 & 11 & 21 & 14 & 19 & 19
\end{array}$$

The first column, $x_n := \{c_{F_n}\}$ apparently satisfies the Padovan recurrence: $x_{n+2} = x_n + x_{n-1}$. Moreover, every column is apparently a "dying rabbit" sequence: $x_{n+1} = x_n + x_{n-1} - x_{n-3}$ or, more precisely, if $x_n := c_{F_n+k} + c_k$, then $x_{n+2} = x_n + x_{n-1}$. This is indeed the case which we now prove.

Theorem 5.1. *For $k \leq F_{n-2}$, $c_{F_{n+2}+k} = c_{F_n+k} + c_k + c_{F_{n-1}+k}$.*

Proof. Given a string X of integers, let $\overline{X}$ denote the reverse of string X, let X^+ denote the string X with 1 added to every integer, and let X^- denote the string X with 1 subtracted from every integer (e.g., if $X = 1223$, then $\overline{X} = 3221$, $X^+ = 2334$, and $\overline{X}^- = 2110$). If $X := t_0 \ldots t_{k-1}$, let $G(X) := \sum_{j=0}^{k-1} \sigma^{t_j}$. Finally, given strings X, Y, we let XY denote the concatenation of the two strings and $X - Y$ denote the pointwise difference (e.g., if $X = 457$ and $Y = 123$ then $XY = 457123$ and $X - Y = 334$).

Let $s_n := s_F(n)$ be the number of terms in the Zeckendorf representation of n. For any interval I, let s_I denote the *string* $s_{i_1} s_{i_2} \ldots s_{i_k}$ where $i_1 < i_2 < \ldots < i_k$ and $\{i_1, i_2, \ldots, i_k\} = I \cap \mathbb{N}$. Then $c_n = G((s_{[0,n)} - \overline{s}_{[0,n)}))$ where the difference of two strings is the string of differences. Since we will use this formula, we let $\Delta_I = s_I - \overline{s}_I$ so that $c_n = G(\Delta_{[0,n)})$.

By the definition of s, it's clear that $s_{[F_n,F_n+k)} = s^+_{[0,k)}$ if $k \le F_{n-1}$. Since

$$s_{[0,F_n+k)} = s_{[0,k)}s_{[k,F_n)}s_{[F_n,F_n+k)} = s_{[0,k)}s_{[k,F_n)}s^+_{[0,k)},$$

it follows that

$$\overline{s}_{[0,F_n+k)} = \overline{s}^+_{[0,k)}\overline{s}_{[k,F_n)}\overline{s}_{[0,k)},$$

and thus

$$\Delta_{[0,F_n+k)} = \Delta^-_{[0,k)}\Delta_{[k,F_n)}\Delta^+_{[0,k)}.$$

Hence, because $\sigma^{-1} + \sigma = 1$,

$$c_{F_n+k} = \sigma^{-1}c_k + G(\Delta_{[k,F_n)}) + \sigma c_k = c_k + G(\Delta_{[k,F_n)}). \qquad (*)$$

Assuming $k \le F_{n-2}$, we see that

$$\begin{aligned} s_{[0,F_{n+2}+k)} &= s_{[0,k)}s_{[k,F_n)}s_{[F_n,F_n+k)}s_{[F_n+k,F_{n+1})}s_{[F_{n+1},F_{n+1}+k)}s_{[F_{n+1}+k,F_{n+2})}s_{[F_{n+2},F_{n+2}+k)} \\ &= s_{[0,k)}s_{[k,F_n)}s^+_{[0,k)}s^+_{[k,F_{n-1})}s^+_{[0,k)}s^+_{[k,F_n)}s^+_{[0,k)} \end{aligned}$$

and thus

$$\overline{s}_{[0,F_{n+2}+k)} = \overline{s}^+_{[0,k)}\overline{s}^+_{[k,F_n)}\overline{s}^+_{[0,k)}\overline{s}^+_{[k,F_{n-1})}\overline{s}^+_{[0,k)}\overline{s}_{[k,F_n)}\overline{s}_{[0,k)}.$$

Hence,

$$\Delta_{[0,F_{n+2}+k)} = \Delta^-_{[0,k)}\Delta^-_{[k,F_n)}\Delta_{[0,k)}\Delta_{[k,F_{n-1})}\Delta_{[0,k)}\Delta^+_{[k,F_n+k)}\Delta^+_{[0,k)}.$$

Applying G:

$$c_{F_{n+2}+k} = \sigma^{-1}c_k + \sigma^{-1}G(\Delta_{[k,F_n)}) + c_k + G(\Delta_{[k,F_{n-1})}) + c_k + \sigma G(\Delta_{[k,F_n)}) + \sigma c_k.$$

Again, since $\sigma^{-1} + \sigma = 1$, and by (*), we have

$$c_{F_{n+2}+k} = 3c_k + G(\Delta_{[k,F_n)}) + G(\Delta_{[k,F_{n-1})}) = c_k + c_{F_n+k} + c_{F_{n-1}+k}.$$

□

There are many patterns in the crushed array. Two such patterns can be proven by induction based on the previous theorem.

Corollary 5.2. *$c_{F_n} + c_{F_{n-1}+2} = c_{F_n+1}$ and $c_{F_n+1} = c_{F_{n+1}+2}$ for all n.*

We have many other questions or *apparent* properties, all waiting for a proof (though, of course, of varying difficulty).

- Five inequalities: $c_{\sigma_2(n)+1} \ge c_{\lfloor n\phi^2 \rfloor} \ge c_{\lfloor n\phi \rfloor} \ge c_{\sigma(n)} \ge c_n \ge 0$.
- The minimum of each row in the crushed array is the leftmost element. (If true, then the last inequality above, $c_n \ge 0$, is true).
- If $c_n \ge 0$ for all n, then what do these numbers count?
- The following sequences have crushed arrays with columns satisfying $x_{n+1} = x_n + x_{n-1} - x_{n-j}$ for given j:

  $$\{s_F(n)\} \text{ has } j = 1,$$
  $$\{R_n\} \text{ has } j = 2,$$
  $$\{c_n\} \text{ has } j = 3.$$

  Is there a general principle at work in this progression? Is there a similarly defined sequence with $j = 4$ for example?

Acknowledgement

I thank an anonymous referee for several helpful suggestions and for a considerably improved proof of Lemma 3.5.

References

[1] M. Bicknell-Johnson, *Stern diatomic array applied to Fibonacci representations*, Fibonacci Quart. **41** (2003) 169-180.

[2] M. Bicknell-Johnson, *The Fibonacci diatomic array applied to Fibonacci representations*, in Applications of Fibonacci Numbers, vol. 9, Kluwer Academic, Dordrecht, 2004, 29–38.

[3] M. Coons and J. Tyler, *The maximal order of Stern's diatomic sequence*, preprint, ArXiv:1307.1521v2 [math.NT] 17 Jun 2014.

[4] N. Calkin and H. S. Wilf, *Recounting the rationals*, Amer. Math. Monthly **107** (2000) 360-367.

[5] H. M. Edwards, *Riemann's Zeta Function*, Pure and Applied Mathematics, vol. 58, Academic Press, New York, 1974.

[6] R. Graham, D. Knuth, O. Patashnik, *Concrete Mathematics, 2nd ed.*, Addison-Wesley, New York (1994).

[7] D. A. Klarner, *Partitions of N into Distinct Fibonacci Numbers*. Fibonacci Quart. 6 1968 no. 4, 235-244.

[8] D. H. Lehmer, *On Stern's diatomic sequence*, Amer. Math. Monthly **36** (1929) 59-67.

[9] S. Northshield, *Stern's diatomic sequence 0,1,1,2,1,3,2,3,1,4,...*. Amer. Math. Monthly 117 (2010), no. 7, 581-598.

[10] S. Northshield, *Sums across Pascal's triangle mod 2*, Congr. Numer. **200** (2010), 35-52.

[11] S. Northshield, *Ford Circles and Spheres*, preprint.

[12] OEIS Foundation Inc. (2011), The On-Line Encyclopedia of Integer Sequences, `http://oeis.org`.

[13] R. P. Stanley and H. S. Wilf, *Refining the Stern Diatomic Sequence*, preprint http://www-math.mit.edu/ rstan/papers/stern.pdf

[14] P. Stockmeyer, *A smooth tight upper bound for the Fibonacci representation function R(N)*. Fibonaci Quart. 46/47 (2008/2009), no. 2, 103-106.

MSC2010: 11B83

Department of Mathematics, SUNY-Plattsburgh, Plattsburgh, NY 12901
E-mail address: `northssw@plattsburgh.edu`

BALANCING-LIKE SEQUENCES ASSOCIATED WITH INTEGRAL STANDARD DEVIATIONS OF CONSECUTIVE NATURAL NUMBERS

G. K. PANDA AND A. K. PANDA

ABSTRACT. The variance of first n natural numbers is $\frac{n^2-1}{12}$ and is a natural number if n is odd, $n > 1$ and is not a multiple of 3.The values of n corresponding to integral standard deviations constitute a sequence behaving like the sequence of Lucas-balancing numbers and the corresponding standard deviations constitute a sequence having some properties identical with balancing numbers. The factorization of the standard deviation sequence results in two other interesting sequences sharing important properties with the two original sequences.

1. INTRODUCTION

The concept of balancing numbers was first given by Behera and Panda [1] in connection with the Diophantine equation $1+2+\cdots+(n-1) = (n+1)+(n+2)+\cdots+(n+r)$, wherein, they call n a balancing number and r the balancer corresponding to n. The n^{th} balancing number is denoted by B_n and the balancing numbers satisfy the binary recurrence $B_{n+1} = 6B_n - B_{n-1}$ with $B_0 = 0$ and $B_1 = 1$ [1]. In [3], Panda explored many fascinating properties of balancing numbers, some of them are similar to the corresponding results on Fibonacci numbers, while some others are more exciting.

A detailed study of balancing and some related number sequences is available in [5]. In a latter paper [4], as a generalization of the sequence of balancing numbers, Panda and Rout studied a class of binary recurrences defined by $x_{n+1} = Ax_n - Bx_{n-1}$ with $x_0 = 0$ and $x_1 = 1$ where A and B are any natural numbers. They proved that when $B = 1$ and $A \notin \{1, 2\}$, sequences arising out of these recurrences have many important and interesting properties identical to those of balancing numbers. We, therefore, prefer to call this class of sequences as balancing-like sequences.

For each natural number n, $8B_n^2 + 1$ is a perfect square and $C_n = \sqrt{8B_n^2 + 1}$ is called a Lucas-balancing number [5]. We can, therefore, call $\{C_n\}$, the Lucas-balancing sequence. In a similar manner, if x_n is a balancing-like sequence with $kx_n^2 + 1$ is a perfect square for some natural number k and for all n and $y_n = \sqrt{kx_n^2 + 1}$, we call $\{y_n\}$ a Lucas-balancing-like sequence.

Khan and Kwong [2] called sequences arising out of the above class of recurrences corresponding to $B = 1$ as generalized natural number sequences because of their similarity with natural numbers with respect to certain properties. Observe that, the sequence of balancing numbers is a member of this class corresponding to $A = 6, B = 1$. In this paper, we establish the close association of another sequence of this class to an interesting Diophantine problem of basic statistics.

The variance of the real numbers x_1,x_2,$\cdots$,x_n is given by $\frac{1}{n}\Sigma_{i=1}^{n}(x_i-\bar{x})^2$, where $\bar{x} = \frac{1}{n}\Sigma_{i=1}^{n}x_i$ is the mean of x_1,x_2,$\cdots$ x_n. Using the above formula, it can be checked that the variance of first n natural numbers (and hence the variance of any n consecutive natural numbers) is $s_n^2 = \frac{n^2-1}{12}$. It is easy to see that this variance is a natural number if and only if n is odd but not a multiple of 3. Our focus is on those values of n that correspond to integral values of the standard deviation s_n. Observe that for some N, s_N is a natural number say, $s_N = \sigma$ if $N^2 - 1 = 12\sigma^2$ which is equivalent to the Pell's equation $N^2 - 12\sigma^2 = 1$. The fundamental solution corresponds to $N_1 = 7$ and $\sigma_1 = 2$. Hence, the totality of solutions is given by

$$N_k + 2\sqrt{3}\sigma_k = (7 + 4\sqrt{3})^k; k = 1, 2, \cdots . \tag{1.1}$$

This gives

$$N_k = \frac{(7 + 4\sqrt{3})^k + (7 - 4\sqrt{3})^k}{2} \tag{1.2}$$

and

$$\sigma_k = \frac{(7 + 4\sqrt{3})^k - (7 - 4\sqrt{3})^k}{4\sqrt{3}}. \tag{1.3}$$

Because (N_k, σ_k) is a solution of the Pell's equation $N^2 - 12\sigma^2 = 1$, both N_k and σ_k are natural numbers for each k.

2. RECURRENCE RELATIONS FOR N_k AND σ_k

In the last section, we obtained the Binet forms for N_k and σ_k where σ_k is the standard deviation of N_k consecutive natural numbers. Notice that the standard deviation of a single number is zero and hence we may assume that $N_0 = 1$ and $\sigma_0 = 0$, and indeed, from the last section, we already have $N_1 = 7$ and $\sigma_1 = 2$. Observe that $u_n = (7+4\sqrt{3})^n$ and $v_n = (7-4\sqrt{3})^n$ both satisfy the binary recurrences

$$u_{n+1} = 14u_n - u_{n-1}, v_{n+1} = 14v_n - v_{n-1};$$

hence, the linear binary recurrences for both $\{N_k\}$ and $\{\sigma_k\}$ are given by

$$N_{k+1} = 14N_k - N_{k-1}; N_0 = 1, N_1 = 7$$

and

$$\sigma_{k+1} = 14\sigma_k - \sigma_{k-1}; \sigma_0 = 0, \sigma_1 = 2.$$

The first five terms of both sequences are thus $N_1 = 7, N_2 = 97, N_3 = 1351, N_4 = 18817, N_5 = 262087$ and $\sigma_1 = 2, \sigma_2 = 28, \sigma_3 = 390, \sigma_4 = 5432, \sigma_5 = 75658$. Using the above binary recurrences for N_k and σ_k, some useful results can be obtained. The following theorem deals with two identities in which N_k and σ_k behave like hyperbolic functions.

