RAMANUJAN-NAGELL TYPE EQUATIONS AND PERFECT NUMBERS

PHILIPPE ELLIA AND PAOLO MENEGATTI

Abstract. We prove that if δ is a triangular number congruent to 3 modulo 4, then the equation $x - y = \delta$ has a finite number of solutions with x, y both perfect numbers. We outline a general approach to determine the exact number of solutions and show that there is none for $\delta = 3, 15$.

Introduction

An integer $n \in \mathbb{N}$ is said to be perfect if $\sigma(n) = 2n$ where σ is the sum of the divisors function. By results of Euler every even perfect number has the form $n = 2^{p-1}(2^p - 1)$ where $2^p - 1$ is prime, whereas every odd perfect number is of the form $n = q^{4b+1}. \prod p_i^{2a_i}$, q, p_i distinct primes, $q \equiv 1 \pmod 4$; in particular if n is an odd perfect number, then $n \equiv 1 \pmod 4$. It is still unknown if odd perfect numbers exist (for some recent results see [8, 9]).

In [6], Luca and Pomerance have proved, *assuming* the *abc*-conjecture, that the equation $x - y = \delta$ has a finite number of solutions with x, y perfect, if δ is odd. Our interest in the distance between two perfect numbers comes from this result and the following obvious remark: if one could prove that an odd integer cannot be the distance between two perfect numbers, then it would follow that every perfect number is even.

From Touchard's Theorem [2] it follows that an integer $\delta \equiv \pm 1 \pmod{12}$ cannot be the distance between two perfect numbers. In [1], it has been shown that there exist infinitely many odd (triangular) numbers ($\not\equiv \pm 1 \pmod{12}$) which cannot be the distance between perfect numbers. In this note, by using results on generalized Ramanujan-Nagell equations, we prove that if δ is a triangular number congruent to 3 modulo 4, then $x - y = \delta$ has a finite number of solutions with x, y perfect numbers. We also outline a general approach to determine the exact number of solutions. For example, we show that $\delta = 3, 15$ cannot be the distance between two perfect numbers.

1. Ramanujan-Nagell Equations and Perfect Numbers

Let $D_1, D_2 \in \mathbb{Z}$ be nonzero integers, then the equation (in x, n)

$$D_1 x^2 + D_2 = 2^n \tag{1.1}$$

is a *generalized Ramanujan-Nagell* equation. Recall the following result of Thue [4].

Theorem 1.1. *Let $a, b, c, d \in \mathbb{Z}$ such that $ad \neq 0$, $b^2 - 4ac \neq 0$. Then the equation*

$$ax^2 + bx + c = dy^n \tag{1.2}$$

has only a finite number of solutions in integers x and y when $n \geq 3$.

Applying this result to $D_1 x^2 + D_2 = dy^3$, $d = 1, 2, 4$, we conclude with the following corollary.

Corollary 1.2. *For $n \geq 3$, equation (1.1) has a finite number of solutions (x, n).*

An odd perfect number n is congruent to 1 modulo 4, while an even one, m, is congruent to 0 modulo 4 except if $m = 6$. It follows, for odd δ, that $\delta \equiv 1 \pmod 4$ if $\delta = n - m$ or $\delta \equiv 3 \pmod 4$ if $\delta = m - n$ or $\delta + 6 = n$. Using the result above we have the following theorem.

Theorem 1.3. *Let $\delta = b(b-1)/2$ be a triangular number such that $\delta \equiv 3 \pmod 4$. The equation $x - y = \delta$ has at most finitely many solutions x, y both perfect.*

Proof. We may assume $m - n = \delta = b(b-1)/2$, with m, n perfect numbers and $m = 2^{p-1}(2^p - 1)$. Then we have:

$$2n = (2^p - 1 + b)(2^p - b). \tag{1.3}$$

Moreover by Euler's Theorem $n = q^{4b+1}\prod p_i^{2a_i}$, q, p_i distinct primes, $q \equiv 1 \pmod 4$.

Since $(2^p - 1 + b, 2^p - b) = (2^p - 1 + b, 2b - 1)$, if a prime p divides both $A = 2^p - 1 + b$ and $B = 2^p - b$, it must divide $2b - 1$. In any case we can write $A = p^{\varepsilon}.A'$, $\varepsilon \in \{0, 1\}$ and $p^{2\alpha} \parallel A'$. Similarly $B = p^e.B'$, $e \in \{0, 1\}$, $p^{2\beta} \parallel B'$. It turns out that A or B is of the form: d times a square, where d is a (square free) divisor of $2(2b-1)$. So $2^p = dC^2 - b + 1$ or $2^p = dD^2 + b$. By Corollary 1.2 each equation $dx^2 + D_2 = 2^n$ ($D_2 = -b + 1$ or b) has a finite number of solutions. Since $2(2b - 1)$ has a finite number of divisors we are done. □

