A PROPERTY OF A FIBONACCI STAIRCASE

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Abstract. In this article we construct, by way of the Fibonacci numbers, a geometrical object resembling an infinite staircase. We go on to demonstrate an interesting property of this Fibonacci staircase.

1. introduction

A well-known combinatorial situation in which Fibonacci numbers are associated with a staircase is as follows:

> Suppose that you are walking up a staircase comprising n steps, and climbing either one or two steps at a time. In how many ways could you reach the top?

This is in fact a rather straightforward problem, the solution to which is F_{n+1} [3]. Note that it is not necessary to know the precise height and depth of each step in order to solve this problem; we simply need to be able to assume that all the ways of getting to the top by climbing either one or two steps at a time are actually possible. By way of a contrast, we consider here a scenario involving the Fibonacci numbers and a staircase in which the dimensions of the latter are indeed crucial. As will be shown below, the Fibonacci numbers are used to construct an infinite staircase possessing an interesting geometrical, as opposed to a combinatorial, property.

Let us demonstrate how the staircase is built. Assuming that the smallest two squares each have side length 1 unit, the Fibonacci rectangle given in Figure 1 is composed of squares of side lengths $F_1, F_2, F_3, \ldots, F_8$. Readers may have seen pictures of nautilus shells drawn using just such a rectangle. Now imagine unraveling this rectangle to give a staircase comprising these Fibonacci squares, as is shown (with the largest square omitted) in Figure 2. Let $\mathcal{S}$ denote the infinite Fibonacci staircase that results on allowing this construction to continue ad infinitum, so that the nth step consists of a $F_n \times F_n$ square. In this article we consider what happens when we choose two distinct corners of $\mathcal{S}$, join them with a line, and then extend this line as far as the 1×1 square on the left and indefinitely to the right, with the aim of obtaining a result giving, for each corner in $\mathcal{S}$, conditions on whether it is above, on, or below the line.

2. Some Initial Definitions and Calculations

For the sake of notational convenience, we set the coordinates of the top left-hand corner of the left-most square in $\mathcal{S}$ as $(1,1)$. The coordinates of successive corners of $\mathcal{S}$ are then given by $(1,1),(2,1),(3,2),(5,3),\ldots$, and so on, with the nth such corner being located at the point (F_{n+1}, F_n).

In Figure 3 we see that the corners $A\,(F_6, F_5)$ and $B\,(F_7, F_6)$ have been connected with a line which has subsequently been extended in both directions. In this example the corners are consecutive, but we cover here the general case in which the corners are given by $A\,(F_{m+1}, F_m)$

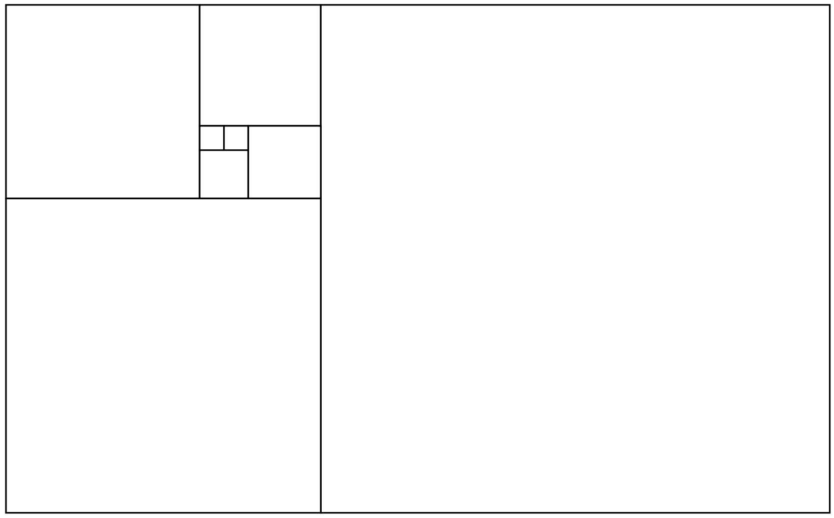

FIGURE 1. A Fibonacci rectangle.

