CONGRUENCES FOR PARTIAL SUMS OF RECIPROCALS

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Abstract. We systematically derive congruences for the sums $\sum_{j=1}^{\lfloor kp/M\rfloor} 1/j^2$ modulo p and for the sums $\sum_{j=1}^{\lfloor kp/M\rfloor} 1/j$ modulo p^2, for all integers $M \geq 1$ that divide 24 and integers k with $1 \leq k \leq M$ and $\gcd(M,k) = 1$. While many of these congruences are well-known, others are new in the forms given. The congruences involve Fermat quotients, Euler numbers, Bernoulli polynomials, and some particular classes of generalized Bernoulli numbers belonging to quadratic characters.

1. Introduction

Congruences for sums of reciprocals of consecutive integers have been studied extensively, beginning with a paper of Eisenstein in 1850. Let us first consider the easiest such congruence. Noting that $\{1, 1/2, \dots, 1/(p-1)\}$ forms a reduced residue system modulo an odd prime p since $\{1, 2, \dots, p-1\}$ does, we immediately get

$$\sum_{j=1}^{p-1} \frac{1}{j} \equiv 0 \pmod{p}. \tag{1.1}$$

Now an obvious question to ask is what can be said about *partial* sums of reciprocals. In [10], Eisenstein proved what amounts to the congruence

$$\sum_{j=1}^{\frac{p-1}{2}} \frac{1}{j} \equiv -2\, q_p(2) \pmod{p}, \tag{1.2}$$

where $q_p(a)$, for an odd prime p and an integer $a \geq 2$ with $p \nmid a$, is the *Fermat quotient to base* a defined by

$$q_p(a) := \frac{a^{p-1}-1}{p}; \tag{1.3}$$

by Fermat's Little Theorem this is an integer. The congruence (1.2) can be, and has been, generalized in various directions; see the final section for further details.

Historically, these sums and congruences were of interest in connection with the classical theory of Fermat's Last Theorem; see, e.g., [16, p. 358ff.], [17, p. 155ff.], or [9]. More recent applications of such congruences include extensions of the binomial coefficient theorems of Gauss and Jacobi to higher (especially mod p^3) congruences; see [6].

It is the purpose of this paper to derive some new congruences for sums of reciprocals that are required for proving other congruences modulo p^3 for binomial coefficients; see [1]. In the process we give a systematic treatment of congruences for the sums

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$$\sum_{j=1}^{\lfloor kp/M \rfloor} \frac{1}{j^r} \pmod{p^s}, \qquad r = 1, 2, \quad s = 3 - r, \tag{1.4}$$

where $M \geq 1$ is a divisor of 24 and k is an integer with $1 \leq k \leq M$ and $\gcd(k, m) = 1$. This last condition presents no loss of generality since other pairs (k, M) can be clearly reduced to this case. Also, sums over intervals other than those beginning with $j = 1$ can be obtained by subtracting appropriate sums of the type (1.4) from each other.

The cases $M = 1$, 2, 3, 4, and 6 are well-known for $r = 1$ and 2 and modulo (at least) p^2, or are easy to derive from known cases. The relevant congruences will be collected in Section 2. Recent work of Kuzumaki and Urbanowicz (see [12, 14, 15]) dealt with all M that are divisors of 24, thus giving the new cases $M = 8$, 12, and 24. This was done in a very general setting, with the drawback that the congruences obtained are rather cumbersome and difficult to use. In this paper we extract the most important special cases and supplement them with congruences for other parameters k with $\gcd(k, M) = 1$. This will be done in Sections 4 and 5, after some background on generalized Bernoulli numbers is given in Section 3.

To achieve the greatest possible clarity and usefulness of our results, we made the conscious decision to restrict our attention to what we consider the most important cases, namely congruences modulo p^2 when $r = 1$ and modulo p when $r = 2$. Some references to more general cases will be given in the final Section 7, which follows the brief Section 6 on double sums.