Theorem 2.1. *For natural numbers k and l, $\sigma_{k+l} = \sigma_k N_l + N_k\sigma_l$ and $N_{k+l} = N_k N_l + 12\sigma_k\sigma_l$.*

Proof. Since the identity

$$N_k + 2\sqrt{3}\sigma_k = (7 + 4\sqrt{3})^k$$

holds for each natural number k, it follows that

$$\begin{aligned} N_{k+l} + 2\sqrt{3}\sigma_{k+l} &= (7 + 4\sqrt{3})^{k+l} = (7 + 4\sqrt{3})^k(7 + 4\sqrt{3})^l \\ &= (N_k + 2\sqrt{3}\sigma_k)(N_l + 2\sqrt{3}\sigma_l) \\ &= (N_k N_l + 12\sigma_k\sigma_l) + 2\sqrt{3}(\sigma_k N_l + N_k\sigma_l) \end{aligned}$$

Comparing the rational and irrational parts,the desired follows. □

The following corollary is a direct consequence of Theorem 2.1

Corollary 2.2. *If $k \in N$, $\sigma_{k+1} = 7\sigma_k + 2N_k$, $N_{k+1} = 7N_k + 24\sigma_k$, $\sigma_{2k} = 2\sigma_k N_k$, $N_{2k} = N_k^2 + 12\sigma_k^2$.*

Theorem 2.1 can be used for the derivation of another similar result. The following theorem provides formulas for σ_{k-l} and N_{k-l} in terms of N_k, N_l, σ_k and σ_l.

Theorem 2.3. *If k and l are natural numbers with $k > l$, then $\sigma_{k-l} = \sigma_k N_l - N_k \sigma_l$ and $N_{k-l} = N_k N_l - 12\sigma_k \sigma_l$.*

Proof. By virtue of Theorem 2.1,

$$\sigma_k = \sigma_{(k-l)+l} = \sigma_{k-l}N_l + N_{k-l}\sigma_l$$

and

$$N_k = N_{(k-l)+l} = 12\sigma_{k-l}\sigma_l + N_{k-l}N_l.$$

Solving these two equations for σ_{k-l} and N_{k-l}, we obtain

$$\sigma_{k-l} = \frac{\begin{vmatrix} \sigma_k & \sigma_l \\ N_k & N_l \end{vmatrix}}{\begin{vmatrix} \mathbb{N}_l & \sigma_l \\ 12\sigma_l & N_l \end{vmatrix}} = \frac{\sigma_k N_l - N_k \sigma_l}{N_l^2 - 12\sigma_l^2}$$

and

$$N_{k-l} = \frac{\begin{vmatrix} N_k & \sigma_k \\ 12\sigma_l & N_k \end{vmatrix}}{\begin{vmatrix} \mathbb{N}_l & \sigma_l \\ 12\sigma_l & N_l \end{vmatrix}} = \frac{N_k N_l - 12\sigma_k \sigma_l}{N_l^2 - 12\sigma_l^2}.$$

Since for each natural number l, (N_l, σ_l) is a solution of the Pell equation $N^2 - 12\sigma^2 = 1$, the proof is complete. □

The following corollary follows from Theorem 2.3 in the exactly same way Corollary 2.2 follows from Theorem 2.1.

Corollary 2.4. *For any natural number $k > 1$, $\sigma_{k-1} = 7\sigma_k - 2N_k$ and $N_{k-1} = 7N_k - 24\sigma_k$.*

Theorems 2.1 and 2.3 can be utilized to form interesting higher order non-linear recurrences for both $\{N_k\}$ and $\{\sigma_k\}$ sequences. The following theorem is crucial in this regard.

Theorem 2.5. *If k and l are natural numbers with $k > l$, $\sigma_{k-1}\sigma_{k+1} = \sigma_k^2 - \sigma_l^2$ and $N_{k-l}N_{k+l} + 1 = N_k^2 + N_l^2$.*

Proof. By virtue of Theorems 2.1 and 2.3,

$$\sigma_{k-l}\sigma_{k+l} = \sigma_k^2 N_l^2 - N_k^2 \sigma_l^2$$

and since for each natural number r, $N_r^2 = 12\sigma_r^2 + 1$,

$$\sigma_{k-l}\sigma_{k+l} = \sigma_k^2(12\sigma_l^2 + 1) - \sigma_l^2(12\sigma_k^2 + 1) = \sigma_k^2 - \sigma_l^2.$$

Further,

$$N_{k-l}N_{k+l} = N_k^2 N_l^2 - 144\sigma_k^2\sigma_l^2 = N_k^2 N_l^2 - 144.\frac{N_k^2 - 1}{12}.\frac{N_l^2 - 1}{12}$$

implies

$$N_{k-l}N_{k+l} + 1 = N_k^2 + N_l^2.$$

□

The following corollary is a direct consequence of Theorem 2.5.

Corollary 2.6. *For any natural number $k > 1$, $\sigma_{k-1}\sigma_{k+1} = \sigma_k^2 - 4$ and $N_{k-1}N_{k+1} = N_k^2 + 48$.*

In view of Theorem 2.5, we also have $\sigma_{k+1}^2 - \sigma_k^2 = 2\sigma_{2k+1}$. Adding this identity for $k = 0, 1, \cdots, l-1$, we get the identity

$$2(\sigma_1 + \sigma_3 + \cdots + \sigma_{2l-1}) = \sigma_l^2.$$

This proves

Corollary 2.7. *Twice the sum first l odd ordered terms of the standard deviation sequence is equal to the variance of first N_l natural numbers.*

Again from Theorem 2.5,

$$7N_{2k+1} + 1 = N_{k+1}^2 + N_k^2$$

Summing over $k = 0$ to $k = l-1$, we find

Corollary 2.8. *For each natural number l, $7(N_1 + N_3 + \cdots + N_{2l-1}) + (l-1) = 2(N_1^2 + N_2^2 + \cdots + N_{l-l}^2) + N_l^2$.*

3. BALANCING-LIKE SEQUENCES DERIVED FROM $\{N_k\}$ AND $\{\sigma_k\}$

The linear binary recurrences for the sequences $\{N_k\}$ and $\{\sigma_k\}$ along with their properties suggest that$\{\frac{\sigma_k}{2}\}$ is a balancing-like sequence whereas $\{N_k\}$ is the corresponding Lucas-balancing-like sequence [3]. In addition, these sequences are closely related to two other sequences that can also be described by similar binary recurrences.

The following theorem deals with a sequence derived from $\{N_k\}$, the terms of which are factors of corresponding terms of the sequence $\{\sigma_k\}$.

Theorem 3.1. *For each natural number k, $\frac{N_k+1}{2}$ is a perfect square. Further, $M_k = \sqrt{\frac{N_k+1}{2}}$ divides σ_k.*

Proof. By virtue of Theorem 2.1 and the Pell's equation $N^2 - 12\sigma^2 = 1$

$$\frac{N_{2k}+1}{2} = \frac{N_k^2 + 12\sigma_k^2 + 1}{2} = N_k^2$$

implying that $M_{2k} = N_k$. Since $\sigma_{2k} = 2\sigma_k N_k$, M_{2k} divides σ_{2k} for each natural number k. Further

$$\begin{aligned}\frac{N_{2k+1}+1}{2} &= \frac{7N_{2k} + 24\sigma_k + 1}{2} = \frac{7(N_k^2 + 12\sigma_k^2) + 48\sigma_k N_k + 1}{2} \\ &= 84\sigma_k^2 + 24\sigma_k N_k + 4 = 36\sigma_k^2 + 24\sigma_k N_k + 4N_k^2 = (6\sigma_k + 2N_k)^2 = (7\sigma_k + 2N_k - \sigma_k)^2 \\ &= (\sigma_{k+1} - \sigma_k)^2\end{aligned}$$

from which we obtain $M_{2k+1} = \sigma_{k+1} - \sigma_k$. By virtue of Theorem 2.5, $\sigma_{k+1}^2 - \sigma_k^2 = 2\sigma_{2k+1}$ and thus

$$\sigma_{2k+1} = \frac{\sigma_{k+1} + \sigma_k}{2}(\sigma_{k+1} - \sigma_k) = \delta_k(\sigma_{k+1} - \sigma_k)$$

where $\delta_k = \frac{\sigma_{k+1}+\sigma_k}{2}$ is a natural number since σ_k is even for each k and hence M_{2k+1} divides σ_{2k+1}. □

We have shown while proving Theorem 3.1 that $M_{2k+1} = \sigma_{k+1} - \sigma_k$. Thus, we have

Corollary 3.2. *The sum of first l odd terms of the sequence $\{M_k\}$ is equal to the standard deviation of the first N_l natural numbers.*

By virtue of Theorem 3.1, M_k divides σ_k for each natural number k. Therefore, it is natural to study the sequence $L_k = \frac{\sigma_k}{M_k}$. From the proof of Theorem 3.1, it follows that $L_{2k} = 2\sigma_k$ and $L_{2k+1} = \frac{(\sigma_{k+1}+\sigma_k)}{2}$.

Our next objective is to show that the sequence $\{L_k\}_{k=1}^{\infty}$ is a balancing-like sequence and $\{M_k\}_{k=1}^{\infty}$ is the corresponding Lucas-balancing-like sequence. This claim is validated by the following theorem.

Theorem 3.3. *For each natural number k, $M_k^2 = 3L_k^2 + 1$. Further, the sequences $\{L_k\}_{k=1}^{\infty}$ and $\{M_k\}_{k=1}^{\infty}$ satisfy the binary recurrences $L_{k+1} = 4L_k - L_{k-1}, k \geq 1$ with $L_0 = 0$ and $L_1 = 1$ and $M_{k+1} = 4M_k - M_{k-1}, k \geq 1$ with $M_0 = 1$ and $M_1 = 2$.*

Proof. In view of the Pell's equation $N^2 - 12\sigma^2 = 1$, Corollary 2.4 and the discussion following Corollary 3.2,

$$3L_{2k}^2 + 1 = 3(2\sigma_k)^2 + 1 = N_k^2 = M_{2k}^2$$

and

$$\begin{aligned} 3L_{2k-1}^2 + 1 &= 3(\frac{\sigma_k + \sigma_{k-1}}{2})^2 + 1 = 3(4\sigma_k - N_k)^2 + 1 \\ &= (6\sigma_k - 2N_k)^2 = (\sigma_k - \sigma_{k-1})^2 = M_{2k-1}^2. \end{aligned}$$

To this end, using Corollary 2.2, we get

$$4M_{2k+1} - M_{2k} = 4(\sigma_{k+1} - \sigma_k) - N_k = 4(6\sigma_k + 2N_k) - N_k = N_{k+1} = M_{2k+2}$$

and

$$\begin{aligned} 4M_{2k} - M_{2k-1} &= 4N_k - (\sigma_{k+1} - \sigma_k) = 4N_k - (-6\sigma_k + 2N_k) \\ &= 6\sigma_k + 2N_k = \sigma_{k+1} - \sigma_k = M_{2k+1}. \end{aligned}$$

Thus, the sequence M_k satisfies the binary recurrence

$$M_{k+1} = 4M_k - M_{k-1}.$$

Similarly, the identities

$$4L_{2k+1} - L_{2k} = 2(\sigma_{k+1} + \sigma_k) - 2\sigma_k = 2\sigma_{k+1} = L_{2k+2}$$

and

$$4L_{2k} - L_{2k-1} = 8\sigma_k - \frac{\sigma_k + \sigma_{k-1}}{2} = 8\sigma_k - (4\sigma_k - N_k = 4\sigma_k + N_k = \frac{\sigma_k + \sigma_k}{2} = L_{2k+1}$$

confirm that the sequence L_k satisfies the binary recurrences $L_{k+1} = 4L_k - L_{k-1}$. □

It is easy to check that the Binet forms of the sequences $\{L_k\}$ and $\{M_k\}$ are respectively

$$L_k = \frac{(2+\sqrt{3})^k - (2-\sqrt{3})^k}{2\sqrt{3}}$$

and

$$M_k = \frac{(2+\sqrt{3})^k + (2-\sqrt{3})^k}{2} k = 1, 2, \cdots .$$

Using the Binet forms or otherwise, the interested reader is invited the following identities.

(1) $L_1 + L_3 + \cdots + L_{2n-1}) = L_n^2$,
(2) $M_1 + M_3 + \cdots + M_{2n-1} = \frac{L_{2n}}{2}$,
(3) $L_2 + L_4 + \cdots + L_{2k} = L_k L_{k+1}$,
(4) $M_2 + M_4 + \cdots + M_{2k} = \frac{(L_{2k+1} - 1)}{2}$,

(5) $L_{x+y} = L_x M_y + M_x L_y$,
(6) $M_{x+y} = M_x M_y + 3L_x L_y$.

4. ACKNOWLEDGEMENT

It is a pleasure to thank the anonymous referee for his valuable suggestions and comments which resulted in an improved presentation of this paper.

References

[1] A. Behera and G. K. Panda, *On the square roots of triangular numbers*, Fib. Quart., **37**(2)(1999), 98–105.
[2] M. A. Khan and H. Kwong, *Some Binomial Identities Associated with the Generalized Natural Number sequence*, Fib. Quart., **49**(1)(2011), 57–65.
[3] G. K. Panda, *Some fascinating properties of balancing numbers*, In Proc. Eleventh Internat. Conference on Fibonacci Numbers and their Applications, Cong Numerantium, **194**(2009), 185–189.
[4] G. K. Panda and S. S. Rout, *A class of recurrent sequences exhibiting some exciting properties of balancing numbers*, Int. J. Math. Comp. Sci., **6**(2012), 4-6.
[5] P. K. Ray,*Balancing and cobalancing numbers*, Ph.D thesis, submitted to National Institute of Technology, Rourkela, 2009.

MSC2010(A.M.S.): 11 B 37, 11 B 83

Department of Mathematics, National Institute of Technology Rourkela, Odisha, 769 008, India
E-mail address: gkpanda_nit@rediffmail.com

Department of Mathematics, National Institute of Technology Rourkela, Odisha, 769 008, India
E-mail address: akpandamath@gmail.com

NORMIC CONTINUED FRACTIONS IN TOTALLY AND TAMELY RAMIFIED EXTENSIONS OF LOCAL FIELDS

PANTELIMON STĂNICĂ

Abstract. The goal of this paper is to introduce a new way of constructing continued fractions in a Galois, totally and tamely ramified extension of local fields. We take a set of elements of a special form using the norm of that extension and we show that the set such defined is dense in the field by the means of continued fractions.