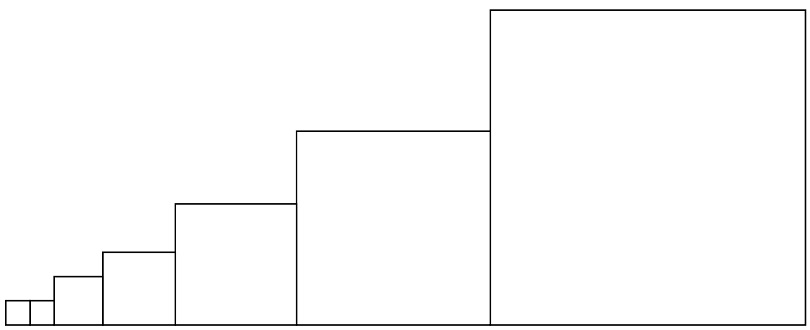

FIGURE 2. A Fibonacci staircase.

and $B\left(F_{n+1}, F_n\right)$ for any $m, n \in \mathbb{N}$ such that $n > m \geq 1$. We denote the extended line by $\mathcal{L}(m, n)$.

The gradient of $\mathcal{L}(m, n)$ is given by

$$\frac{F_n - F_m}{F_{n+1} - F_{m+1}},$$

and its Cartesian equation is thus of the form

$$y = \left(\frac{F_n - F_m}{F_{n+1} - F_{m+1}}\right) x + c$$

for some constant c. Since $\mathcal{L}(m, n)$ passes through the point (F_{n+1}, F_n), we see that

$$c = F_n - \frac{F_{n+1}\left(F_n - F_m\right)}{F_{n+1} - F_{m+1}}.$$

From this it follows that the difference in the y coordinate of the corner (F_{j+1}, F_j) and that of $\mathcal{L}(m,n)$ when $x = F_{j+1}$ is given by

$$\begin{aligned} D(j,m,n) &= F_j - \left(\frac{F_{j+1}\left(F_n - F_m\right)}{F_{n+1} - F_{m+1}} + F_n - \frac{F_{n+1}\left(F_n - F_m\right)}{F_{n+1} - F_{m+1}} \right) \\ &= F_j - F_n - \frac{\left(F_{j+1} - F_{n+1}\right)\left(F_n - F_m\right)}{F_{n+1} - F_{m+1}} \\ &= \frac{\left(F_j - F_n\right)\left(F_{n+1} - F_{m+1}\right) - \left(F_n - F_m\right)\left(F_{j+1} - F_{n+1}\right)}{F_{n+1} - F_{m+1}}, \end{aligned}$$

noting that the corner (F_{j+1}, F_j) is above $\mathcal{L}(m,n)$ if, and only if, $D(j,m,n) > 0$.

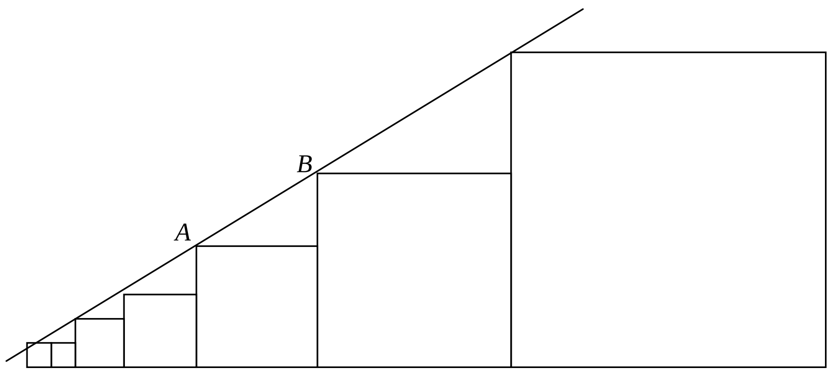

FIGURE 3. A line constructed on a Fibonacci staircase.

We are interested here only in whether a given corner is above or below $\mathcal{L}(m,n)$, and not in the actual numerical differences in their heights. Since $n > m \geq 1$, it follows that $F_{n+1} - F_{m+1} > 0$, and we thus need only to consider the sign of $G(j,m,n)$ given by

$$G(j,m,n) = \left(F_j - F_n\right)\left(F_{n+1} - F_{m+1}\right) - \left(F_n - F_m\right)\left(F_{j+1} - F_{n+1}\right). \tag{2.1}$$