2. The Cases $M = 1$, 2, 3, 4, and 6

The theory of sums of reciprocals is closely related to Bernoulli numbers and polynomials, as well as to Euler numbers. This connection can be explicitly seen, e.g., in the beginning of Emma Lehmer's seminal paper [16]. Here we will only give some basic definitions, beginning with the *Bernoulli polynomials* $B_n(x), n \geq 0$, which can be defined by the generating function

$$\frac{te^{tx}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} \qquad (|t| < 2\pi). \tag{2.1}$$

Similarly, the *Euler numbers* $E_n, n \geq 0$, can be defined by

$$\frac{2}{e^t + e^{-t}} = \sum_{n=0}^{\infty} \frac{E_n}{n!} t^n \qquad (|t| < \pi). \tag{2.2}$$

The first few Euler numbers are $1, 0, -1, 0, 5, 0, -61, 0, 1385$. We also require the Legendre symbol

$$\left(\frac{p}{3}\right) = \begin{cases} 1 & \text{if } p \equiv 1 \pmod 3, \\ -1 & \text{if } p \equiv 2 \pmod 3. \end{cases}$$

We are now ready to state the main results of this section.

Proposition 1 ($M = 1, 2, 3, 4$ and $r = 2$)**.** *For all primes $p \geq 5$ we have*

$$\sum_{j=1}^{p-1} \frac{1}{j^2} \equiv 0 \pmod{p}, \tag{2.3}$$

$$\sum_{j=1}^{(p-1)/2} \frac{1}{j^2} \equiv 0 \pmod{p}, \tag{2.4}$$

$$\sum_{j=1}^{\lfloor p/3 \rfloor} \frac{1}{j^2} \equiv \frac{1}{2} \left(\frac{p}{3}\right) B_{p-2}(\tfrac{1}{3}) \pmod{p}, \tag{2.5}$$

$$\sum_{j=1}^{\lfloor 2p/3 \rfloor} \frac{1}{j^2} \equiv -\frac{1}{2} \left(\frac{p}{3}\right) B_{p-2}\left(\tfrac{1}{3}\right) \pmod{p}, \tag{2.6}$$

$$\sum_{j=1}^{\lfloor p/4 \rfloor} \frac{1}{j^2} \equiv 4(-1)^{\frac{p-1}{2}} E_{p-3} \pmod{p}, \tag{2.7}$$

$$\sum_{j=1}^{\lfloor 3p/4 \rfloor} \frac{1}{j^2} \equiv -4(-1)^{\frac{p-1}{2}} E_{p-3} \pmod{p}. \tag{2.8}$$

The congruence (2.3) is a special case of a more general one in [16, p. 353], and (2.4) is a special case of a congruence in [19, p. 296]. Equation (2.5) can be found in [19, p. 302] and (2.7) is a special case of Corollary 3.8 in [19, p. 296]. Finally, the congruences (2.6) and (2.8) are obtained by subtracting (2.5) and (2.7), respectively, from (2.3).

Proposition 2 ($M = 1, 2, 3, 4$ and $r = 1$)**.** *For all primes $p \geq 5$ we have*

$$\sum_{j=1}^{p-1} \frac{1}{j} \equiv 0 \pmod{p^2}, \tag{2.9}$$

$$\sum_{j=1}^{(p-1)/2} \frac{1}{j} \equiv -2q_p(2) + pq_p(2)^2 \pmod{p^2}, \tag{2.10}$$

$$\sum_{j=1}^{\lfloor p/3 \rfloor} \frac{1}{j} \equiv -\frac{3}{2}q_p(3) + p\left(\frac{3}{4}q_p(3)^2 - \frac{1}{6}\left(\frac{p}{3}\right) B_{p-2}(\tfrac{1}{3})\right) \pmod{p^2}, \tag{2.11}$$

$$\sum_{j=1}^{\lfloor 2p/3 \rfloor} \frac{1}{j} \equiv -\frac{3}{2}q_p(3) + p\left(\frac{3}{4}q_p(3)^2 + \frac{1}{3}\left(\frac{p}{3}\right) B_{p-2}(\tfrac{1}{3})\right) \pmod{p^2}, \tag{2.12}$$

$$\sum_{j=1}^{\lfloor p/4 \rfloor} \frac{1}{j} \equiv -3q_p(2) + p\left(\frac{3}{2}q_p(2)^2 + (-1)^{\frac{p+1}{2}} E_{p-3}\right) \pmod{p^2}, \tag{2.13}$$

$$\sum_{j=1}^{\lfloor 3p/4 \rfloor} \frac{1}{j} \equiv -3q_p(2) + \left(\frac{3}{2}q_p(2)^2 + 3(-1)^{\frac{p-1}{2}} E_{p-3}\right) \pmod{p^2}. \tag{2.14}$$

