RAMANUJAN'S LAST PROBLEM

MICHAEL D. HIRSCHHORN

1. Introduction

In his first letter to Hardy, Ramanujan [1] made a claim that turned out to be among the last of his claims to be settled. With some poetic licence, I dub this claim "Ramanujan's Last Problem".

Before I state this claim, I would like to set the scene. Consider the series

$$e^n = 1 + \frac{n}{1!} + \frac{n^2}{2!} + \cdots + \frac{n^{n-1}}{(n-1)!} + \frac{n^n}{n!} + \cdots .$$

The terms increase in size, then decrease. There are two equally large, largest terms, namely

$$\frac{n^{n-1}}{(n-1)!} = \frac{n^n}{n!} = M.$$

We can split the series into two "halves", the terms up to and including $\frac{n^{n-1}}{(n-1)!}$, and the terms from $\frac{n^n}{n!}$ onwards. In terms of M, the "left-hand half" can be written

$$A = M\left\{1 + \left(1-\frac{1}{n}\right) + \left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + \left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\left(1-\frac{3}{n}\right) + \cdots\right\}$$

while the "right-hand half" can be written

$$B = M\left\{1 + 1\Big/\left(1+\frac{1}{n}\right) + 1\Big/\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right) + 1\Big/\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\left(1+\frac{3}{n}\right) + \cdots\right\}.$$

It is easy to see that $B > A$: for $k = 1, \cdots, n-1$, the kth term in B is greater than the kth term in A,

$$1\Big/\left(1+\frac{1}{n}\right)\cdots\left(1+\frac{k}{n}\right) > \left(1-\frac{1}{n}\right)\cdots\left(1-\frac{k}{n}\right),$$

and there are more terms in B, all positive.

Now let us define θ by

$$B - A = 2\theta M.$$

If we transfer θM from B to A, the two quantities $A + \theta M$ and $B - \theta M$ are equal, and

$$A + \theta M = B - \theta M = \frac{e^n}{2}.$$

That is,

$$1 + \frac{n}{1!} + \frac{n^2}{2!} + \cdots + \frac{n^{n-1}}{(n-1)!} + \frac{n^n}{n!}\theta = \frac{e^n}{2},$$

where

$$2\theta = \left\{1\Big/\left(1+\frac{1}{n}\right) + 1\Big/\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right) + \cdots\right\}$$
$$- \left\{\left(1-\frac{1}{n}\right) + \left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + \cdots\right\}.$$

The claim that Ramanujan made in his letter to Hardy was that

$$\theta = \frac{1}{3} + \frac{4}{135(n+k)} \quad \text{where } \frac{2}{21} < k < \frac{8}{45}.$$

Ramanujan's claim has only recently been proved [2]. The proof required considerable ingenuity.

What is surprising, amazing even, is that θ has a limit as $n \to \infty$, and indeed that

$$\theta \to \frac{1}{3} \text{ as } n \to \infty. \tag{1.1}$$

Ramanujan [3] later indicated that

$$\theta = \frac{1}{3} + \frac{4}{135n} - \frac{8}{2835n^2} - \frac{16}{8505n^3} + \cdots \text{ as } n \to \infty.$$

The object of this note is to take a naive approach and prove (1.1).

2. In Which We Show That $\theta \to \frac{1}{3}$ as $n \to \infty$

We start by writing

$$\theta = \sum_{k=1}^{\infty} \theta_k$$

where

$$2\theta_k = \prod_{l=1}^{k}\left(1+\frac{l}{n}\right)^{-1} - \prod_{l=1}^{k}\left(1-\frac{l}{n}\right).$$

If we choose large values of n, and plot θ_k against k, up to say, $k = n$, we obtain graphs of the same shape, with a maximum which appears to occur at $k = \sqrt{3n}$. This gives us a clue as to how we might proceed: we graph θ_k against $x = \dfrac{k}{\sqrt{n}}$. But because of this fore-shortening by a factor of $\sqrt{n}$, we correspondingly increase the height by a factor of $\sqrt{n}$, and plot $\theta_k\sqrt{n} = f_n(x)$ against x.

If we do this, we find that the curves for various values of n are virtually identical! And of course θ is essentially equal to the area under this curve, whatever it may be. So we wish to determine the equation of this curve, which has a maximum at $x = \sqrt{3}$, and which is to all intents and purposes equal to 0 for $x > 5$.

We have

$$\begin{aligned} 2\theta_k &= \frac{n^k n!}{(n+k)!} - \frac{(n-1)!}{n^k(n-k-1)!} \\ &= \frac{n^{x\sqrt{n}} n!}{(n+x\sqrt{n})!} - \frac{(n-1)!}{n^{x\sqrt{n}}(n-x\sqrt{n}-1)!} \\ &= \frac{n^{x\sqrt{n}}\Gamma(n+1)}{\Gamma(n+x\sqrt{n}+1)} - \frac{\Gamma(n)}{n^{x\sqrt{n}}\Gamma(n-x\sqrt{n})}. \end{aligned} \tag{2.1}$$

We use Stirling's formula, in the form

$$\log \Gamma(x+1) = \left(x + \frac{1}{2}\right) \log x - x + \frac{1}{2} \log 2\pi + \mathrm{O}\left(\frac{1}{x}\right) \quad \text{as } x \to \infty.$$