It is possible to give $G(j,m,n)$ in a slightly more amenable form. Indeed, on expanding and simplifying the right-hand side of (2.1) we obtain

$$\begin{aligned} G(j,m,n) &= \left(F_jF_{n+1} - F_{j+1}F_n\right) + \left(F_nF_{m+1} - F_{n+1}F_m\right) - \left(F_jF_{m+1} - F_{j+1}F_m\right) \\ &= (-1)^n F_{j-n} + (-1)^m \left(F_{n-m} - F_{j-m}\right), \end{aligned} \tag{2.2}$$

where we have used d'Ocagne's identity $F_aF_{b+1} - F_{a+1}F_b = (-1)^b F_{a-b}$ [4]. Note that for any $n \in \mathbb{N}$, we may extend the Fibonacci numbers to negative subscripts by way of the following [2]:

$$F_{-n} = (-1)^{n+1} F_n. \tag{2.3}$$

3. Possible Patterns

For a particular line $\mathcal{L}(m,n)$, we may associate with each corner of $\mathcal{S}$ precisely one of the symbols '+', '0' or '−', depending on whether the corner lies above, lies on, or lies below $\mathcal{L}(m,n)$, respectively. Thus each line $\mathcal{L}(m,n)$ corresponds to a particular infinite string comprising the symbols '+', '0', and '−'. To take two examples, $\mathcal{L}(6,11)$ and $\mathcal{L}(9,10)$ give rise to the strings

$$+ - + - + 0 + + + + 0 - - - - - \ldots$$

and

$$+-+-+-0-00+++++\dots,$$

respectively, as is easily checked. We will use $\mathcal{Q}(m,n)$ to denote the infinite string induced by the staircase and the line $\mathcal{L}(m,n)$. The kth character, counting from the left, of this string is given by $\mathcal{Q}_k(m,n)$. For example, $\mathcal{Q}(6,11) = +-+-+0++++0------\dots$ and $\mathcal{Q}_8(6,11) = +$.

In the following sections we will, by using (2.2) and (2.3), classify all possible such patterns. Incidentally, as may be seen in Figure 3, it is not always easy to determine with the naked eye which of the symbols is associated with a particular corner of $\mathcal{S}$.

4. Some General Results

Note first that by construction we always have $G(m,m,n) = G(n,m,n) = 0$. However, when $n-m \le 3$, $\mathcal{Q}(m,n)$ can contain up to three '0's, as we shall see in Section 5. Thus, for the sake of clarity, we will assume that $n-m \ge 4$ through the current section. We obtain here three lemmas, as follows.

Lemma 4.1. *Let $j > n$. Then either $G(j,m,n) > 0$ or $G(j,m,n) < 0$, depending on whether m is odd or even, respectively.*

Proof. From [1, 2] we have

$$F_{a+b} = F_{a-1}F_b + F_aF_{b+1}. \tag{4.1}$$

On setting $a = j-n$ and $b = n-m$ in (4.1), we obtain

$$\begin{aligned}
G(j,m,n) &= (-1)^n F_{j-n} + (-1)^m \left(F_{n-m} - F_{j-m}\right) \\
&= (-1)^n F_{j-n} + (-1)^m \left(F_{n-m} - F_{(j-n)+(n-m)}\right) \\
&= (-1)^n F_{j-n} + (-1)^m \left(F_{n-m} - F_{j-n-1}F_{n-m} - F_{j-n}F_{n-m+1}\right) \\
&= F_{j-n}\left((-1)^n - (-1)^m F_{n-m+1}\right) + (-1)^m F_{n-m}\left(1 - F_{j-n-1}\right).
\end{aligned}$$

Thus, if m is odd we have

$$G(j,m,n) = F_{j-n}\left((-1)^n + F_{n-m+1}\right) + F_{n-m}\left(F_{j-n-1} - 1\right).$$

Then, since $F_{j-n} \ge F_1 = 1$ and $F_{j-n-1} \ge F_0 = 0$, we obtain the result

$$G(j,m,n) \ge \left(F_{n-m+1} - 1\right) - F_{n-m} > 0,$$

on noting that $F_{n-m+1} - 1 > F_{n-m}$ when $n-m \ge 4$. In a similar manner, it may be shown that $G(j,m,n) < 0$ when m is even. □

Lemma 4.2. *Let $m < j < n$. Then either $G(j,m,n) < 0$ or $G(j,m,n) > 0$, depending on whether m is odd or even, respectively.*

Proof. If m is odd then, on using (2.3), we obtain

$$\begin{aligned}
G(j,m,n) &= (-1)^n(-1)^{n-j+1}F_{n-j} + (-1)^m \left(F_{n-m} - F_{j-m}\right) \\
&= (-1)^{j-1}F_{n-j} - F_{n-m} + F_{j-m}.
\end{aligned}$$

In order to prove what is required here, we need to show that

$$F_{n-m} > (-1)^{j-1}F_{n-j} + F_{j-m}. \tag{4.2}$$

On setting $a = n-j$ and $b = j-m$ in (4.1), (4.2) may be rewritten as

$$F_{j-m}\left(F_{n-j-1} - 1\right) > F_{n-j}\left((-1)^{j-1} - F_{j-m+1}\right).$$

The left-hand side of this inequality is positive when $n-j \geq 4$, while the right-hand side is always non-positive. It is therefore the case that, in order to show that (4.2) is true, we simply need to check the cases $n-j=1$, $n-j=2$ and $n-j=3$.