Proof. The congruence (2.9) is a special case of a congruence in [16, p. 353], while (2.10) and (2.13) follow from congruences in [19, p. 290]. Equation (2.11) follows from Theorem 3.9(iii) in [19, p. 301], using the congruence

$$B_{p-2}(\tfrac{1}{6}) \equiv 5B_{p-2}(\tfrac{1}{3}) \pmod{p}; \tag{2.15}$$

see, e.g., [17, p. 158] or [19, p. 302]. Next, for integers $M \geq 2$ and $1 \leq k < M$, $\gcd(k, M) = 1$, we have

$$\sum_{j=1}^{\lfloor\frac{(M-k)p}{M}\rfloor} \frac{1}{j} = \sum_{j=1}^{p-1} \frac{1}{j} - \sum_{j=1}^{\lfloor kp/M\rfloor} \frac{1}{p-j} \equiv -\sum_{j=1}^{\lfloor kp/M\rfloor} \frac{1}{p-j} \pmod{p^2}, \tag{2.16}$$

where we have used (2.9). Using the congruence

$$\frac{1}{j-p} \equiv \frac{1}{j} + \frac{p}{j^2} \pmod{p^2} \tag{2.17}$$

(see [16, p. 359]), we get with (2.16),

$$\sum_{j=1}^{\lfloor\frac{(M-k)p}{M}\rfloor} \frac{1}{j} \equiv \sum_{j=1}^{\lfloor kp/M\rfloor} \frac{1}{j} + p\sum_{j=1}^{\lfloor kp/M\rfloor} \frac{1}{j^2} \pmod{p^2}. \tag{2.18}$$

Applying this to $M = 3$ and $k = 1$, we immediately get (2.12) from (2.11) and (2.5). Similarly, with $M = 4$ and $k = 1$ we get (2.14) from (2.13) and (2.7). This completes the proof of Proposition 2. □

Proposition 3 ($M = 6$ and $r = 2$). *For all primes $p \geq 5$ we have*

$$\sum_{j=1}^{\lfloor p/6\rfloor} \frac{1}{j^2} \equiv \frac{5}{2}\left(\frac{p}{3}\right) B_{p-2}(\tfrac{1}{3}) \pmod{p}, \tag{2.19}$$

$$\sum_{j=1}^{\lfloor 5p/6\rfloor} \frac{1}{j^2} \equiv -\frac{5}{2}\left(\frac{p}{3}\right) B_{p-2}(\tfrac{1}{3}) \pmod{p}. \tag{2.20}$$

The congruence (2.19) follows from Theorem 3.9(i) in [19, p. 301] together with (2.15), and then (2.20) is obtained by subtracting (2.19) from (2.3).

Proposition 4 ($M = 6$ and $r = 1$). *For all primes $p \geq 5$ we have*

$$\sum_{j=1}^{\lfloor p/6\rfloor} \frac{1}{j} \equiv -2q_p(2) - \frac{3}{2}q_p(3) + p\left(q_p(2)^2 + \frac{3}{4}q_p(3)^2 - \frac{5}{12}\left(\frac{p}{3}\right) B_{p-2}(\tfrac{1}{3})\right) \pmod{p^2}, \tag{2.21}$$

$$\sum_{j=1}^{\lfloor 5p/6\rfloor} \frac{1}{j} \equiv -2q_p(2) - \frac{3}{2}q_p(3) + p\left(q_p(2)^2 + \frac{3}{4}q_p(3)^2 + \frac{25}{12}\left(\frac{p}{3}\right) B_{p-2}(\tfrac{1}{3})\right) \pmod{p^2}. \tag{2.22}$$

Similarly to the above, (2.21) follows from Theorem 3.9(ii) in [19, p. 301] together with (2.15), and (2.22) is obtained from (2.21) and (2.11), with (2.18).

3. Generalized Bernoulli Numbers

The results for $M = 8$, 12, and 24 in the next section involve certain generalized Bernoulli numbers belonging to residue class characters. In this section we will give some basic definitions, results, and tables. For further details see, e.g., [11] or [20].