1. Introduction

A ring A is a discrete valuation ring (DVR) if it has a unique maximal ideal $\mathbb{m}_A$, it is a principal ideal domain, but not a field. The residue field of A is the quotient field $\overline{\mathbb{k}}_A = A/\mathbb{m}_A$. Recall that a complete discrete valuation ring is a DVR that is complete with respect to the topology in which $\{\mathbb{m}_A^n\}_{n\geq 1}$ forms a basis of open neighborhoods of 0; that is, every series $\sum_{j=0}^{\infty} a_j\pi^j$ converges to an element of A, where π is a generator (often called *uniformizer*) of the (principal) maximal ideal $\mathbb{m}_A$.

Throughout this paper, $\mathbb{k}$ denotes a local field with a discrete valuation $v_{\mathbb{k}}$, which is a field of fractions of a complete discrete valuation ring $A_{\mathbb{k}}$ [7, §2, P.3], with finite residue class fields. Its maximal ideal is $\pi_{\mathbb{k}}$, its finite residue field is $\overline{\mathbb{k}} = A_{\mathbb{k}}/\pi_{\mathbb{k}}$, and $U_{\mathbb{k}} = A_{\mathbb{k}} - \pi_{\mathbb{k}}$ is the multiplicative group of invertible elements of $A_{\mathbb{k}}$. The local fields are the p-adic fields, which are finite extensions of the field $\mathbb{Q}_p$ of p-adic numbers (characteristic char $= 0$), and the finite extensions of the power series field $\mathbb{F}_p((x))$ (case char $= p > 0$); these are also locally compact, but we do not need that here. We refer to [7, 3], for example, for more on this topic.

If $\mathbb{K}$ is a finite extension of $\mathbb{k}$ (here, we write this as $\mathbb{k} \hookrightarrow \mathbb{K}$), we denote by $A_{\mathbb{K}}$ the integral closure of $A_{\mathbb{k}}$ in $\mathbb{K}$. We define $v_{\mathbb{K}}$, $\Pi_{\mathbb{K}}$, $U_{\mathbb{K}}$, $\overline{\mathbb{K}}$ as before. We will always assume that $\mathbb{k} \hookrightarrow \mathbb{K}$ is Galois, totally and tamely ramified extension. The ramification index of $\mathbb{K}/\mathbb{k}$, which is the degree of this extension will be denoted by e. We also assume that $v_{\mathbb{k}}$ is the restriction to $\mathbb{k}$ of $v_{\mathbb{K}}$, so we will use the same notation v for both of them. Choose $\Pi \in \mathbb{K}$, $\pi \in \mathbb{k}$ prime elements, such that $\Pi^e = \pi$ (see [5, Theorem 5.11]). Denoting the norm of $\mathbb{K}/\mathbb{k}$ by $N_{\mathbb{K}/\mathbb{k}}$, it is known that

$$v(x) = \frac{1}{e} v\left(N_{\mathbb{K}/\mathbb{k}}(x)\right),\ \forall\ x \in \mathbb{K}.$$

and we may assume that $v(\Pi) = 1$ and $v(\pi) = e$.

For easy writing, we use the notation $[\alpha, \beta, \gamma, \ldots]$ to mean

$$\alpha + \cfrac{1}{\beta + \cfrac{1}{\gamma + \cfrac{1}{\ddots}}}.$$

We want to mention that there are several nonequivalent definitions of continued fractions in the the field $\mathbb{Q}_p$ of p-adic numbers (see [1, 2] and the references therein). There are similarities as well as differences between these definitions and the classical real continued fractions.

Among other continued fractions approaches, we want to mention the expansion of $\alpha \in \mathbb{Q}_p$ in the form

$$\alpha = [b_0, b_1, \ldots],$$

where $b_j \in \mathbb{Z}\left[\frac{1}{p}\right] \cap (0, p)$ (see Ruban [6]), and $b_j \in \mathbb{Z}\left[\frac{1}{p}\right] \cap \left(-\frac{p}{2}, \frac{p}{2}\right)$ (see Browkin [1, 2] and the references therein).

The groups of norms in such extensions play a very important role in class field theory. The goal of this paper is to introduce a new way of constructing continued fractions in a Galois, totally and tamely ramified extension of local fields $\mathbb{K}/\mathbb{k}$. We take a set of elements of a special form using the norm of that extension and we show that the defined set is dense in the larger field $\mathbb{K}$ by the means of continued fractions. This will give a glance to the "topological distance" between the set of norms of $\mathbb{K}/\mathbb{k}$ and $\mathbb{K}$. The approximation will be exact, and we will give the degree of the approximation as exact as we can by our method. In the last section we solve an equation in two variables using our continued fraction expansion.

We take $A \cup \{0\}$ to be a complete system of representatives of $\overline{\mathbb{K}} = \overline{\mathbb{k}}$, such that $A \subset A_{\mathbb{k}}$, $A^p = A$ where p is the (prime) characteristic of the residue field $\overline{\mathbb{k}}$. A has the structure of a group that is isomorphic to $\overline{\mathbb{k}} - \{0\} = \overline{\mathbb{K}} - \{0\}$ [4, Theorem 4.10]. Put

$$\begin{aligned}\Re := \big\{ [\Pi^{p_1} N_0 c_0, \ldots, \Pi^{p_s} N_s c_s] \mid p_i \in (1-e)\mathbb{Z},\ c_i \in A^{1-e} \\ \text{and } \exists x_i \in \mathbb{K},\ N_{\mathbb{K}/\mathbb{k}}(x_i) = N_i,\ i = 1, \ldots, s \big\}.\end{aligned}$$

We define the *choice map* $c : U_{\mathbb{K}} \to A^* = A - \{0\}$ by $c(u) := a$, where a is the unique element of A such that $u \equiv a \pmod{\Pi}$ [4, Theorem 4.10]. The map c has the following properties:

(*i*) c is surjective and $c_{|A} = 1_A$.
(*ii*) $c(u_1 u_2) = c(u_1) c(u_2)$.
(*iii*) $c(u^{-1}) = c(u)^{-1}$.

2. The normic continued fractions approach

We shall need the following lemma.

Lemma 2.1. *We have*

$$v\left(1 + \Pi x - N_{\mathbb{K}/\mathbb{k}}(1 + \Pi x)\right) \geq 1 + v(x), \ \textit{whenever } v(x) \geq 0.$$

Proof. We have

$$\begin{aligned} 1 + \Pi x - N_{\mathbb{K}/\mathbb{k}}(1 + \Pi x) &= 1 + \Pi x - (1 + \Pi x)(1 + \Pi^{(1)} x) \cdots (1 + \Pi^{(e-1)} x) \\ &= \Pi x - Tr_{\mathbb{K}/\mathbb{k}}(\Pi x) - \sum \Pi^{(i)} \Pi^{(j)} x^{(i)} x^{(j)} - \cdots \end{aligned}$$

where $x^{(i)}, \Pi^{(i)}$ are the conjugates of x, Π in the extension. Since $v(\Pi^{(i)}) = v(\Pi)$ and $v(x^{(i)}) = v(x)$ for all conjugates $\Pi^{(i)}$ of Π and $x^{(i)}$ of x, we get

$$v\left(1 + \Pi x - N_{\mathbb{K}/\mathbb{k}}(1 + \Pi x)\right) \geq \min\left(v(\Pi x), v\left(Tr_{\mathbb{K}/\mathbb{k}}(\Pi x)\right), \ldots\right) = 1 + v(x)$$

when $v(x) \geq 0$. We have used here the fact that we deal with local fields, hence with Henselian fields (fields where Hensel's lemma holds, that is, a simple root in a residue field can be lifted in the field above). □

Take an element $\alpha \in \mathbb{K} - \{0\}$, and define the (finite or infinite) sequences $\{\alpha_n\}_n$, $\{a_n\}_n$, $\{u_n\}_n$ as follows:

$$\alpha_0 := \alpha, a_0 := N_{\mathbb{K}/\mathbb{k}}(\alpha), u_0 := \alpha \Pi^{-v(\alpha)}$$

If α_n, a_n, u_n are defined, then

$$\alpha_{n+1} := \left(\alpha_n - c(u_n)^{1-e}\Pi^{(1-e)v(\alpha_n)}N_{\mathbb{K}/\mathbb{k}}(\alpha_n)\right)^{-1}, \tag{2.1}$$

(if the inverse exists, otherwise the sequence "*terminates*" at n)

$$a_{n+1} := N_{\mathbb{K}/\mathbb{k}}(\alpha_{n+1}), u_{n+1} := \alpha_{n+1}\Pi^{-v(\alpha_{n+1})},$$

where c is the choice map defined in Section 1. Putting

$$\alpha_n = \Pi^{v(\alpha_n)}u_n = \Pi^{v(\alpha_n)}c(u_n)u'_n$$

where u'_n is a unit in $U_{\mathbb{K}}$ which starts with 1 in the canonical expansion after powers of Π and coefficients in A, that is, $u'_n = 1 + \Pi x_n$ and $v(x_n) \geq 0$, we see that (2.1) can be rewritten in the following form:

$$\alpha_{n+1} = (c(u_n))^{-1}\Pi^{-v(\alpha_n)}\left(u'_n - N_{\mathbb{K}/\mathbb{k}}(u'_n)\right)^{-1}. \tag{2.2}$$

Thus, the sequence terminates if $u'_n - N_{\mathbb{K}/\mathbb{k}}(u'_n) = 0$ (we will deal with this condition in Theorem 3.5).

Our intuition tells us that $\alpha \neq 0$ can be expanded as

$$c(u_0)^{1-e}a_0\Pi^{(1-e)v(\alpha_0)} + \cfrac{1}{c(u_1)^{1-e}a_1\Pi^{(1-e)v(\alpha_1)} + \cfrac{1}{c(u_2)^{1-e}a_2\Pi^{(1-e)v(\alpha_2)} + \cfrac{1}{\ddots}}}$$

and proving this and other basic properties will be our goal in the main section of this paper.

3. The results

We start with a lemma on the valuation of α_n.

Lemma 3.1. *With the notations of the previous section, let*

$$t_n := v\left({u_n}' - N_{\mathbb{K}/\mathbb{k}}({u_n}')\right).$$

We assume that $N_{\mathbb{K}/\mathbb{k}}({u_n}') \neq {u_n}'$, hence $t_n < \infty$. Then

$$v(\alpha_n\alpha_{n+1}) = -t_n < 0, \text{ for all } n \in \mathbb{N}. \tag{3.1}$$

Furthermore,

$$v(\alpha_{n+1}) = -t_n + t_{n-1} + \cdots + (-1)^n t_0 + (-1)^n v(\alpha_0), \text{ for all } n \in \mathbb{N}.$$

Proof. We first observe that α_{n+1} exists since $N_{\mathbb{K}/\mathbb{k}}({u_n}') \neq {u_n}'$. The first claim is immediate from Lemma 2.1 and equation (2.2). The last claim follows by induction. □

We will define now the approximation of elements of $\mathbb{K}$ with elements of $\Re$. Take

$$p_{-1} := 1,\ q_{-1} := 0,\ p_0 := a_0c(u_0)^{1-e}\Pi^{(1-e)v(\alpha_0)},\ q_0 := 1, \tag{3.2}$$

and

$$\begin{aligned} p_{n+1} :=& a_{n+1}c(u_{n+1})^{1-e}\Pi^{(1-e)v(\alpha_{n+1})}p_n + p_{n-1},\\ q_{n+1} :=& a_{n+1}c(u_{n+1})^{1-e}\Pi^{(1-e)v(\alpha_{n+1})}q_n + q_{n-1}, \end{aligned} \tag{3.3}$$

assuming that α_{n+1} defined by (2.1) exists. We will call $\left\{\frac{p_n}{q_n}\right\}_{n\in\mathbb{N}\cup\{-1\}}$ the *convergents* of α and we observe that they belong to the set $\Re$.

Lemma 3.2. *We have*

$$q_{n+1}p_n - p_{n+1}q_n = (-1)^n.$$

Proof. Follows from the definitions (3.2) and (3.3) of p_n and q_n. $\square$

Theorem 3.3. *Let $\alpha_0 \in \mathbb{K}^*$. We have $v(q_0) = 0$ and*

$$\begin{aligned} v(p_n) &= v(\alpha_0\alpha_1 \cdots \alpha_n) \\ v(q_n) &= v(\alpha_1\alpha_2 \ldots \alpha_n) \leq -\left[\frac{n+1}{2}\right] - \varepsilon\ v(\alpha_0),\ \textit{for all } n > 0, \end{aligned} \tag{3.4}$$

where $\varepsilon = 0,\ 1,$ if n is even, respectively, odd.

Proof. The first assertion follows from (3.2) and the second claim will be proved by induction. Obviously, from (3.2) and (3.3) we get $v(p_0) = v(\alpha_0)$ and $v(p_1) = v(\alpha_0\alpha_1)$. Now we show that

$$v(p_{n+1}) = v(\alpha_0\alpha_1 \cdots \alpha_{n+1}),$$

using the induction assumption. So,

$$\begin{aligned} v(p_{n+1}) &= v(a_{n+1}c(u_{n+1})^{(1-e)}\Pi^{(1-e)v(\alpha_{n+1})}p_n + p_{n-1}) \\ &= \min\{v(\alpha_0 \cdots \alpha_{n+1}), v(\alpha_0 \cdots \alpha_{n-1})\} \\ &= v(\alpha_0\alpha_1 \cdots \alpha_{n+1}), \end{aligned}$$

since $v(\alpha_n\alpha_{n+1}) = -t_n < 0$, according to the Lemma 3.1.

The second claim of (3.3) will also be proved by induction. From (3.3), for $n = 1$ we have

$$v(q_1) = (1-e)v(\alpha_1) + v(a_1) + v(q_0) = v(\alpha_1) + v(q_0) = v(\alpha_1).$$

Suppose that the assertion is true for $q_1, \ldots, q_n,$ for $n \geq 2$. Then,

$$\begin{aligned} v(q_{n+1}) &= v\left(c(u_{n+1})^{1-e}a_{n+1}\Pi^{(1-e)v(\alpha_{n+1})}q_n + q_{n-1}\right) \\ &= v(\alpha_{n+1}) + v(q_n) = v(\alpha_1\alpha_2 \cdots \alpha_{n+1}), \end{aligned}$$

since

$$\begin{aligned} v(a_{n+1}c(u_{n+1})^{1-e}\Pi^{(1-e)v(\alpha_{n+1})}q_n) &= (1-e)v(\alpha_{n+1}) + v(a_{n+1}) + v(q_n) \\ &= v(\alpha_{n+1}) + v(q_n) = v(\alpha_1\alpha_2 \cdots \alpha_{n+1}) \\ &< v(\alpha_1\alpha_2 \cdots \alpha_{n-1}) = v(q_{n-1}), \end{aligned}$$

using (3.1).