When $n-j=1$ the right-hand side of (4.2) is given by

$$(-1)^n + F_{n-m-1}.$$

Note that

$$F_{n-m} > (-1)^n + F_{n-m-1}$$

when $n-m \geq 5$, so we just need to check the case $n-m=4$. Since m is odd then, in this particular case, n must also be odd, giving

$$(-1)^n + F_{n-m-1} = -1 + F_3 = 1 < 3 = F_4 = F_{n-m}.$$

Next, if $n-j=2$ then

$$(-1)^{j-1} F_{n-j} + F_{j-m} = (-1)^{n-3} F_2 + F_{n-m-2} \leq 1 + F_{n-m-2} < F_{n-m}.$$

Finally, with $n-j=3$, we have

$$(-1)^{j-1} F_{n-j} + F_{j-m} = (-1)^{n-4} F_3 + F_{n-m-3} = 2(-1)^{n-4} + F_{n-m-3}.$$

If $n-m \geq 5$ then

$$F_{m-n} > 2 + F_{n-m-3} \geq 2(-1)^{n-4} + F_{n-m-3},$$

as required, so we just need once more to check the special case $n-m=4$. When m is odd in this case, n is also odd, so we have

$$(-1)^{j-1} F_{n-j} + F_{j-m} = (-1)^{n-4} F_3 + F_{n-m-3} = -2 + F_1 = -1 < F_4 = F_{n-m},$$

thereby completing the proof of the lemma. □

Lemma 4.3. *Let $j < m$. Then either $G(j,m,n) > 0$ or $G(j,m,n) < 0$, depending on whether j is odd or even, respectively.*

Proof. Using (2.3), we have

$$\begin{aligned} G(j,m,n) &= (-1)^n (-1)^{n-j+1} F_{n-j} + (-1)^m \left(F_{n-m} - (-1)^{m-j+1} F_{m-j} \right) \\ &= (-1)^{j-1} F_{n-j} + (-1)^m F_{n-m} - (-1)^{j-1} F_{m-j}. \end{aligned}$$

If j is odd, this gives

$$\begin{aligned} G(j,m,n) &= F_{n-j} + (-1)^m F_{n-m} - F_{m-j} \\ &\geq F_{n-j} - F_{n-m} - F_{m-j} \\ &= F_{(n-m)+(m-j)} - F_{n-m} - F_{m-j} \\ &= F_{n-m-1} F_{m-j} + F_{n-m} F_{m-j+1} - F_{n-m} - F_{m-j} \\ &= F_{m-j} \left(F_{n-m-1} - 1 \right) + F_{n-m} \left(F_{m-j+1} - 1 \right), \end{aligned}$$

where we have used (4.1) once more. This implies, since $F_{n-m-1} - 1 \geq F_3 - 1 = 1$ and $F_{m-j+1} \geq F_2 = 1$, that $G(j,m,n) > 0$. It may also be shown that $G(j,m,n) < 0$ when j is even. □

5. Locating the Zeros

Since $G(m,m,n) = G(n,m,n) = 0$, it follows from Lemmas 4.1, 4.2, and 4.3 that $\mathcal{Q}(m,n)$ contains exactly two '0's when $n-m \geq 4$, namely $\mathcal{Q}_m(m,n) = \mathcal{Q}_n(m,n) = 0$. On lifting the restriction $n-m \geq 4$, however, we may obtain strings containing exactly three '0's. When $n-m=3$ we have, in addition to $G(n-3,n-3,n) = G(n,n-3,n) = 0$,

$$\begin{aligned} G(n-1,n-3,n) &= (-1)^n F_{-1} + (-1)^{n-3}F_3 - (-1)^{n-3}F_2 \\ &= (-1)^n + 2(-1)^{n-3} - (-1)^{n-3} \\ &= (-1)^n - 2(-1)^n + (-1)^n \\ &= 0. \end{aligned} \tag{5.1}$$

Similarly, when $n-m=2$ and $n-m=1$, we obtain

$$G(n+1,n-2,n) = (-1)^n F_1 + (-1)^{n-2}F_2 - (-1)^{n-2}F_3 = 0 \tag{5.2}$$

and

$$G(n-3,n-1,n) = (-1)^n F_{-3} + (-1)^{n-1}F_1 - (-1)^{n-1}F_{-2} = 0, \tag{5.3}$$

respectively.