Let χ be a primitive residue class character with conductor $f = f_\chi$. The complex numbers B_χ^n, $n \geq 0$, defined by the generating function

$$\sum_{a=1}^{f} \frac{\chi(a) t e^{at}}{e^{ft}-1} = \sum_{n=0}^{\infty} B_\chi^n \frac{t^n}{n!}, \tag{3.1}$$

are called *generalized Bernoulli numbers belonging to the character* χ. If $f = 1$, i.e., if χ is the principal character ($\chi(a) = 1$ for all a), then $B_\chi^n = B_n$, the ordinary Bernoulli number, for all $n \neq 1$, while $B_\chi^1 = \frac{1}{2} = -B_1$. The B_χ^n are elements of the smallest algebraic number field containing all $\chi(a)$. In particular, if χ is a quadratic character, i.e., if $\chi(a)^2 = 1$ whenever $\gcd(a, f) = 1$, then $B_\chi^n \in \mathbb{Q}$ for all n.

To be able to state a few more elementary properties, we define $\delta = \delta_\chi$ to be 0 or 1 according as χ is even (i.e., $\chi(-1) = 1$) or odd (i.e., $\chi(-1) = -1$). Then we have

$$B_\chi^n = 0 \qquad \text{if} \qquad n \not\equiv \delta \pmod 2,$$

and also $B_\chi^0 = 0$ for all non-principal characters χ. Furthermore, when χ is a quadratic character, then the B_χ^n with $n \equiv \delta \pmod 2$ are nonzero and their signs are determined by

$$(-1)^{1+(n-\delta)/2} B_\chi^n > 0 \qquad (n \equiv \delta \pmod 2).$$

The entries in Table 2 below may serve as examples for these properties.

With the notation of [15], Table 1 shows the characters that occur in Sections 4 and 5. χ_{-3} and χ_{-4} are the unique quadratic characters with $f = 3$ and $f = 4$, respectively, while χ_{-8} and χ_8 are the two quadratic characters with $f = 8$. Only half of the values of the product characters $\chi_{-3}\chi_8$ and $\chi_{-3}\chi_{-8}$ are shown; the remaining values are clear from the first character being odd and the second character being even.

Table 1: The characters occurring in Sections 4 and 5.

a	parity	f	1	2	3	4	5	6	7	8	9	10	11	12
$\chi_{-3}(a)$	odd	3	1	-1	0									
$\chi_{-4}(a)$	odd	4	1	0	-1	0								
$\chi_{-8}(a)$	odd	8	1	0	1	0	-1	0	-1	0				
$\chi_8(a)$	even	8	1	0	-1	0	-1	0	1	0				
$\chi_{-3}\chi_{-4}(a)$	even	12	1	0	0	0	-1	0	-1	0	0	0	1	0
$\chi_{-3}\chi_8(a)$	odd	24	1	0	0	0	1	0	1	0	0	0	1	0
$\chi_{-3}\chi_{-8}(a)$	even	24	1	0	0	0	1	0	-1	0	0	0	-1	0

Using these character values and (3.1), we can write down explicit generating functions; for instance,

$$\frac{-te^t}{e^{2t}+1} = \sum_{n=0}^{\infty} B_{\chi_{-4}}^n \frac{t^n}{n!}. \tag{3.2}$$

With the help of a computer algebra system, these generating functions can be used to compute the corresponding generalized Bernoulli numbers; see Table 2 for the first few values for each of the characters in question.

For the computation of further values it is convenient to use the following well-known recurrence relation:

$$\sum_{j=0}^{n-1} \binom{n}{j} f^{n-j} B_\chi^j = n \sum_{a=1}^{f} \chi(a) a^{n-1}. \tag{3.3}$$

Table 2: The first few values of generalized Bernoulli numbers.

n	1	2	3	4	5	6	7	8
$3B_{\chi_{-3}}^n$	-1	0	2	0	-10	0	98	0
$2B_{\chi_{-4}}^n$	-1	0	3	0	-25	0	427	0
$B_{\chi_{-8}}^n$	-1	0	9	0	-285	0	19 341	0
$B_{\chi_8}^n$	0	2	0	-44	0	2 166	0	$-196\,888$
$B_{\chi_{-3}\chi_{-4}}^n$	0	4	0	-184	0	20 172	0	$-4\,120\,688$
$B_{\chi_{-3}\chi_8}^n$	-2	0	138	0	$-39\,850$	0	24 410 722	0
$B_{\chi_{-3}\chi_{-8}}^n$	0	6	0	$-2\,088$	0	912 996	0	$-745\,928\,016$

When $f > 1$, this immediately gives

$$B_\chi^0 = 0, \qquad B_\chi^1 = \frac{1}{f}\sum_{a=1}^{f}\chi(a)a,$$

which is consistent with Table 2.