Remark 1.4. *As far as $\delta = b(b-1)/2$ is congruent to 3 modulo 4 in order to show that δ can't be the distance between two perfect numbers one has:*

(1) *to show that $\delta + 6$ is not perfect.*
(2) *for any square free divisor d of $2(2b - 1)$ to solve the equations: $dx^2 + D_2 = 2^n$ ($D_2 \in \{-b + 1, b\}$ (see proof of Theorem 1.3)). For any solution (x, n) such that $n = p$ is prime, check if $2^p - 1$ is prime. If it is, check if $2^{p-1}(2^p - 1) - \delta$ is perfect.*

Since a great deal is known on the generalized Ramanujan-Nagell equations (see [11] for a survey), in practice, for a given δ, the above procedure should allow to conclude (see also [7, 10] for an algorithmic approach). Sometimes it is possible to go faster, for example we have the following proposition.

Proposition 1.5. *The equation $x - y = 15$ has no solutions with x, y both perfect numbers.*

Proof. This is the case $b = 6$ of Theorem 1.3. If the Euler prime, q, of n divides both A and B it must divide $2b - 1 = 11$, so $q = 11$ which is impossible since $q \equiv 1 \pmod 4$. If $q \mid B$, then $A = 2^p + 5 = dx^2$, with $d = 1$ or 11. Since x is odd we get $5 \equiv d \pmod 8$ which is absurd. We conclude that $q \mid A$ and $B = 2^p - 6 = dx^2$, where $d \mid 22$. Reducing modulo 3 we see that $d = 1$ is impossible. Reducing modulo 4 we exclude the cases $d = 11, 22$. Finally it is easy to see that the unique solution of $2x^2 + 6 = 2^n$ is $(x, n) = (\pm 1, 3)$. □

To conclude let us see another example, the case $\delta = 3$ which is still open.

Lemma 1.6. *Assume m, n are perfect numbers such that $m - n = 3$. Then $m = 2^{p-1}(2^p - 1)$ with $2^p - 1$ prime and $2^p = 5u^2 + 3$ for some integer u.*

Proof. Since $\delta = b(b-1)/2$ with $b = 3$ and since $\delta + 6 = 9$ is not perfect, we see that m is even and n is odd. So $m = 2^{p-1}(2^p - 1)$ with $2^p - 1$ prime. Moreover, $n = (2^{p-1} + 1)(2^p - 3)$ (equation (1.3) in the proof of Theorem 1.3). Also $M = gcd(A, B) = 5$ or 1 ($A = 2^{p-1} + 1$, $B = 2^p - 3$).

If $M = 1$, from Euler's Theorem, A or B is a square. Since $A = (2^{(p-1)/2})^2 + 1$, A can't be a square. Since $B = 2^p - 3 \equiv 2 \pmod 3$, B can't be a square. It follows that $M = 5$.

Since $n = m - 3$ and $m \equiv 1 \pmod 3$, we get $n \equiv 1 \pmod 3$. We also have $n = qD^2$ ($q \equiv 1 \pmod 4$, the Euler's prime). It follows that $q \equiv 1 \pmod 3$. In particular $q \neq 5$. Finally if

$q \nmid A$, then $A = 2^{p-1} + 1 = 5C^2$. Since C is odd $C^2 \equiv 1 \pmod 8$. Since we may assume $p \geq 4$, we get a contradiction. So $q \mid A$ and $B = 2^p - 3 = 5u^2$ for some integer u. □

Corollary 1.7. *If $\delta = |x - y|$, with x, y perfect numbers, then $\delta > 3$.*

Proof. The cases $\delta \leq 2$ follow from considerations on congruences (see [5]). If $\delta = 3$, then from Lemma 1.6: $m - n = 3$, where $m = 2^{p-1}(2^p - 1)$ and $2^p = 5u^2 + 3$. So (u, p) is a solution of the equation $5x^2 + 3 = 2^n$. It is known [3] that the only solutions in positive integers of this equations are $(x, n) = (1, 3), (5, 7)$. Since 25 and $2^6(2^7 - 1) - 3 = 8125$ are not perfect numbers, we conclude. □

If δ is not a triangular number congruent to $3 \pmod 4$ we no longer have the factorization $2n = (2^p - 1 + b)(2^p - b)$ and things get harder. The cases $\delta = 5, 7$ can be excluded by congruence considerations. However for odd $\delta \leq 15$, the case $\delta = 9$ is still open, as is the problem to show that a triangular number congruent to 3 modulo 4 can't be the distance between two perfect numbers.

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Dipartimento di Matematica e Informatica, Università di Ferrara, 35 via Machiavelli, 44100 FERRARA (IT.)

E-mail address: phe@unife.it

Dipartimento di Matematica e Informatica, Università di Ferrara, 35 via Machiavelli, 44100 FERRARA (IT.)

E-mail address: paolo.menegatti@student.unife.it