In fact, as will be explained in Section 6, when $n-m \leq 3$, the only pairs (m,n) for which $\mathcal{Q}(m,n)$ contains exactly two '0's are given by $(1,2)$ and $(2,3)$. We are now in a position to be able to give all possible patterns of the infinite string $\mathcal{Q}(m,n)$.

6. A Classification of the Patterns

Let, for $k \geq 0$, $\text{Alt}(k)$ denote the string of length k comprising alternating '+' and '−' signs, where the left-most sign is always a '+' and $\text{Alt}(0)$ is the empty string. So, for example, $\text{Alt}(6) = + - + - + -$ and $\text{Alt}(7) = + - + - + - +$. Next, let $\text{Plus}(k)$ and $\text{Minus}(k)$ be the strings of length k consisting only of '+' signs and '−' signs, respectively. Note that, for the strings of the latter two types, $\text{Plus}(\infty)$ and $\text{Minus}(\infty)$ have obvious interpretations.

Dealing first with the situation in which $n-m \geq 4$, we obtain, with $X \diamond Y$ denoting the concatenation of the strings X and Y, the result

$$\mathcal{Q}(m,n) = \begin{cases} \text{Alt}(m-1) \diamond 0 \diamond \text{Plus}(n-m-1) \diamond 0 \diamond \text{Minus}(\infty) & \text{if } m \text{ is even;} \\ \text{Alt}(m-1) \diamond 0 \diamond \text{Minus}(n-m-1) \diamond 0 \diamond \text{Plus}(\infty) & \text{if } m \text{ is odd.} \end{cases}$$

This follows directly from Lemmas 4.1, 4.2, and 4.3.

We now consider the remaining cases, starting with $n-m=1$. It is straightforward to check that $G(j,1,2) > 0$ when $j \geq 3$, $G(j,2,3) < 0$ when $j \geq 4$, and $G(1,2,3) > 0$, giving rise to

$$\mathcal{Q}(1,2) = 00 \diamond \text{Plus}(\infty)$$

and

$$\mathcal{Q}(2,3) = \text{Alt}(1) \diamond 00 \diamond \text{Minus}(\infty).$$

When $n \geq 4$ an extra '0' is introduced into $\mathcal{Q}(m,n)$ by way of (5.3). Simple calculations then reveal that for n even we have $G(n-2,n-1,n) < 0$ and $G(j,n-1,n) > 0$ when $j > n$, while if n is odd it is the case that $G(n-2,n-1,n) > 0$ and $G(j,n-1,n) < 0$ when $j > n$.

Furthermore, when $n > 4$ and $1 \leq j \leq n-4$ we have $G(j, n-1, n) > 0$ and $G(j, n-1, n) < 0$ when j is odd and even, respectively. This leads to the following result for $n \geq 4$:

$$\mathcal{Q}(n-1, n) = \begin{cases} \mathrm{Alt}(n-4) \diamond 0 \diamond \mathrm{Minus}(1) \diamond 00 \diamond \mathrm{Plus}(\infty) & \text{if } n \text{ is even;} \\ \mathrm{Alt}(n-4) \diamond 0 \diamond \mathrm{Plus}(1) \diamond 00 \diamond \mathrm{Minus}(\infty) & \text{if } n \text{ is odd.} \end{cases}$$

Similarly, when $n - m = 2$, we have, on using (5.2),

$$\mathcal{Q}(n-2, n) = \begin{cases} \mathrm{Alt}(n-3) \diamond 0 \diamond \mathrm{Minus}(1) \diamond 00 \diamond \mathrm{Plus}(\infty) & \text{if } n \text{ is odd;} \\ \mathrm{Alt}(n-3) \diamond 0 \diamond \mathrm{Plus}(1) \diamond 00 \diamond \mathrm{Minus}(\infty) & \text{if } n \text{ is even.} \end{cases}$$

Finally, when $n - m = 3$, we obtain, using (5.1),

$$\mathcal{Q}(n-3, n) = \begin{cases} \mathrm{Alt}(n-4) \diamond 0 \diamond \mathrm{Minus}(1) \diamond 00 \diamond \mathrm{Plus}(\infty) & \text{if } n \text{ is even;} \\ \mathrm{Alt}(n-4) \diamond 0 \diamond \mathrm{Plus}(1) \diamond 00 \diamond \mathrm{Minus}(\infty) & \text{if } n \text{ is odd.} \end{cases}$$

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