Note that all nonzero numbers in the first row of Table 2 have denominator 3 and those in the second row have denominator 2, while all others are integers. This is an instance of the character analogue of the theorem of von Staudt and Clausen; see, e.g., [5]. While this theorem will not be needed in what follows, the character analogue of the equally well-known and important Kummer congruence in its basic form will be useful in the next sections. We quote a special case; see, e.g., [5].

Let p be an odd prime and χ a character whose conductor f is not a power of p. If $w \in \mathbb{N}$ is such that $p-1 \mid w$, then

$$\frac{B_\chi^{n+w}}{n+w} \equiv \frac{B_\chi^n}{n} \pmod{p} \qquad (n \geq 2). \tag{3.4}$$

To conclude this section, we mention some connections between the generalized Bernoulli numbers and the Euler numbers and Bernoulli polynomials. First, by comparing the generating functions (2.2) and (3.2), we immediately get

$$E_n = \frac{-2}{n+1}B_{\chi_{-4}}^{n+1} \qquad (n \geq 0), \tag{3.5}$$

and consequently, for odd primes p,

$$B_{\chi_{-4}}^{p-2} \equiv E_{p-3} \pmod{p}. \tag{3.6}$$

Next, comparing the generating functions (2.1) and (3.1), we get the identity

$$B_\chi^n = f^{n-1}\sum_{a=1}^{f}\chi(a)B_n(\tfrac{a}{f}). \tag{3.7}$$

For $\chi = \chi_{-3}$ we immediately get

$$B_{\chi_{-3}}^n = 3^{n-1}\left(B_n(\tfrac{1}{3}) - B_n(\tfrac{2}{3})\right),$$

and then the well-known reflection formula $B_n(1-x) = (-1)^n B_n(x)$ yields

$$B_{\chi_{-3}}^n = 2\cdot 3^{n-1}B_n(\tfrac{1}{3}) \qquad (n \text{ odd}). \tag{3.8}$$

When $p \geq 5$ is a prime then, since $3^{p-1} \equiv 1 \pmod{p}$, we have

$$B_{\chi_{-3}}^{p-2} \equiv \frac{2}{9}B_{p-2}(\tfrac{1}{3}) \pmod{p}. \tag{3.9}$$

The identities (3.5), (3.8) and the congruences (3.6), (3.9) will be useful in the next section.

4. The Cases $M = 8$ and 12

In this section we will derive the relevant congruences for $M = 8$ and 12 and present them in the same order and format as we did in Section 2.

Proposition 5 ($M = 8$ and $r = 2$)**.** *For all primes $p \geq 5$ we have*

$$\sum_{j=1}^{\lfloor p/8 \rfloor} \frac{1}{j^2} \equiv 8\left((-1)^{\frac{p-1}{2}} E_{p-3} + (-1)^{\frac{(p-1)(p+5)}{8}} B_{\chi_{-8}}^{p-2}\right) \pmod{p}, \tag{4.1}$$

$$\sum_{j=1}^{\lfloor 3p/8 \rfloor} \frac{1}{j^2} \equiv 8\left(-(-1)^{\frac{p-1}{2}} E_{p-3} + (-1)^{\frac{(p-1)(p+5)}{8}} B_{\chi_{-8}}^{p-2}\right) \pmod{p}, \tag{4.2}$$

$$\sum_{j=1}^{\lfloor 5p/8 \rfloor} \frac{1}{j^2} \equiv 8\left((-1)^{\frac{p-1}{2}} E_{p-3} - (-1)^{\frac{(p-1)(p+5)}{8}} B_{\chi_{-8}}^{p-2}\right) \pmod{p}, \tag{4.3}$$

$$\sum_{j=1}^{\lfloor 7p/8 \rfloor} \frac{1}{j^2} \equiv 8\left(-(-1)^{\frac{p-1}{2}} E_{p-3} - (-1)^{\frac{(p-1)(p+5)}{8}} B_{\chi_{-8}}^{p-2}\right) \pmod{p}. \tag{4.4}$$