We now show the inequality (3.3) satisfied by $v(q_n)$. From Lemma 3.1 and the previous result of this theorem we have

$$\begin{aligned} v(q_{2m}) &= v(\alpha_1\alpha_2) + \cdots + v(\alpha_{2m-1}\alpha_{2m}) \\ &= -t_1 - t_2 - \cdots t_{2m-1} \leq -m \end{aligned}$$

and

$$\begin{aligned} v(q_{2m+1}) &= v(\alpha_0\alpha_1) + \cdots + v(\alpha_{2m}\alpha_{2m+1}) - v(\alpha_0) \\ &= -t_0 - t_1 - \cdots - t_{2m} - v(\alpha_0) \leq -(m+1) - v(\alpha_0). \end{aligned}$$

The theorem is shown. $\square$

Now we will study the behavior of the sequence $\left\{\frac{p_n}{q_n}\right\}_{n\in\mathbb{N}\cup\{-1\}}$. We shall prove now that our sequence is Cauchy and, consequently, it has a limit.

Theorem 3.4. *The sequence* $\left\{\frac{p_n}{q_n}\right\}_{n\in\mathbb{N}\cup\{-1\}}$ *is convergent and its limit is* α.

Proof. First observe that

$$v\left(\frac{p_{n+1}}{q_{n+1}}-\frac{p_n}{q_n}\right)=v\left(\frac{(-1)^{n+1}}{q_nq_{n+1}}\right)=-v(q_nq_{n+1})$$
$$=v(\alpha_0)+t_0+t_1+\cdots+t_n\geq n+1+v(\alpha_0)$$

and

$$v\left(\frac{p_s}{q_s}-\frac{p_r}{q_r}\right)\geq\min\left(v\left(\frac{p_s}{q_s}-\frac{p_{s-1}}{q_{s-1}}\right),\ldots,v\left(\frac{p_{r+1}}{q_{r+1}}-\frac{p_r}{q_r}\right)\right)$$
$$=v(\alpha_0)+t_0+t_1+\cdots+t_r\to\infty \text{ as } \; s,r\to\infty$$

assuming, without loss of generality, that $s\geq r$.

Next, take

$$v\left(\alpha-\frac{p_n}{q_n}\right)=v\left(\frac{(-1)^n}{q_n(\alpha_{n+1}q_n+q_{n-1})}\right)$$
$$=-v(q_n)-v(\alpha_{n+1}q_n+q_{n-1})$$

since

$$\alpha=\frac{\alpha_{n+1}p_n+p_{n-1}}{\alpha_{n+1}q_n+q_{n-1}},$$

which follows from our definition (2.1) of α_n. Now set $w_{n+1}:=\alpha_{n+1}q_n+q_{n-1}$ and estimate

$$\begin{aligned}w_{n+1}&=\alpha_{n+1}(q_n+\alpha_{n+1}^{-1}q_{n-1})\\&=\alpha_{n+1}\left(q_n+\left(\alpha_n-a_nc(u_n)^{1-e}\Pi^{(1-e)v(\alpha_n)}\right)q_{n-1}\right)\\&=\alpha_{n+1}\left(q_n+\alpha_nq_{n-1}-a_nc(u_n)^{1-e}\Pi^{(1-e)v(\alpha_n)}q_{n-1}\right)\\&=\alpha_{n+1}(\alpha_nq_{n-1}+q_{n-2})=\alpha_{n+1}w_n=\alpha_1\cdots\alpha_{n+1}.\end{aligned}\tag{3.5}$$

Hence

$$v\left(\alpha-\frac{p_n}{q_n}\right)=-v(q_n)-v(\alpha_1\cdots\alpha_{n+1})=-v(q_nq_{n+1})\to\infty,$$

as $n\to\infty$, so α is the limit of our sequence. □

It is known that in the classical case, finite continued fractions with integer terms represent rational numbers. We investigate the same problem next for our continued fraction expansion.

Theorem 3.5. *The sequence* $\{\alpha_n\}_n$ *is finite if and only if there exists* n *such that*

$$\alpha_n=a\xi_{e-1}\Pi^{v(\alpha_n)},\tag{3.6}$$

where $a\in A$ *and* ξ_{e-1} *is an* $(e-1)$*–root of unity in* $\Bbbk$.

Proof. Our sequence terminates if and only if there exists n such that

$$u_n-c(u_n)^{1-e}N_{\mathbb{K}/\Bbbk}(u_n)=0.$$

This is the same as saying that

$$1+\Pi x_n=N_{\mathbb{K}/\Bbbk}\left(1+\Pi x_n\right)\in\Bbbk$$

where $u_n = c(u_n)\,(1+\Pi x_n)$, for an element $x_n \in \mathbb{K}$ with $v(x_n) \geq 0$. So there exists an element $x_n' \in \Bbbk$ such that

$$x_n = x_n'\Pi^{e-1} \quad \text{and} \quad v(x_n') \geq 0.$$

We also must have the condition

$$N_{\mathbb{K}/\Bbbk}(1+\pi x_n') = 1+\pi x_n'$$

fulfilled, which is equivalent to (knowing that $(1+\pi x_n') \in \Bbbk$)

$$N_{\mathbb{K}/\Bbbk}(1+\pi x_n') = (1+\pi x_n')^e = 1+\pi x_n'. \tag{3.7}$$

Obviously, $1+\pi x_n'$ can never be zero, so the only case we could have (3.7) is when

$$(1+\pi x_n')^{e-1} = 1,$$

hence u_n must be of the form

$$u_n = c(u_n)\xi_{e-1} \quad \text{and} \quad \alpha_n = c(u_n)\xi_{e-1}\Pi^{v(\alpha_n)} \tag{3.8}$$

where $\xi_{e-1} = 1+\Pi x_n \in \Bbbk$ is an $(e-1)$-root of unity. $\square$

Remark 3.6. *In the p-adic field $\mathbb{Q}_p$, the condition* (3.6) *could be re-written as* $\mathrm{Log_p}(\alpha_\mathrm{n}) = 0$, *in terms of the analytic continuation of the usual logarithm, called the Iwasawa logarithm* $\mathrm{Log_p}$, *(for example, if $x \in \mathbb{Z}_p^*$, then* $\mathrm{Log_p(x)} = \frac{1}{\mathrm{p}-1}\mathrm{Log_p}(\mathrm{x}^{\mathrm{p}-1}) = \frac{1}{1-\mathrm{p}}\sum_{\mathrm{k}\geq 1}\frac{(1-\mathrm{x}^{\mathrm{p}-1})^\mathrm{k}}{\mathrm{k}}$*), but this gives no other indication on the set of elements of the form* (3.6).

4. An application

We will use our continued fraction process to solve an equation, namely

$$ax+by+d=0 \tag{4.1}$$

where

$$\gcd(a,b)=1 \quad \text{and} \quad a,b,d \in A_{\mathbb{K}}$$

are such that

$$\frac{a}{b} - c\left(\frac{a}{b}\Pi^{v(\frac{b}{a})}\right)^{(1-e)} \;\Pi^{(1-e)v(\frac{a}{b})}\; N_{\mathbb{K}/\Bbbk}\left(\frac{a}{b}\right) = \xi_{e-1}$$

is an $(e-1)$-root of unity in a Galois, totally and tamely ramified extension $\Bbbk \hookrightarrow \mathbb{K}$ of degree e and

$$v(d) \geq v\left(\frac{b}{a}\right).$$

We are looking for solutions in $A_{\mathbb{K}}$. Suppose that we found a solution of (4.1), say (x_0, y_0). Thus

$$ax_0+by_0+d=0. \tag{4.2}$$

Subtracting (4.2) from (4.1) we get

$$a(x-x_0)+b(y-y_0)=0$$

or

$$y-y_0 = \frac{a}{b}(x_0-x).$$

Since $\gcd(a,b)=1$ we must have $b\big|(x-x_0)$ in $A_{\mathbb{K}}$, so

$$\begin{aligned} x =& x_0 - bt \\ y =& y_0 + at \end{aligned} \tag{4.3}$$

for some $t \in A_{\mathbb{K}}$. So we have showed that if (x, y) is solution of (4.1), then it must satisfies (4.3) for some $t \in A_{\mathbb{K}}$. Conversely, we take (x_1, y_1) of the form (4.3) and we show that it is a solution of (4.1). We have

$$ax_1 + by_1 + d = ax_0 + by_0 + d + abt_1 - abt_1 = ax_0 + by_0 + d = 0.$$

We must find now a particular solution of (4.1). This can be done using our continued fraction expansion for $\alpha_0 = a/b$. We will use the notations of Section 2. Since

$$\alpha_1 = \left(\frac{a}{b} - c \left(\frac{a}{b} \Pi^{v(\frac{b}{a})} \right)^{(1-e)} \ \Pi^{(1-e)v(\frac{a}{b})} N_{\mathbb{K}/\mathbb{k}} \ \left(\frac{a}{b} \right) \right)^{-1}$$

is an $(e-1)$-root of unity this implies that α_2 does not exist. Hence

$$\frac{p_1}{q_1} = \frac{a}{b}$$

and

$$\frac{p_1}{q_1} - \frac{p_0}{q_0} = \frac{1}{q_1 q_0}$$

or

$$\frac{a}{b} - \frac{p_0}{q_0} = \frac{1}{bq_0}.$$

Furthermore, $aq_0 - bp_0 = 1$ or $aq_0 - bp_0 - 1 = 0$. Multiplying the previous relation by $-d$ we get

$$-adq_0 + bdp_0 + d = 0$$

and taking

$$\begin{aligned} x_0 &= -\, dq_0 = -d \\ y_0 &= dp_0 = d \ a_0 \ c \left(\frac{a}{b} \ \Pi^{v(\frac{a}{b})} \right)^{1-e} \ \Pi^{(1-e)v(\frac{a}{b})} \end{aligned} \tag{4.4}$$

we have produced a particular solution of (4.1) and consequently, we have found all the solution of our equation in algebraic integers of the extension $\mathbb{k} \hookrightarrow \mathbb{K}$. However we must make sure that our particular solution is in $A_{\mathbb{K}}$, so we have to check that both $v(x_0)$ and $v(y_0)$ are positive. We have no trouble with x_0 since $q_0 = 1$ and $d \in A_{\mathbb{K}}$. For y_0 we get

$$v(y_0) = v(d) + v(p_0) = v(d) + v \left(a_0 c (u_0)^{1-e} \Pi^{(1-e)v(\frac{a}{b})} \right) = v(d) + v \left(\frac{a}{b} \right) \geq 0$$

and we have solved the problem.

Acknowledgement. We thank the referee for a careful reading of the paper and for comments which improved its quality.

References

[1] J. Browkin, "Continued fractions in local fields", I. *Demonstratio Math.* **11** (1978), 67–82.
[2] J. Browkin, "Continued fractions in local fields", II. *Math. Comp.* **70:235** (2001), 1281–1292.
[3] K. Iwasawa, *Local Class Field Theory*, Oxford Univ. Press, New York, 1986.
[4] S. Iyanaga, *The Theory of Numbers*, Amsterdam-Oxford-New York, 1969.
[5] W. Narkiewicz, *Elementary and Analytic Theory of Algebraic Numbers*, Springer-Verlag, Warszawa, 1990.
[6] A. A. Ruban, "Certain metric properties of p-adic numbers", (Russian), Sibirsk Mat. Zh. **11** (1970), 222–227; English translation: Siberian Math. J. **11**, 176–180.
[7] J. P. Serre, *Local Fields*, Springer-Verlag, New York, 1979.
[8] J. Tate, "p-divisible groups", Proc. Conf. Local Fields (Driebergen, 1966), Springer–Berlin, 1967, 158–183.

Naval Postgraduate School, Applied Mathematics Department,, Monterey, CA 93943–5216, USA;, and *Institute of Mathematics "Simion Stoilow" of the Romanian Academy*, Bucharest, Romania
E-mail address: pstanica@nps.edu

COMPOSITIONS AND RECURRENCES

WILLIAM WEBB AND NATHAN HAMLIN

Abstract. If a_n denotes the number of compositions of n into parts in a set S, we show that a_n satisfies a recurrence equation if and only if $S = S_1 \cup S_2$ where S_1 is a finite set and $S_2 = \{k \geq k_0 : k \equiv r_1, r_2, ..., r_h \pmod{m}\}$.

1. Introduction

Let a_n denote the number of compositions of n subject to some system of constraints C. If the constraint is using only odd parts, then $a_n = F_n$ (the n^{th} Fibonacci number). Thus, $5 = 3+1+1 = 1+3+1 = 1+1+3 = 1+1+1+1+1$ has $5 = F_5$ compositions. If the constraint is using only parts ≥ 2, then $a_n = F_{n-1}$, and if only parts 1 and 2 are allowed $a_n = F_{n+1}$. All of these are mentioned in the OEIS for the Fibonacci sequence A000045 [10]. In [6] problem 5.4.13 asks to show that a_n is the n^{th} Padovan number, satisfying the recurrence $a_{n+3} = a_{n+1} + a_n$, if only odd parts ≥ 3 are allowed. The Padovan numbers also count the number of compositions into parts congruent to 2 (mod 3). These results are also mentioned in the OEIS for the Padovan sequence A000931.

In some recent papers other constraints on the allowed compositions include: no part of a specified size t [3] [4] [7], at least one part of size t [1], parts of size 1 and t [2], and no parts divisible by 3 [9]. Some of these papers deal with the recurrence satisfied by a_n, others with expressions of a_n as sums of other quantities.

In all the examples described above, except for "at least one part of size t", the type of constraint C is of the form requiring all parts to be chosen from a specified set S. This leads naturally to the question: for which such sets S does a_n satisfy a linear, homogeneous, constant coefficient recurrence equation? Our goal is to answer this question.

2. Generating Functions

We will approach this problem using ordinary generating functions (OGF). One of the key properties of a recurrence sequence a_n is that its OGF is a rational function $P(x)/Q(x)$ where $\deg P(x) < \deg Q(x)$ [6]. If $\deg P(x) \geq \deg Q(x)$, then a_n satisfies a recurrence equation except for a finite number of initial terms.