The first term on the right of (2.1) is

$$\begin{aligned}
\exp\Bigg\{ & x\sqrt{n} \log n + \left(n + \frac{1}{2}\right) \log n - n \\
& - \left(n + x\sqrt{n} + \frac{1}{2}\right)\left(\log n + \log\left(1 + \frac{x}{\sqrt{n}}\right)\right) + (n + x\sqrt{n}) + \mathrm{O}\left(\frac{1}{n}\right)\Bigg\} \\
& = \exp\left\{-\frac{x^2}{2} + \frac{x^3 - 3x}{6\sqrt{n}} + \mathrm{O}\left(\frac{1}{n}\right)\right\} \\
& = e^{-x^2/2}\left\{1 + \frac{x^3 - 3x}{6\sqrt{n}} + \mathrm{O}\left(\frac{1}{n}\right)\right\}
\end{aligned}$$

while the second term on the right of (2.1) is

$$\begin{aligned}
\exp\Bigg\{ & \left(n - \frac{1}{2}\right)\left(\log n + \log\left(1 - \frac{1}{n}\right)\right) - (n-1) - x\sqrt{n} \log n \\
& - \left(n - x\sqrt{n} - \frac{1}{2}\right)\left(\log n + \log\left(1 - \frac{x\sqrt{n} + 1}{n}\right)\right) \\
& + (n - x\sqrt{n} - 1) + \mathrm{O}\left(\frac{1}{n}\right)\Bigg\} \\
& = \exp\left\{-\frac{x^2}{2} - \frac{x^3 + 3x}{6\sqrt{n}} + \mathrm{O}\left(\frac{1}{n}\right)\right\} \\
& = e^{-x^2/2}\left\{1 - \frac{x^3 + 3x}{6\sqrt{n}} + \mathrm{O}\left(\frac{1}{n}\right)\right\}.
\end{aligned}$$

Thus (2.1) becomes

$$2\theta_k = e^{-x^2/2}\left\{\frac{x^3}{3\sqrt{n}} + \mathrm{O}\left(\frac{1}{n}\right)\right\}$$

and hence,

$$f_n(x) = \theta_k \sqrt{n} = \frac{x^3 e^{-x^2/2}}{6} + \mathrm{O}\left(\frac{1}{\sqrt{n}}\right).$$

So we see that the limiting curve is

$$\frac{x^3 e^{-x^2/2}}{6},$$

and

$$\lim_{n \to \infty} \theta = \int_0^\infty \frac{x^3 e^{-x^2/2}}{6}\, dx = \frac{1}{3}.$$

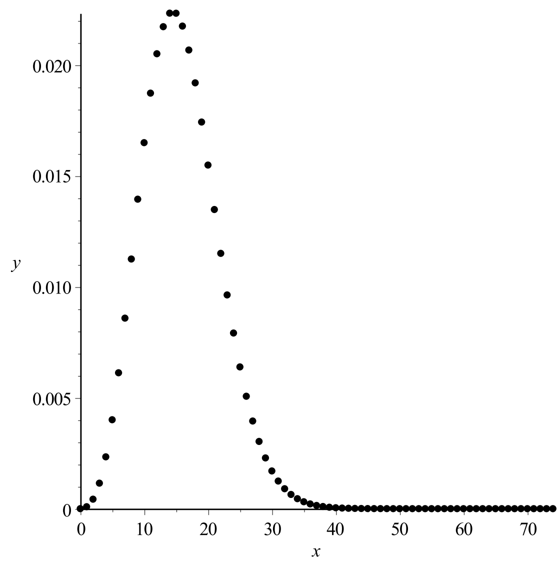

FIGURE 1. The case $n = 75$. θ_k for $k = 0, \cdots, n$.

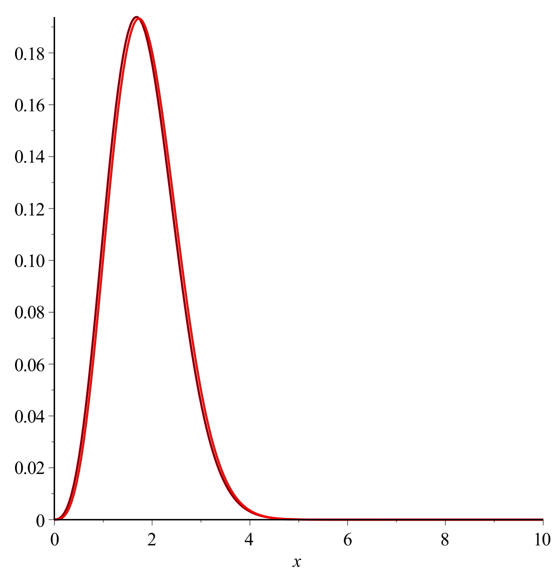

FIGURE 2. The case $n = 75$. $f_n(x)$ and the limiting curve plotted together.

REFERENCES

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MSC2010: 41A60, 97I50

School of Mathematics and Statistics, UNSW, Sydney, Australia 2052
E-mail address: m.hirschhorn@unsw.edu.au