Proof. (4.1) is obtained by specializing the second congruence in Theorem 2.1(v) in [15] and then using (3.6). To obtain (4.2), we note that more generally we have, for integers $M \geq 2$ and $1 \leq k \leq M-1$, not divisible by the odd prime p and with $\gcd(k, M) = 1$,

$$\begin{aligned}\sum_{j=1}^{\lfloor \frac{(M-k)p}{2M} \rfloor} \frac{1}{j^2} &= \sum_{j=1}^{\frac{p-1}{2}} \frac{1}{j^2} - \sum_{j=\lfloor \frac{(M-k)p}{2M} \rfloor + 1}^{\frac{p-1}{2}} \frac{1}{j^2} \equiv 0 - \sum_{j=1}^{\lfloor \frac{kp}{2M} \rfloor} \frac{1}{(\frac{p+1}{2} - j)^2} \pmod{p} \\ &\equiv -4 \sum_{j=1}^{\lfloor \frac{kp}{2M} \rfloor} \frac{1}{(2j-1)^2} = -4\left(\sum_{j=1}^{\lfloor \frac{kp}{M} \rfloor} \frac{1}{j^2} - \sum_{j=1}^{\lfloor \frac{kp}{2M} \rfloor} \frac{1}{(2j)^2}\right) \pmod{p},\end{aligned}$$

where we have used (2.4). Hence we have

$$\sum_{j=1}^{\lfloor \frac{(M-k)p}{2M} \rfloor} \frac{1}{j^2} \equiv \sum_{j=1}^{\lfloor \frac{kp}{2M} \rfloor} \frac{1}{j^2} - 4 \sum_{j=1}^{\lfloor \frac{kp}{M} \rfloor} \frac{1}{j^2} \pmod{p}. \tag{4.5}$$

We use this with $M = 4$, $k = 1$ and see that (4.1) together with (2.7) yields (4.2). The congruences (4.3) and (4.4) now follow from subtracting (4.2) and (4.1), respectively, from (2.3). □

Proposition 6 ($M = 8$ and $r = 1$)**.** *For all primes $p \geq 5$ we have*

$$\begin{aligned}\sum_{j=1}^{\lfloor p/8 \rfloor} \frac{1}{j} &\equiv -4q_p(2) + 2(-1)^{\frac{p^2-1}{8}} \frac{B_{\chi_8}^{p^2-p}}{p^2-p} + 2pq_p(2)^2 \\ &\quad - p(-1)^{\frac{p-1}{2}} E_{p-3} - p(-1)^{\frac{(p-1)(p+5)}{8}} B_{\chi_{-8}}^{p-2} \pmod{p^2},\end{aligned} \tag{4.6}$$

$$\sum_{j=1}^{\lfloor 3p/8\rfloor}\frac{1}{j}\equiv -4q_p(2)-2(-1)^{\frac{p^2-1}{8}}\frac{B_{\chi_8}^{p^2-p}}{p^2-p}+2pq_p(2)^2$$

$$+3p(-1)^{\frac{p-1}{2}}E_{p-3}-3p(-1)^{\frac{(p-1)(p+5)}{8}}B_{\chi_{-8}}^{p-2}\pmod{p^2},\tag{4.7}$$

$$\sum_{j=1}^{\lfloor 5p/8\rfloor}\frac{1}{j}\equiv -4q_p(2)-2(-1)^{\frac{p^2-1}{8}}\frac{B_{\chi_8}^{p^2-p}}{p^2-p}+2pq_p(2)^2$$

$$-5p(-1)^{\frac{p-1}{2}}E_{p-3}+5p(-1)^{\frac{(p-1)(p+5)}{8}}B_{\chi_{-8}}^{p-2}\pmod{p^2},\tag{4.8}$$

$$\sum_{j=1}^{\lfloor 7p/8\rfloor}\frac{1}{j}\equiv -4q_p(2)+2(-1)^{\frac{p^2-1}{8}}\frac{B_{\chi_8}^{p^2-p}}{p^2-p}+2pq_p(2)^2$$

$$+7p(-1)^{\frac{p-1}{2}}E_{p-3}+7p(-1)^{\frac{(p-1)(p+5)}{8}}B_{\chi_{-8}}^{p-2}\pmod{p^2}.\tag{4.9}$$

Proof. The congruence (4.6) is obtained by specializing the first congruence of Theorem 2.1(v) in [15] to modulus p^2 and then using (3.6) and the Kummer congruence (3.4).