Theorem 2.1. *The number of compositions of n into parts from a set S of positive integers satisfies a linear, homogeneous, constant coefficient recurrence equation, except possibly for finitely many terms, if and only if $S = S_1 \cup S_2$ where S_1 is a finite set, and there are residues $r_1, r_2, ..., r_h$ modulo m such that $S_2 = \{k \geq k_0 : k \equiv r_1, r_2, ..., r_h \pmod{m}\}$.*

Proof. The number of compositions of n into exactly p parts from a set S, is the number of solutions of: $y_1 + y_2 + ... + y_p = n$, $n \geq 1$, where $y_i \in S$.

The corresponding OGF is:

$$f_p(x) = (\sum_{s\in S} x^s)^p = f(x)^p \tag{2.1}$$

If a_n counts the number of such compositions into any number of parts, then its OGF is:

$$\sum_{n\geqslant 0} a_n x^n = \sum_{p=1}^{\infty} f_p(x) = \sum_{p=1}^{\infty} f(x)^p = \frac{f(x)}{1-f(x)} \tag{2.2}$$

Thus, a_n satisfies a recurrence equation if and only if: $\frac{f(x)}{1-f(x)} = \frac{P(x)}{Q(x)}$ is a rational function, which is true if and only if $f(x) = \frac{P(x)}{P(x)+Q(x)}$ is a rational function, which is true if and only if $f(x)$ is the OGF of a sequence which satisfies a recurrence equation except possibly for finitely many terms. Suppose this recurrence equation is of order t. Since all of the coefficients of $f(x)$ are 0 or 1, there are only finitely many different blocks of length t. Hence the coefficients of $f(x)$ must be periodic, but not necessarily purely periodic. That is, S must be of the form described in the theorem. □

3. Some Implications

Theorem 2.1 shows that all of the examples in the introduction satisfy a recurrence equation. Note that the number of compositions with at least one part of size t is the same as all compositions minus those with no parts of size t. The OGF for no parts of size t uses $f(x) = \sum_{i\geqslant 1} x^i - x^t = \frac{x}{1-x} - x^t = \frac{x-x^t+x^{t+1}}{1-x}$, which is a rational function.

Example 1. If a_n counts the number of compositions with no parts of sizes t_1, t_2, ... , t_k, then from the proof of Theorem 2.1, the OGF for a_n is $\frac{f(x)}{1-f(x)}$ where

$$f(x) = \sum_{j\geqslant 1} x^j - x^{t_1} - x^{t_2} - ... - x^{t_k} = \frac{x - x^{t_1} + x^{t_1+1} - x^{t_2} + x^{t_2+1} - ... + x^{t_k+1}}{1-x}. \tag{3.1}$$

Hence, the OGF for a_n is

$$\frac{x - x^{t_1} + x^{t_1+1} - ... + x^{t_k+1}}{1 - 2x + x^{t_1} - x^{t_1+1} + ... - x^{t_k+1}}. \tag{3.2}$$

Thus, a_n satisfies the recurrence equation

$$a_{n+t_k+1} - 2a_{n+t_k} + a_{n+t_k-t_1+1} - ... - a_n = 0. \tag{3.3}$$

Theorem 2.1 also proves that many types of compositions do no satisfy a recurrence equation. For example, the number of compositions into prime numbers, squares, or Fibonacci numbers do not satisfy a recurrence, since these sets are not of the type described in Theorem 2.1. However, compositions into numbers which are either primes or Fibonacci numbers less than 100 are counted by a recurrence since this is a finite set.

Example 2. If b_n counts the number of compositions of n into parts which are congruent to r_1, r_2, ..., r_h modulo m, $0 \le r_i \le m-1$, then $S = \{s\colon\ s \equiv r_1, r_2, ..., r_h \pmod{m}\}$ and

$$f(x) = \sum_{s\in S} x^s = \sum_{i=1}^{h}\sum_{j=0}^{\infty} x^{r_i+jm} = \sum_{i=1}^{h} \frac{x^{r_i}}{1-x^m} = \frac{x^{r_1} + x^{r_2} + ... + x^{r_h}}{1-x^m}. \tag{3.4}$$

By Theorem 2.1, the OGF of the sequence b_n is

$$\frac{f(x)}{1-f(x)} = \frac{x^{r_1} + ... + x^{r_h}}{1 - x^{r_1} - ... - x^{r_h} - x^m}. \tag{3.5}$$

Hence, b_n satisfies the recurrence equation

$$b_{n+m} - b_{n+m-r_1} - ... - b_{n+m-r_h} - b_n = 0. \tag{3.6}$$

Example 3. If c_n counts the number of compositions of n into parts of size 2 or 3 or numbers congruent to 2 or 4 modulo 7 and greater than 14, the function $f(x)$ in Theorem 2.1 is

$$f(x) = x^2 + x^3 + \frac{x^{16} + x^{18}}{1 - x^7}. \tag{3.7}$$

Then the OGF for c_n is

$$\frac{f(x)}{1 - f(x)} = \frac{\frac{x^2+x^3-x^9-x^{10}+x^{16}+x^{18}}{1-x^7}}{\frac{1-x^7-x^2-x^3+x^9+x^{10}-x^{16}-x^{18}}{1-x^7}} = \frac{x^2 + x^3 - x^9 - x^{10} + x^{16} + x^{18}}{1 - x^2 - x^3 - x^7 + x^9 + x^{10} - x^{16} - x^{18}}. \tag{3.8}$$

Thus, c_n satisfies the recurrence

$$c_{n+18} - c_{n+16} - c_{n+15} - c_{n+11} + c_{n+9} + c_{n+8} - c_{n+2} - c_n = 0. \tag{3.9}$$

Suppose we are given a recurrence sequence a_n and ask if there is a type of composition which is counted by a_n. As in Theorem 2.1 if $\sum_{n\geqslant 0} a_n x^n = \frac{P(x)}{Q(x)}$ then $\sum_{s\in S} x^s = f(x) = \frac{P(x)}{P(x)+Q(x)}$.

Example 4. Is there a composition counted by the Fibonacci sequence so that $a_n = F_n$ for $n \geq 1$? Since $\sum_{n\geqslant 1} F_n x^n = \frac{x}{1-x-x^2} = \frac{P(x)}{Q(x)}$, $f(x) = \frac{x}{1-x^2} = \sum_{i\geqslant 0} x^{2i+1}$. Hence, a_n counts compositions into odd parts. Similarly, if we want $a_n = F_{n+1}$, since $\sum_{n\geqslant 1} F_{n+1} x^n = \frac{x+x^2}{1-x-x^2} = \frac{P(x)}{Q(x)}$, $f(x) = x + x^2$ so a_n counts compositions into parts of size 1 or 2. If $a_n = F_{n-1}$, since $\sum_{n\geqslant 1} F_{n-1} x^n = \frac{x^2}{1-x-x^2} = \frac{P(x)}{Q(x)}$, $f(x) = \frac{x^2}{1-x} = \sum_{n\geqslant 2} x^n$ so a_n counts compositions into parts greater than or equal to 2. However, for $a_n = F_{n+2}$, a similar calculation leads to $f(x) = \frac{2x+x^2}{1+x} = 2x - x^2 + x^3 - x^4 + \cdots$, which is not of the form $\sum_{s\in S} x^s$.

Example 5. Is there a composition counted by the tribonacci sequence? In this case the sequence satisfies the recurrence equation $a_{n+3} - a_{n+2} - a_{n+1} - a_n = 0$ with the usual initial values $a_1 = 0$, $a_2 = 1$, $a_3 = 1$. The OGF is $\frac{x^2}{1-x-x^2-x^3}$ so $f(x) = \frac{x^2}{1-x-x^3} = x^2 + x^3 + x^4 + 2x^5 + 3x^6 + \cdots$. Since this series has coefficients other that 0 or 1 it cannot equal $\sum_{s\in S} x^s$. Thus, there is no composition of the desired kind. However, if we change the initial values but keep the tribonacci recurrence, so that the OGF is $\frac{x+x^2}{1-x-x^2-x^3}$, i.e., $a_1 = 1$, $a_2 = 2$, $a_3 = 3$, then $f(x) = \frac{x+x^2}{1-x^3} = (x + x^2) \sum_{i\geqslant 0} x^{3i} = \sum_{i\geqslant 0}(x^{3i+1} + x^{3i+2})$ so S is the set of positive integers congruent to 1 or 2 (mod 3).

There are other types of constraints that are not of the kind described in Theorem 2.1, such as restricting the number of times specific parts can be used, or if the choice for one part restricts the choice for another part. We hope to address compositions of such types in the future.

References

[1] A. Benjamin, P. Chinn, J. Scott, G. Simay, *Combinations of Two-Toned Tilings*, The Fibonacci Quarterly **49** (2011), 290-297.

[2] Chinn, Phyllis and Silvia Heubach, "(l,k)- Compositions," *Congressus Numerantium*, v.164, December 2003, p 183-194.

[3] Chinn, Phyllis and Silvia Heubach, "Compositions of n with no occurrence of k," *Congressus Numerantium*, v.164, December 2003, p 33-52.

[4] Chinn, Phyllis and Silvia Heubach, "Integer Sequences Related to Compositions without 2's," *Journal of Integer Sequences*, Article 03.2.3: Volume 6, 2003.

[5] Chinn, Phyllis and Heinrich Niederhausen, "Compositions into Powers of 2," *Congressus Numerantium*, v.168 (2004), p 215-220.
[6] D. DeTemple, and W. Webb, *Combinatorial Reasoning: An Introduction to the Art of Counting*, First Edition, John Wiley and Sons, Inc., Hoboken, New Jersey, 2014.
[7] Grimaldi, Ralph, "Compositions without the summand 1," *Congressus Numerantium*, 152 (2001), pp. 33-43.
[8] Odlyzko, A. and B. Richmond, "On the Compositions of an Integer," *Lecture Notes in Mathematics* 829, Springer-Verlag, (1980), p. 199-210.
[9] Robbins, Neville, *On Tribonacci Numbers and 3-Regular Compositions*, The Fibonacci Quarterly, **52** (2014), 16-19.
[10] OEIS Foundation Inc. (2011), The On-Line Encyclopedia of Integer Sequences, `http://oeis.org`.

Department of Mathematics, Washington State University, Pullman, Washington, 99163, USA
E-mail address: `webb@math.wsu.edu`

Department of Mathematics, Washington State University, Pullman, Washington, 99163, USA
E-mail address: `nghamlin@wsu.edu`

SYMMETRIES OF STIRLING NUMBER SERIES

PAUL THOMAS YOUNG

Abstract. We consider Dirichlet series generated by weighted Stirling numbers, focusing on a symmetry of such series which is reminiscent of a duality relation of negative-order poly-Bernoulli numbers. These series are connected to several types of zeta functions and this symmetry plays a prominent role. We do not know whether there are combinatorial explanations for this symmetry, as there are for the related poly-Bernoulli identity.

1. Introduction

This paper is concerned with the Dirichlet series

$$S_{j,r}(s,a) = \sum_{m=j}^{\infty} \frac{(-1)^{m+j} s(m,j|r)}{m!(m+a)^s} \tag{1.1}$$

where $s(m,j|r)$ denotes the *weighted Stirling number of the first kind* [4, 5] defined for non-negative integers m, j and $r \in \mathbb{C}$ by the vertical generating function

$$(1+t)^{-r}(\log(1+t))^j = j! \sum_{m=j}^{\infty} s(m,j|r) \frac{t^m}{m!} \tag{1.2}$$

or by the horizontal generating function

$$(x)_m = \sum_{j=0}^{m} s(m,j|r)(x+r)^j \tag{1.3}$$

where $(x)_m = x(x-1)\cdots(x-m+1)$ denotes the falling factorial. If j is a nonnegative integer, $S_{j,r}(s,a)$ converges for $r, s, a \in \mathbb{C}$ such that $\Re(s) > \Re(r)$ and $\Re(a) > -j$; when $r \in \mathbb{Z}^+$ it has poles of order $j+1$ at $s = 1, 2, .., r$ and of order at most j at nonpositive integers s. When $j = 0$ we recover the *Barnes multiple zeta functions*, and when $j = 1$ we obtain special values of *non-strict multiple zeta functions*, also known as *zeta-star values* (see section 3). We will focus on the symmetric identity

$$S_{j,r}(k+1, 1-t) = S_{k,t}(j+1, 1-r), \tag{1.4}$$

valid for integers $r \le k$ and $t \le j$, which bears a striking resemblance to a symmetric identity of *poly-Bernoulli polynomials* (Theorem 6.1 below). Since this poly-Bernoulli identity has known combinatorial interpretations in the case where $r = t = 0$, we find it interesting to ask whether the symmetry (1.4) may be proved or interpreted in terms of counting arguments.

2. Stirling and r-Stirling numbers

The weighted Stirling numbers of the first kind $s(n,k|r)$ may be defined by either (1.2) or (1.3), or by the recursion

$$s(n+1,k|r) = s(n,k-1|r) - (n+r)s(n,k|r) \tag{2.1}$$

with initial conditions $s(n, n|r) = 1$, $s(n, 0|r) = (-r)_n$. Their dual companions [8] are the *weighted Stirling numbers of the second kind* $S(n, k|r)$ [4, 5] which may be defined by the vertical generating function

$$e^{rt}(e^t - 1)^m = m! \sum_{n=m}^{\infty} S(n, m|r) \frac{t^n}{n!}, \tag{2.2}$$

the horizontal generating function

$$x^n = \sum_{k=0}^{n} S(n, k|r)(x - r)_k, \tag{2.3}$$

or by the recursion

$$S(n + 1, k|r) = S(n, k - 1|r) + (k + r)S(n, k|r) \tag{2.4}$$

with initial conditions $S(n, n|r) = 1$, $S(n, 0|r) = r^n$. It is clear that both $s(n, k|r)$ and $S(n, k|r)$ are polynomials in r with integer coefficients of degree $n - k$ whose derivatives are given by

$$s'(n, k|r) = (k + 1)s(n, k + 1|r) \qquad \text{and} \qquad S'(n, k|r) = nS(n - 1, k|r). \tag{2.5}$$

For combinatorial interpretations, when the "weight" r is a nonnegative integer we may write

$$(-1)^{m+j} s(m, j|r) = \begin{bmatrix} m + r \\ j + r \end{bmatrix}_r \tag{2.6}$$

in terms of *r-Stirling numbers* $\left[{n \atop k} \right]_r$, which count the number of permutations of $\{1, 2, ..., n\}$ having k cycles, with the elements $1, 2, ..., r$ restricted to appear in different cycles [3, 1]. When $r = 0$ these definitions reduce to those of the usual Stirling numbers, and in that case the parameter r is often suppressed in the notation. Furthermore if $j = 1$ and $r \geq 0$ the coefficients $(-1)^{m+1}s(m, 1|r)/m!$ are called *hyperharmonic numbers* $H_m^{[r]}$ defined by $H_m^{[0]} = \frac{1}{m}$ for $m > 0$, $H_0^{[r]} = 0$, and

$$H_m^{[r]} = \sum_{i=1}^{m} H_i^{[r-1]} \tag{2.7}$$

(cf. [1, 14, 9]). Thus $H_n = H_n^{[1]}$ denotes the usual harmonic number.