To obtain (4.7), we begin by proving a more general auxiliary result. If $M\geq 2$ is an integer, then

$$\sum_{j=1}^{\lfloor\frac{(M-1)p}{2M}\rfloor}\frac{1}{j}=\sum_{j=1}^{(p-1)/2}\frac{1}{j}-\sum_{j=\lfloor\frac{(M-1)p}{2M}\rfloor+1}^{(p-1)/2}\frac{1}{j}\equiv -2q_p(2)+pq_p(2)^2-\sum_{j=1}^{\lfloor\frac{p}{2M}\rfloor}\frac{1}{\frac{p+1}{2}-j}\pmod{p^2},\tag{4.10}$$

where we have used (2.10). Applying (2.17) with j replaced by $2j-1$, we get

$$-\sum_{j=1}^{\lfloor\frac{p}{2M}\rfloor}\frac{1}{\frac{p+1}{2}-j}\equiv 2\sum_{j=1}^{\lfloor\frac{p}{2M}\rfloor}\frac{1}{2j-1}+2p\sum_{j=1}^{\lfloor\frac{p}{2M}\rfloor}\frac{1}{(2j-1)^2}\pmod{p^2}.$$

Writing each of the two sums on the right as a difference of the corresponding sum over all integers, minus those over only the even integers, we get with (4.10),

$$\sum_{j=1}^{\lfloor\frac{(M-1)p}{2M}\rfloor}\frac{1}{j}\equiv -2q_p(2)+2\sum_{j=1}^{\lfloor\frac{p}{M}\rfloor}\frac{1}{j}-\sum_{j=1}^{\lfloor\frac{p}{2M}\rfloor}\frac{1}{j}+p\left(q_p(2)^2+2\sum_{j=1}^{\lfloor\frac{p}{M}\rfloor}\frac{1}{j^2}-\frac{1}{2}\sum_{j=1}^{\lfloor\frac{p}{2M}\rfloor}\frac{1}{j^2}\right)\pmod{p^2}.\tag{4.11}$$

We apply this with $M=4$ and use (2.13), (4.6), (2.7), and (4.1). Combining everything and simplifying, we obtain (4.7).

Next, (4.8) is obtained by applying (2.18) with $M=8$ and $k=3$, and using (4.7) and (4.2). Similarly, we get (4.9) from (2.18) with $M=8$ and $k=1$, and using (4.6) and (4.1). ☐

Proposition 7 ($M=12$ and $r=2$)**.** *For all primes $p\geq 5$ we have*

$$\sum_{j=1}^{\lfloor p/12\rfloor}\frac{1}{j^2}\equiv 5\left(\frac{p}{3}\right)B_{p-2}(\tfrac{1}{3})+20(-1)^{\frac{p-1}{2}}E_{p-3}\pmod{p},\tag{4.12}$$

$$\sum_{j=1}^{\lfloor 5p/12\rfloor}\frac{1}{j^2}\equiv -5\left(\frac{p}{3}\right)B_{p-2}(\tfrac{1}{3})+20(-1)^{\frac{p-1}{2}}E_{p-3}\pmod{p},\tag{4.13}$$

$$\sum_{j=1}^{\lfloor 7p/12\rfloor}\frac{1}{j^2}\equiv 5\left(\frac{p}{3}\right)B_{p-2}(\tfrac{1}{3})-20(-1)^{\frac{p-1}{2}}E_{p-3}\pmod{p},\tag{4.14}$$

$$\sum_{j=1}^{\lfloor 11p/12\rfloor}\frac{1}{j^2}\equiv -5\left(\frac{p}{3}\right)B_{p-2}(\tfrac{1}{3})-20(-1)^{\frac{p-1}{2}}E_{p-3}\pmod{p}.\tag{4.15}$$