3. Dirichlet series Identities

Our interest in the series (1.1) is derived from the fact that they specialize to known multiple zeta functions when $j = 0, 1$. First, the series $S_{0,1}(s, 1)$ is the Riemann zeta function $\zeta(s)$; more generally for $r \in \mathbb{Z}^+$ the series $S_{0,r}(s, a)$ is a *Barnes multiple zeta function* $\zeta_r(s, a)$ [15, 16] defined for $\Re(s) > r$ and $\Re(a) > 0$ by

$$\zeta_r(s, a) = \sum_{t_1=0}^{\infty} \cdots \sum_{t_r=0}^{\infty} (a + t_1 + \cdots + t_r)^{-s}. \tag{3.1}$$

If we view $\zeta_r(s, a)$ as an analytic function of its order r as in [15, 16], then we can view $S_{j,r}(s, a) = j! D_r^j \zeta_r(s, a)$ by means of (2.5), where D_r denotes the derivative d/dr. From this identification we deduce from ([16], Corollary 2) that the series $S_{j,r}(s, a)$ is convergent when $\Re(s) > \Re(r)$ and $\Re(a) > -j$.

For $r \in \mathbb{Z}^+$ the series $S_{1,r}(s,0)$ is also a specialization of a *non-strict multiple zeta function*, namely $S_{1,r}(s,0) = \zeta^\star(s, \underbrace{0,...,0}_{r-1}, 1)$, where

$$\zeta^\star(s_1,...,s_m) := \sum_{n_1 \geq n_2 \geq \cdots \geq n_m \geq 1} \frac{1}{n_1^{s_1} n_2^{s_2} \cdots n_m^{s_m}} \tag{3.2}$$

([9], Prop. 2.1). The zeta-star values are related to Arakawa-Kaneko zeta functions, whose values at negative integers are given by the poly-Bernoulli numbers $\mathbb{B}_n^{(k)}$ ([9, 6]).

The series (1.1) satisfies several identities.

Theorem 3.1. *The following identities hold where defined.*

i. *We have* $S_{j,r}(s,a) = S_{j,r}(s,a+1) + S_{j,r-1}(s,a)$.
ii. *For* $r \in \mathbb{Z}^+$ *we have* $S_{j,r}(s,a) = S_{j,0}(s,a) + \sum_{t=1}^{r} S_{j,t}(s,a+1)$.
iii. *For* $0 \leq m \leq r$ *we have* $S_{j,r}(s,a) = \sum_{t=0}^{m} \binom{m}{t} S_{j,r-t}(s,a+m-t)$.
iv. *We have*

$$S_{j,r}(s,a) - aS_{j,r}(s+1,a) = S_{j-1,r+1}(s+1,a+1) + rS_{j,r+1}(s+1,a+1).$$

v. *(Symmetry relation.) For integers* $r \leq k$ *and* $t \leq j$ *we have*

$$S_{j,r}(k+1,1-t) = S_{k,t}(j+1,1-r).$$

Thus when it converges, the series $S_{j,r}(k+1,1-t)$ *is invariant under* $(j,k,r,t) \mapsto (k,j,t,r)$.

Proof. Identity (i) follows from the Stirling number recurrence (2.1), or equivalently from the difference equation

$$\zeta_r(s,a) - \zeta_r(s,a+1) = \zeta_{r-1}(s,a) \tag{3.3}$$

([15], eq. (2.1)) of the Barnes multiple zeta functions. Identities (ii) and (iii) may be obtained by induction from (i), or from Identity 5 and Identity 7 in [1]. To obtain (iv), we differentiate the generating function (1.2) with respect to r and equate coefficients of $t^n/n!$ to obtain

$$s(n+1,j|r) = s(n,j-1|r+1) - r\, s(n,j|r+1). \tag{3.4}$$

Dividing by $(n+1)!(n+a)^s$ and summing over n then yields (iv). By means of (2.5) we have $S_{j,r}(s,a) = j! D_r^j \zeta_r(s,a)$, and therefore the symmetry relation (v) follows from the identity

$$(k-1)! D_t^{j-1} \zeta_t(k,1-r) = (j-1)! D_r^{k-1} \zeta_r(j,1-t) \tag{3.5}$$

([16], Corollary 2). □

4. Combinatorial interpretation

Restricting our attention to the case where r is a nonnegative integer, the symmetry relation Theorem 3.1(v) may be written as

$$\sum_{m=j}^{\infty} \frac{\begin{bmatrix} m+r \\ j+r \end{bmatrix}_r}{m!(m+1-t)^{k+1}} = \sum_{m=k}^{\infty} \frac{\begin{bmatrix} m+t \\ k+t \end{bmatrix}_t}{m!(m+1-r)^{j+1}} \tag{4.1}$$

for integers $0 \leq r \leq k$ and $0 \leq t \leq j$, where the r-Stirling number $\begin{bmatrix} n \\ k \end{bmatrix}_r$ = the number of permutations of $\{1,2,...,n\}$ having k cycles, with the elements $1,2,...,r$ restricted to appear in different cycles. When $r, t \in \{0, 1\}$ this gives series identities for the usual Stirling numbers of the first kind; for example, in

$$\sum_{m=j}^{\infty} \frac{\begin{bmatrix} m \\ j \end{bmatrix}}{m!(m+1)^{k+1}} = \sum_{m=k}^{\infty} \frac{\begin{bmatrix} m \\ k \end{bmatrix}}{m!(m+1)^{j+1}} \tag{4.2}$$

we have $\left[{m \atop k}\right]/m!$ equal to the proportion of permutations of $\{1, ..., m\}$ which have k cycles. Thus the left side of (4.2) may be viewed as a sum over permutations which have j cycles and the right side as a sum over permutations which have k cycles.

Question 1: Can the identities (4.2) or (4.1) be proved by combinatorial means?

5. Values at positive integers

The identities of section 3 may be used to demonstrate a large class of values of $S_{j,r}(s, a)$ which may be expressed as polynomials in values of the Riemann zeta function.

Theorem 5.1. *When $j \in \{0, 1\}$ or $s \in \{1, 2\}$ we have $S_{j,r}(s, a) \in \mathbb{Q}[\zeta(2), \zeta(3), \zeta(5), ...]$ for integers $r < s$ and $a > -j$.*

Proof. Write $R = \mathbb{Q}[\zeta(2), \zeta(3), \zeta(5), ...]$. When $j = 0$ and $r \leq 0$ the sum for $S_{j,r}(s, a)$ is finite, and therefore rational, so the theorem is therefore true in that case. For $j = 0$ and $r > 0$ we have $S_{0,r}(s, a) = \zeta_r(s, a)$ and we use the identity

$$\zeta_r(s, a) = \frac{1}{(r-1)!} \sum_{k=0}^{r-1} s(r-1, k|a+1-r)\, \zeta_1(s-k, a) \tag{5.1}$$

([16], eq. (3.3)) to prove the theorem in that case, since $\zeta_1(s, a) \in R$ for integers $s > 1$ and $a > 0$. The theorem is therefore established for $j = 0$.

In the case $j = 1$ the theorem generalizes Euler's classical identity

$$S_{1,1}(s, 0) = \sum_{n=1}^{\infty} \frac{H_n}{n^s} = \frac{s+2}{2}\zeta(s+1) - \frac{1}{2}\sum_{j=1}^{s-2} \zeta(s-j)\zeta(j+1) \in R. \tag{5.2}$$

Kamano [9] proved that

$$(r-1)! S_{1,r}(s, 0) = \sum_{k=1}^{r} \begin{bmatrix} r \\ k \end{bmatrix} S_{1,1}(s, 0) + \left(k \begin{bmatrix} r \\ k+1 \end{bmatrix} - \begin{bmatrix} r \\ k \end{bmatrix} H_{r-1}\right) \zeta(s+1-k) \tag{5.3}$$

which, together with (5.2), implies that $S_{1,r}(s, 0) \in R$ when $r > 0$. (Alternatively one can use the recursion

$$S_{1,r}(s, 0) = S_{1,1}(s, 0) + \sum_{k=1}^{r-1} \frac{1}{k} \left(S_{1,k}(s-1, 0) + B(k, s)\right) \tag{5.4}$$

([14], Theorem 6), where $B(k, s)$ is a linear polynomial in $\{\zeta(j)\}_{m \geq 2}$, to show this). When $j = 1$ and $r = 0$ we observe that $S_{1,0}(1, a) = H_a/a \in \mathbb{Q}$ for $a \in \mathbb{Z}^+$; induction using Theorem 3.1(iv) then shows $S_{1,0}(s, a) \in R$ for all $s > r$ and $a \geq 0$. So $S_{1,r}(s, a) \in R$ when either $a = 0$ or $r = 0$; an induction argument using Theorem 3.1(i) shows that $S_{1,r}(s, a) \in R$ when $r \geq 0$ and $a \geq 0$.

A similar induction argument, using Theorem 3.1(i) and (iv), shows that $S_{1,r}(s, a) \in R$ for $a \geq 0$ when r is a negative integer and $s > r$. This completes the proof of the theorem for

$j \in \{0, 1\}$. The statement concerning $s \in \{1, 2\}$ then is obtained by the symmetry relation Theorem 3.1(v). □

6. Poly-Bernoulli polynomials

In this final section we prove a finite sum symmetric identity which bears a striking resemblance to the infinite sum symmetric identity of Theorem 3.1(v). The *weighted shifted poly-Bernoulli numbers* $\mathbb{B}_n^{(k)}(a, r)$ of order k are defined by

$$\Phi(1 - e^{-t}, k, a)e^{-rt} = \sum_{n=0}^{\infty} \mathbb{B}_n^{(k)}(a, r)\frac{t^n}{n!} \tag{6.1}$$

$$\text{where} \qquad \Phi(z, s, a) = \sum_{m=0}^{\infty} \frac{z^m}{(m + a)^s} \qquad (|z| < 1) \tag{6.2}$$

is the Lerch transcendent. (The generalization (6.1) was communicated to me by Mehmet Cenkci, to whom I am grateful). When $a = 1$ and $r = 0$ we obtain the usual poly-Bernoulli numbers $\mathbb{B}_n^{(k)} = \mathbb{B}_n^{(k)}(1, 0)$ defined and studied by Kaneko [10], since in that case the Lerch transcendent reduces to the usual order k polylogarithm function

$$\mathrm{Li}_k(z) = \sum_{m=1}^{\infty} \frac{z^m}{m^k}. \tag{6.3}$$

The $\mathbb{B}_n^{(k)}(a, r)$ are polynomials of degree n in r and they are polynomials of degree $-k$ in a when $-k \in \mathbb{Z}^+$. When $k = 1$ and $a = 0$ we have

$$\mathbb{B}_n^{(1)}(0, r) = (-1)^n B_n(r) \tag{6.4}$$

in terms of the usual Bernoulli polynomials $B_n(x)$. The weighted Lerch poly-Bernoulli numbers may also be expressed in terms of weighted Stirling numbers of the second kind as

$$\mathbb{B}_n^{(k)}(a, r) = (-1)^n \sum_{m=0}^{n} \frac{(-1)^m m! S(n, m|r)}{(m + a)^k}. \tag{6.5}$$

Therefore in the case $r = 0$ these polynomials agree with the shifted poly-Bernoulli numbers of ([12], §6). The weighted shifted poly-Bernoulli polynomials satisfy the following symmetric identity.

Theorem 6.1. *For all nonnegative integers n and k we have*

$$\mathbb{B}_n^{(-k)}(1 - t, r) = \mathbb{B}_k^{(-n)}(1 - r, t).$$

Proof. This result was proved by Kaneko [10] in the case $r = 0$, $t = 0$, and the proof is adapted from Kaneko's proof. Straightforward calculation shows that

$$\begin{aligned}
\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\mathbb{B}_n^{(-k)}(1-a,x)\frac{t^n}{n!}\frac{u^k}{k!} &= \sum_{k=0}^{\infty}\Phi(1-e^{-t},-k,1-a)e^{-xt}\frac{u^k}{k!} \\
&= \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{(1-e^{-t})^m e^{-xt}u^k}{(m+1-a)^{-k}k!} \\
&= e^{-xt}\sum_{m=0}^{\infty}(1-e^{-t})^m e^{(m+1-a)u} \\
&= e^{-xt}e^{(1-a)u}\sum_{m=0}^{\infty}((1-e^{-t})e^u)^m \\
&= \frac{e^{-xt}e^{(1-a)u}}{1-(1-e^{-t})e^u} \\
&= \frac{e^{(1-x)t}e^{(1-a)u}}{e^t+e^u-e^{t+u}} \qquad (6.6)
\end{aligned}$$

is invariant under $(t,u,a,x) \mapsto (u,t,x,a)$. □

This theorem says that the expression $\mathbb{B}_n^{(-k)}(1-t,r)$ is a polynomial in r and t which is invariant under $(n,k,r,t) \mapsto (k,n,t,r)$. In terms of weighted Stirling numbers it reads

$$\sum_{m=0}^{n}(-1)^{m+n}m!S(n,m|r)(m+1-t)^k = \sum_{m=0}^{k}(-1)^{m+k}m!S(k,m|t)(m+1-r)^n. \qquad (6.7)$$

We find this identity to be strikingly similar to the symmetric identity, for $r \le k$ and $t \le j$,

$$\sum_{m=j}^{\infty}\frac{(-1)^{m+j}s(m,j|r)}{m!(m+1-t)^{k+1}} = \sum_{m=k}^{\infty}\frac{(-1)^{m+k}s(m,k|t)}{m!(m+1-r)^{j+1}}, \qquad (6.8)$$

given by Theorem 3.1(v). The two identities appear to share a kind of duality, but it is curious that one identity is for finite sums and the other is for infinite series.

In the case $r = t = 0$, the poly-Bernoulli numbers $\mathbb{B}_n^{(-k)}$ have found at least two important combinatorial interpretations. In [2] it is shown that $\mathbb{B}_n^{(-k)}$ equals the number of distinct $n \times k$ lonesum matrices, where a *lonesum matrix* is a matrix with entries in $\{0,1\}$ which is uniquely determined by its row and column sums. In [13] it is shown that the number of permutations σ of the set $\{1,2,\ldots,n+k\}$ which satisfy $-k \le \sigma(i) - i \le n$ for all i is the poly-Bernoulli number $\mathbb{B}_n^{(-k)}$. Either of these two combinatorial interpretations make the $r = t = 0$ case of the symmetry relation of Theorem 6.1 obvious.

Question 2. Can the symmetric identity of Theorem 6.1 be proved by a counting argument in cases where r and t are nonzero integers?

References

[1] A. Benjamin, D. Gaebler, R. Gaebler, *A combinatorial approach to hyperharmonic numbers*, INTEGERS **3** (2003), article A15, 9pp.

[2] C. Brewbaker, *A combinatorial interpretation of the poly-Bernoulli numbers and two Fermat analogues*, INTEGERS **8** (2008), article A02, 9pp.
[3] A. Broder, *The r-Stirling numbers*, Disc. Math. **49** (1984), 241-259.
[4] L. Carlitz, *Weighted Stirling numbers of the first and second kind - I*, The Fibonacci Quarterly **18** (1980), 147-162.
[5] L. Carlitz, *Weighted Stirling numbers of the first and second kind - II*, The Fibonacci Quarterly **18** (1980), 242-257.
[6] M.-A. Coppo and B. Candelpergher, *The Arakawa-Kaneko zeta functions*, Ramanujan J. **22** (2010), 153-162.
[7] F. T. Howard, *Congruences and recurrences for Bernoulli numbers of higher order*, The Fibonacci Quarterly **32** (1994), 316-328.
[8] L. Hsu and P. Shiue, *A unified approach to generalized Stirling numbers*, Adv. Appl. Math. **20** (1998), 366-384.
[9] K. Kamano, *Dirichlet series associated with hyperharmonic numbers*, Mem. Osaka Inst. Tech. Ser. A **56** (2011), 11-15.
[10] M. Kaneko, *Poly-Bernoulli numbers*, J. Théorie Nombres Bordeaux **9** (1997), 199-206.
[11] M. Kaneko, *A note on poly-Bernoulli numbers and multiple zeta values*, in: Diophantine Analysis and Related Fields, AIP Conference Proceedings vol. 976, 118-124, AIP, New York, 2008.
[12] T. Komatsu and L. Szalay, *Shifted poly-Cauchy numbers*, Lithuanian Math. J. **54** (2014), 166-181.
[13] S. Launois, *Rank t $\mathcal{H}$-primes in quantum matrices*, Comm. Alg. **33** (2005), 837-854.
[14] I. Mezö and A. Dil, *Hyperharmonic series involving Hurwitz zeta function*, J. Number Theory **130** (2010), 360-369.
[15] P. T. Young, *Rational series for multiple zeta and log gamma functions*, J. Number Theory **133** (2013), 3995-4009.
[16] P. T. Young, *Symmetries of Bernoulli polynomial series and Arakawa-Kaneko zeta functions*, J. Number Theory **143** (2014), 142-161.

MSC2010: 11B73, 11M41, 11M32

Department of Mathematics, College of Charleston, Charleston, SC 29424
E-mail address: `paul@math.cofc.edu`

DIOPHANTINE TRIPLES AND EXTENDIBILITY OF $\{1,2,5\}$ AND $\{1,5,10\}$

YIFAN ZHANG AND GEORGE GROSSMAN

Abstract. In this paper we consider Diophantine triples, (denoted $D(n)$-3-tuples,) $\{1,2,5\}$, $\{1,5,10\}$ for the case $n=-1$. We show using properties of Lucas and Fibonacci numbers that neither of 3-tuples $\{1,2,5\}$, $\{1,5,10\}$ can be extended to a $D(-1)$-4-tuple.

1. INTRODUCTION

Definition 1.1. *A set of m positive integers is called a Diophantine m-tuple with the property $D(n)$ or simply $D(n)$-m-tuple, if the product of any two elements of this set increased by n is a perfect square.*

As a special case, a Diophantine m-tuple is a set of m positive integers with the property: the product of any two of them increased by one unit is a perfect square, for example, {1, 3, 8, 120} is a Diophantine quadruple, since we have

$$1\times 3+1=2^2, 1\times 8+1=3^2, 1\times 120+1=11^2,$$

$$3\times 8+1=5^2, 3\times 120+1=19^2, 8\times 120+1=31^2.$$

The study of Diophantine m-tuple can be traced back to the third century AD, when the Greek mathematician Diophantus discovered that $\left\{\frac{1}{16},\frac{33}{16},\frac{17}{4},\frac{105}{16}\right\}$ is a set of four rationals which has the above property. Then Fermat obtained the first Diophantine quadruple {1, 3, 8, 120}. Astoundingly, $\frac{777480}{8288641}$ was found to extend the Fermat's set to $\left\{1,3,8,120,\frac{777480}{8288641}\right\}$ and then the product of any two elements of this set increased by one unit is a perfect square of a rational number, which was Euler's contribution. Moreover, he acquired the infinite family of Diophantine quadruple $\{a,b,a+b+2r,4r\,(r+a)\,(r+b)\}$, if $ab+1=r^2$. In January 1999, Gibbs [8] found the first set of six positive rationals with the above property. In the integer case, there is a famous conjecture: there does not exist a Diophantine quintuple.

The case $n\neq 1$ also have been studied by several mathematicians, for example, $\{1,2,5\}$ is a $D(-1)$-triple. It is interesting to note that if n is an integer of form $n=4k+2$, then there does not exist a Diophantine quadruple with the property $D(n)$. This theorem has been independently proved by Brown [2], Gupta & Singh [9] and Mohanty & Ramasamy [13] all in 1985. In 1993, Dujella [3] proved that if an integer n does not have the form $n=4k+2$ and $n\notin S=\{-4,-3,-1,3,5,8,12,20\}$, then there exists at least one Diophantine quadruple with the property $D(n)$. In the case $n=-1$, the conjecture—there does not exist a $D(-1)$-quadruple is known as $D(-1)$-quadruple conjecture.

In 1985, Brown [2] proved the nonextendability of the Diophantine $D(-1)$ triple $\{1,2,5\}$. Walsh [15] and Kihel [10] also independently proved that in 1999 and 2000 respectively. In 1984, Mohanty & Ramasamy [12] proved that the Diophantine $D(-1)$ triple $\{1,5,10\}$ can not be extended to a $D\,(-1)$ quadruple. Furthermore, Brown [2] proved that the following triple

$$\left\{n^2+1,(n+1)^2+1,(2n+1)^2+4\right\},$$

can not be extended to a Diophantine quadruple with the property $D(-1)$ if $n \equiv 0 \pmod 4$. $\{17, 26, 85\}$ is an example when $n = 4$. Dujella [4] was the first mathematician who proved the nonextendability for all triples of the form $\{1, 2, c\}$ in 1998. The endeavor in proving that $\{1, 5, c\}$ can not be extended was mostly attributed to Muriefah & Al-Rashed [14]. In 2005, Filipin [7] proved the nonextendability of $\{1, 10, c\}$.

In [2, 4, 7, 10, 12, 14, 15] solution techniques involved the intersection of solutions of systems of certain Pellian equations, including also employing methods such as linear forms in logarithms from the results of Baker and Davenport, [1], and other deep theoretical results from Diophantine analysis. Our paper uses only elementary number theory including use of results related to Legendre symbols, basic properties of Fibonacci and Lucas numbers with congruences, and thus, represents a distinctly original approach, i.e., without use of Pellian equations.

There does not exist a Diophantine quintuple with the property $D(-1)$. This was proved by Dujella & Fuchs [6] in 2005. Moreover, in 2007, Dujella, Filipin & Fuchs [5] proved that there are only exist finitely many quadruples with the property $D(-1)$.

2. NONEXTENDABILITY OF $\{1, 2, 5\}$

We will use the property of Fibonacci and Lucas sequences to prove the nonextendability of Diophantine triple $\{1, 2, 5\}$ with the property $D(-1)$.

Definition 2.1. *F_n is Fibonacci sequence beginning with $F_0 = 0$, $F_1 = 1$ and satisfying the property $F_{n+2} = F_{n+1} + F_n$. L_n is Lucas sequence beginning with $L_0 = 2$, $L_1 = 1$ and satisfying the property $L_{n+2} = L_{n+1} + L_n$.*

It is well-known that if (X, Y) are positive integers such that $X^2 - 5Y^2 = \pm 4$, then $(X, Y) = (L_m, F_m)$ for some positive integer m and the sign on the right is given by $(-1)^m$, also this result can be found in Koshy's [11] book, Theorem 5.4 in page 75 and Theorem 5.10 in page 83. If $1, 5, d$ are in the same $D(-1)$ set, then exists integers A, B such that $d - 1 = A^2$ and $5d - 1 = B^2$, thus we have $B^2 - 5A^2 = 4$ and then $A = F_{2n}$ for some positive integer n.

In order to prove that $\{1, 2, 5, d\}$ and $\{1, 5, 10, d\}$ are not Diophantine quadruple, we need prove $2d - 1$ and $10d - 1$ are not perfect squares, respectively. Since $d = A^2 + 1$ and $A = F_{2n}$ for some positive integer n, we reduce these two questions to prove $2F_{2n}^2 + 1$ and $10F_{2n}^2 + 9$ are not perfect squares for any positive integer n, respectively.

Lemma 2.2. *For any nonnegative integer q,*
$$5\left(F_{3^q}^2 + 2F_{2\cdot 3^q}^2 + 1\right) = \left(L_{2\cdot 3^q} + 1\right)\left(2L_{2\cdot 3^q} - 1\right).$$

Proof. This lemma can be derived by the following calculation:
$$\begin{aligned}
&5\left(F_{3^q}^2 + 2F_{2\cdot 3^q}^2 + 1\right) - \left(L_{2\cdot 3^q} + 1\right)\left(2L_{2\cdot 3^q} - 1\right)\\
&= 5F_{3^q}^2 + 10F_{2\cdot 3^q}^2 + 5 - 2L_{2\cdot 3^q}^2 - L_{2\cdot 3^q} + 1\\
&= \left(5F_{3^q}^2 - 4\right) + 2\left(5F_{2\cdot 3^q}^2 + 4\right) + 2 - 2L_{2\cdot 3^q}^2 - L_{2\cdot 3^q}\\
&= L_{3^q}^2 + 2 - L_{2\cdot 3^q}\\
&= L_{2\cdot 3^q} - L_{2\cdot 3^q}\\
&= 0.
\end{aligned}$$
□

We will use this formula for Lemma 2.3 and Lemma 3.1, $F_{nm}^2 - F_m^2 = F_{(n+1)m}F_{(n-1)m}$ with $m(n-1)$ even, this formula can be found in Koshy's [11] book, the 55th Fibonacci and Lucas identity in page 90 with n replaced by m and $2k$ replaced by $m(n-1)$. Let $\alpha = \frac{\sqrt{5}+1}{2}$, $\beta = \frac{1-\sqrt{5}}{2}$, then $F_m = \frac{1}{\sqrt{5}}\left(\alpha^m - \beta^m\right)$, $L_m = \alpha^m + \beta^m$ and $\alpha\beta = -1$.

Lemma 2.3. *If n is a positive integer not divisible by 3, then $F^2_{2\cdot 3^q\cdot n} \equiv F^2_{2\cdot 3^q} \pmod{(L_{2\cdot 3^q}+1)}$.*

Proof. If $3 \nmid n$, then $3 \mid (n+1)$ or $3 \mid (n-1)$, thus $F_{3m} \mid F_{(n+1)m}$ or $F_{3m} \mid F_{(n-1)m}$. For even integer m, $F_{3m} = \frac{1}{\sqrt{5}}\left(\alpha^{3m} - \beta^{3m}\right) = \frac{1}{\sqrt{5}}\left(\alpha^m - \beta^m\right)\left(\alpha^{2m} + \alpha^m\beta^m + \beta^{2m}\right)$
$= \frac{1}{\sqrt{5}}\left(\alpha^m - \beta^m\right)\left(\left(\alpha^m+\beta^m\right)^2 - (\alpha\beta)^m\right) = F_m\left(L_m^2 - 1\right) = F_m\left(L_m+1\right)\left(L_m-1\right)$, then $(L_m+1) \mid F_{3m} \mid \left(F^2_{nm} - F^2_m\right)$. By letting $m = 2\cdot 3^q$, we get $F^2_{2\cdot 3^q\cdot n} \equiv F^2_{2\cdot 3^q} \pmod{(L_{2\cdot 3^q}+1)}$. □

$\{L_m\}_{m\geq 1}$ is periodic modulo 4 with period 6, then $L_{2\cdot 3^q} \equiv L_0 = 2 \pmod 4$ for $q \geq 1$.

Theorem 2.4. *The Diophantine triple $\{1,2,5\}$ cannot be extended to a Diophantine quadruple $\{1,2,5,d\}$ with the property $D(-1)$, for all integers $d > 5$.*

Proof. We only need to prove $2F^2_{2n} + 1$ is not a perfect square for any positive integer n. Suppose there exists a positive integer l such that $l^2 = 2F^2_{2n} + 1$. Write $2n$ in the form $2n = 2\cdot 3^q \cdot k$ with $q \geq 0$ and $3 \nmid k$.

If $q = 0$, then $F^2_{2n} = F^2_{2\cdot 3^0\cdot k} \equiv F^2_{2\cdot 3^0} = F^2_2 = 1 \pmod{(L_{2\cdot 3^0}+1)}$, then $F^2_{2n} \equiv 1 \pmod 4$ and $l^2 = 2F^2_{2n} + 1 \equiv 3 \pmod 4$, a contradiction to the fact that the square of any integer is congruent to 0 or 1 modulo 4.

If $q \geq 1$, then $L_{2\cdot 3^q} \equiv 2 \pmod 4$, then $L_{2\cdot 3^q} + 1 \equiv 3 \pmod 4$. Therefore, there is a prime number p such that $p \mid (L_{2\cdot 3^q}+1)$ and $p \equiv 3 \pmod 4$.

According to Lemma 2.2, $p \mid \left(5\left(F^2_{3^q} + 2F^2_{2\cdot 3^q} + 1\right)\right)$, since $p \nmid 5$, then $2F^2_{2\cdot 3^q} + 1 \equiv -F^2_{3^q} \pmod p$. Then we have $1 = \left(\frac{l^2}{p}\right) = \left(\frac{2F^2_{2n}+1}{p}\right) = \left(\frac{2F^2_{2\cdot 3^q\cdot k}+1}{p}\right) = \left(\frac{2F^2_{2\cdot 3^q}+1}{p}\right) = \left(\frac{-F^2_{3^q}}{p}\right) = \left(\frac{-1}{p}\right)\left(\frac{F^2_{3^q}}{p}\right) = \left(\frac{-1}{p}\right) = -1$ since $p \equiv 3 \pmod 4$, a contradiction.

In conclusion, the Diophantine triple $\{1,2,5\}$ cannot be extended to a Diophantine quadruple $\{1,2,5,d\}$, for all integers $d > 5$. □

3. NONEXTENDABILITY OF $\{1,5,10\}$

Lemma 3.1. *If q and n are positive integers and n is odd, then $F^2_{2^q\cdot n} \equiv F^2_{2^q} \pmod{L_{2^{q+1}}}$.*

Proof. In formula $F^2_{nm} - F^2_m = F_{(n+1)m}F_{(n-1)m}$, if n is odd, then $4 \mid (n+1)$ or $4 \mid (n-1)$, thus $F_{4m} \mid F_{(n+1)m}$ or $F_{4m} \mid F_{(n-1)m}$. And $F_{4m} = F_{2m}L_{2m}$, then $L_{2m} \mid F_{4m} \mid \left(F^2_{nm} - F^2_m\right)$. By letting $m = 2^q$, we get $F^2_{2^q\cdot n} \equiv F^2_{2^q} \pmod{L_{2^{q+1}}}$. □

Lemma 3.2. *For any positive integer q, $L_{2^{q+1}} \equiv 7 \pmod{10}$.*

Proof. We will proof this lemma by using induction. When $q = 1$, then $L_{2^{q+1}} = L_4 = 7 \pmod{10}$. Suppose that $L_{2^{q+1}} \equiv 7 \pmod{10}$ is true, then $L_{2^{q+2}} = L^2_{2^{q+1}} - 2 \equiv 7^2 - 2 \equiv 7 \pmod{10}$. Therefore, $L_{2^{q+1}} \equiv 7 \pmod{10}$ is true for any positive integer q. □

Theorem 3.3. *The Diophantine triple $\{1,5,10\}$ cannot be extended to a $D(-1)$ quadruple $\{1,5,10,d\}$, for all integers $d > 10$.*

Proof. We only need to prove $10F^2_{2n_0} + 9$ is not a perfect square for any positive integer n_0. Suppose there exists a positive integer l such that $l^2 = 10F^2_{2n_0} + 9$.

If n_0 is odd, then $F^2_{2n_0} \equiv 1 \pmod 7$ by Lemma 3.1 for $q = 1$, then $l^2 = 10F^2_{2n_0} + 9 \equiv 5 \pmod 7$. Thus, $1 = \left(\frac{l^2}{7}\right) = \left(\frac{5}{7}\right) = -1$ gave us a contradiction, therefore n_0 is even. Rewrite $2n_0$ in the form $2n_0 = 2^q \cdot n$ such that $q \geq 2$ and $2 \nmid n$. By Lemma 3.2, $L_{2^{q+1}} \equiv 7 \pmod{10}$, then

$\left(\frac{L_{2^{q+1}}}{5}\right) = \left(\frac{2}{5}\right) = -1$, then there exists an odd prime p such that $p \mid L_{2^{q+1}}$ and $\left(\frac{p}{5}\right) = -1$. Since $10F_{2^q}^2 + 9 = 2\left(5F_{2^q}^2 + 4\right) + 1 = 2L_{2^q}^2 + 1 = 2\left(L_{2^{q+1}} + 2\right) + 1 = 2L_{2^{q+1}} + 5$, then $1 = \left(\frac{l^2}{p}\right) = \left(\frac{10F_{2n_0}^2+9}{p}\right) = \left(\frac{10F_{2^q \cdot n}^2+9}{p}\right) = \left(\frac{10F_{2^q}^2+9}{p}\right) = \left(\frac{2L_{2^{q+1}}+5}{p}\right) = \left(\frac{5}{p}\right) = -1$, a contradiction.

In conclusion, then the Diophantine triple $\{1, 5, 10\}$ cannot be extended to a Diophantine quadruple $\{1, 5, 10, d\}$, for all integers $d > 10$. □

References

[1] A. Baker, *Linear forms in the logarithms of algebraic numbers. I*, Mathematika, A Journal of Pure and Applied Mathematics **13**, (1966) 204-216.

[2] E. Brown, *Sets in which $xy + k$ is always a square.*, Math. Comp. **45** (1985), 613–620.

[3] A. Dujella, *Generalization of a problem of Diophantus*, Acta Arith. **65** (1993), 15–27.

[4] A. Dujella, *Complete solution of a family of simultaneous Pellian equations*, Acta Math. Inform. Univ. Ostraviensis **6** (1998), 59–67.

[5] A. Dujella, A. Filipin and C. Fuchs, *Effective solution of the $D(-1)$-quadruple conjecture*, Acta Math. Inform. Univ. Ostraviensis **128** (2007), no. 4, 319–338.

[6] A. Dujella and C. Fuchs, *Complete solution of a problem of Diophantus and Euler*, Journal of the London Mathematical Society **71** (2005), no. 01, 33–52.

[7] A. Filipin, *Non-extendibility of $D(-1)$-triples of the form $\{1, 10, c\}$*, International journal of mathematics and mathematical sciences **2005** (2005), no. 14, 2217–2226.

[8] P. Gibbs, *A generalised stern-brocot tree from regular Diophantine quadruples*, Arxiv preprint math/9903035 (1999).

[9] H. Gupta and K. Singh, *On k-triad sequences*, International journal of mathematics and mathematical sciences **8** (1985), no. 4, 799–804.

[10] O. Kihel, *On the extendibility of the P_{-1}-set $\{1, 2, 5\}$*, Fibonacci Quart **38** (2000), no. 5, 464–466.

[11] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, John Wiley & Sons, Inc., 2001.

[12] S. P. Mohanty and A. M. S. Ramasamy, *The simultaneous Diophantine equations $5y^2 - 20 = x^2$ and $2y^2 + 1 = z^2$*, Journal of Number Theory **18** (1984), no. 3, 356–359.

[13] S. P. Mohanty and A. M. S. Ramasamy, *On $P_{r,k}$ sequences*, Fibonacci Quarterly **23** (1985), no. 1, 36–44.

[14] F. S. Abu. Muriefah and A. Al-Rashed, *On the extendibility of the Diophantine triple $\{1, 5, c\}$*, International Journal of Mathematics and Mathematical Sciences **2004** (2004), no. 33, 1737–1746.

[15] P. G. Walsh, *On two classes of simultaneous Pell equations with no solutions*, Mathematics of Computation **68** (1999), no. 225, 385–388.

MSC2010: 11B39, 11D09

E-mail address: zhang5y@cmich.edu

Department of Mathematics, Central Michigan University, Mount Pleasant, Michigan

E-mail address: gross1gw@cmich.edu

Department of Mathematics, Central Michigan University, Mount Pleasant, Michigan

CONFERENCE PARTICIPANTS

Emrah Akyar
Anadolu University, Eskisehir, Turkey
eakyar@anadolu.edu.tr

Handan Akyar
Anadolu University, Eskisehir, Turkey
hakyar@anadolu.edu.tr

Jane Anderson
Rochester, NY, USA
jane.b.anderson@gmail.com

Peter G. Anderson
Rochester Institute of Technology, Rochester, NY, USA
pga@cs.rit.edu

Suman Balasubramanian
DePauw University, Greencastle, IN, USA
sumanbalasubramanian@depauw.edu

Christian Ballot
Université of Caen-Basse Normandie, Caen, France
christian.ballot@unicaen.fr

Arthur Benjamin
Harvey Mudd College, Claremont, CA, USA
benjamin@hmc.edu

Gerald Bergum
Brookings, SD, USA
s.bergumsr@mchsi.com

Mirac Cetin Firengiz
Baskent University, Ankara, Turkey, Ankara, Turkey
mcetin@baskent.edu.tr

Charles Cook
Univ of So Carolina Sumter, Sumter, SC, USA
charliecook29150@aim.com

Curtis Cooper
University of Central Missouri, Warrensburg, MO, USA
cooper@ucmo.edu

Janet Cooper
Rockhurst University, Kansas City, MO, USA
Janet.Cooper@rockhurst.edu

Marjorie Cooper
Joplin, MO, USA
(none)

Ashley DeFazio
Fibonacci Association Subscription Manager, Sun City, AZ, USA
fibonacciassociation@gmail.com

Daryl DeFord
Dartmouth College, Hanover, NH, USA
daryl.r.deford.gr@dartmouth.edu

Karl Dilcher
Dalhousie University, Halifax, NS, Canada
dilcher@mathstat.dal.ca

Patrick Dynes
Clemson University, Clemson, SC, USA
pdynes@clemson.edu

Xixi Edelsbrunner
Williams College, Williamstown, MA, USA
xe1@williams.edu

Larry Ericksen
Millville, NJ, USA
LE22@cornell.edu

Rigoberto Florez
The Citadel, The Military College of South Carolina, Charleston, SC, USA
rigo.florez@citadel.edu

Josep Freixas
Technical University of Catalonia, Manresa, Spain
josep.freixas@upc.edu

Heather Gaddy
Wake Forest University, Winston-Salem, NC, USA
Heather.Gaddy@alumni.wfu.edu

Dale Gerdemann
U. Tuebingen, Moessingen, Germany
dale.gerdemann@gmail.com

George Grossman
Central Michigan University, Mount Pleasant, MI, USA
gross1gw@cmich.edu

Susan Gurney
New York, NY, USA
Sgurney@nyc.rr.com

Jaime Gutierrez
University of Cantabria, Santander, Spain
jaime.gutierrez@unican.es

Ashley Hamlin
Pullman, WA, USA
ashleyhamlin@gmail.com

Nathan Hamlin
Washington State University, Pullman, WA, USA
nghamlin@wsu.edu

Heiko Harborth
TU Braunschweig, Braunschweig, Germany
h.harborth@tu-bs.de

Russell Hendel
Towson University, Baltimore, MD, USA
RHendel@Towson.Edu

Dahbia Boukari Hernane
Univ. of Sciences & Tech. Houari Boumediene, Algiers, Algeria
dboukari@yahoo.fr

Mohand Ouamar Hernane
Univ. of Sciences & Tech. Houari Boumediene, Algiers, Algeria
mohernane@yahoo.fr

Marjorie Bicknell Johnson
Santa Clara, CA, USA
marjohnson@mac.com

Virginia Johnson
Columbia College, Greenwood, SC, USA
va4.johnson@gmail.com

Ben Kaufman
Williams College, Williamstown, MA, USA
Benjamin.M.Kaufman@williams.edu

Clark Kimberling
University of Evansville, Evansville, IN, USA
ck6@evansville.edu

Margaret Kimberling
Evansville, IN, USA
morgel@wowway.com

Ron Knott
University of Surrey, Guildford, Bolton, UK
enquiry@ronknott.com

Takao Komatsu
Wuhan University, Wuhan, China
komatsu@cc.hirosaki-u.ac.jp

Jeffrey Lagarias
University of Michigan, Ann Arbor, MI, USA
lagarias@umich.edu

Kálmán Liptai
Esterhazy Karoly College, Eger, Hungary
liptaik@ektf.hu

Lynda Martin
Rubery, Birmingham, UK
lyndajanenitram@blueyonder.co.uk

Brian McDonald
University of Rochester, Rochester, NY, USA
bmcdon11@u.rochester.edu

Karyn McLellan
St. Francis Xavier University, Halifax, NS, Canada
kmclella@stfx.ca

Steven J. Miller
Williams College, Williamstown, MA, USA
sjm1@williams.edu

Peter Moses
Worcestershire, UK
mows@mopar.freeserve.co.uk

Sam Northshield
SUNY-Plattsburgh, Plattsburgh, NY, USA
northssw@plattsburgh.edu

Tomasz Nowicki
IBM, Yorktown Heights, NY, US
tnowicki@us.ibm.com

Gopal Krishna Panda
National Institute of Technology, Rourkela, Rourkela, OR, India
gkpanda_nit@rediffmail.com

Jesse L. Patsolic
Wake Forest University, Winston-Salem, NC, USA
patsjl12@wfu.edu

Chandrakant Nagnath Phadte
G.V.M's College of Commerce, Goa University, Ponda Goa, India
dbyte09@gmail.com

Marc Renault
Shippensburg University, Shippensburg, PA, USA
msrenault@ship.edu

Anthony Shannon
University of Technology, Sydney, Sydney, Australia
tshannon38@gmail.com

Lawrence Somer
Catholic University of America, Prague, Czech Republic
somer@cua.edu

Pantelimon Stănică
NPS, Applied Mathematics, Seaside, CA, USA
pstanica@nps.edu

Paul Stockmeyer
The College of William and Mary, Williamsburg, VA, USA
stockmeyer@cs.wm.edu

Márton Szikszai
University of Debrecen, Debrecen, Hungary
szikszai.marton@science.unideb.hu

Kimsy Tor
Manhattan College, Riverdale, NY, USA
ktor.student@manhattan.edu

Caroline Turnage-Butterbaugh
Williams College & North Dakota State U., Oxford, MS, USA
cturnagebutterbaugh@gmail.com

William Webb
Washington State University, Moscow, ID, USA
webb@math.wsu.edu

Michael Weiner
Penn State Altoona, Altoona, PA, USA
mdw8@psu.edu

Madeleine Weinstein
Harvey Mudd College, Claremont, CA, USA
mweinstein@hmc.edu

Fatma Yeşil
Ankara, Turkey
ftm1618ysl@gmail.com

Paul Young
College of Charleston, Charleston, SC, USA
paul@math.cofc.edu