Proof. Specializing the second congruence in Theorem 2.1(vi) in [15] and applying (3.6) and (3.9), we get (4.12). The congruence (4.13) follows from (4.5) with $M = 6$ and $k = 1$, using (4.12) and (2.19). Finally, (4.14) and (4.15) are obtained by subtracting (4.13) and (4.12), respectively, from (2.3). □

Proposition 8 ($M = 12$ and $r = 1$). *For all primes $p \geq 5$ we have*

$$\begin{aligned}\sum_{j=1}^{\lfloor p/12\rfloor}\frac{1}{j}&\equiv -3q_p(2)-\frac{3}{2}q_p(3)+3(-1)^{\frac{p-1}{2}}\left(\frac{p}{3}\right)\frac{B^{p^2-p}_{\chi_{-3}\chi_{-4}}}{p^2-p}+p\left(\frac{3}{2}q_p(2)^2\right.\\&\left.+\frac{3}{4}q_p(3)^2-\frac{5}{12}\left(\frac{p}{3}\right)B_{p-2}(\tfrac{1}{3})-\frac{5}{3}(-1)^{\frac{p-1}{2}}E_{p-3}\right)\pmod{p^2},\end{aligned}\tag{4.16}$$

$$\begin{aligned}\sum_{j=1}^{\lfloor 5p/12\rfloor}\frac{1}{j}&\equiv -3q_p(2)-\frac{3}{2}q_p(3)-3(-1)^{\frac{p-1}{2}}\left(\frac{p}{3}\right)\frac{B^{p^2-p}_{\chi_{-3}\chi_{-4}}}{p^2-p}+p\left(\frac{3}{2}q_p(2)^2\right.\\&\left.+\frac{3}{4}q_p(3)^2+\frac{25}{12}\left(\frac{p}{3}\right)B_{p-2}(\tfrac{1}{3})-\frac{25}{3}(-1)^{\frac{p-1}{2}}E_{p-3}\right)\pmod{p^2},\end{aligned}\tag{4.17}$$

$$\begin{aligned}\sum_{j=1}^{\lfloor 7p/12\rfloor}\frac{1}{j}&\equiv -3q_p(2)-\frac{3}{2}q_p(3)-3(-1)^{\frac{p-1}{2}}\left(\frac{p}{3}\right)\frac{B^{p^2-p}_{\chi_{-3}\chi_{-4}}}{p^2-p}+p\left(\frac{3}{2}q_p(2)^2\right.\\&\left.+\frac{3}{4}q_p(3)^2-\frac{35}{12}\left(\frac{p}{3}\right)B_{p-2}(\tfrac{1}{3})+\frac{35}{3}(-1)^{\frac{p-1}{2}}E_{p-3}\right)\pmod{p^2},\end{aligned}\tag{4.18}$$

$$\begin{aligned}\sum_{j=1}^{\lfloor 11p/12\rfloor}\frac{1}{j}&\equiv -3q_p(2)-\frac{3}{2}q_p(3)+3(-1)^{\frac{p-1}{2}}\left(\frac{p}{3}\right)\frac{B^{p^2-p}_{\chi_{-3}\chi_{-4}}}{p^2-p}+p\left(\frac{3}{2}q_p(2)^2\right.\\&\left.+\frac{3}{4}q_p(3)^2+\frac{55}{12}\left(\frac{p}{3}\right)B_{p-2}(\tfrac{1}{3})+\frac{55}{3}(-1)^{\frac{p-1}{2}}E_{p-3}\right)\pmod{p^2}.\end{aligned}\tag{4.19}$$

Proof. We proceed as in the proof of Theorem 6. The congruence (4.16) is obtained by taking the first congruence of Theorem 2.1(vi) in [15] modulo p^2 and then using (3.4), (3.6), and (3.9). The congruence (4.17) follows from (4.11) with $M = 6$, by using (2.21), (4.16), (2.19), and (4.12). Next, (4.18) is obtained by applying (2.18) with $M = 12$ and $k = 5$ and using (4.17) and (4.13); finally, we get (4.19) from (2.18) with $M = 12$ and $k = 1$, and using (4.16) and (4.12). □

5. The Case $M = 24$

Because of the increasing complexity of the results in the case $M = 24$, we will not explicity state all the congruences for the various possible parameters k. We begin, as before, with $r = 2$.

Proposition 9 ($M=24$ and $r=2$). *For all primes $p\geq 5$ we have*

$$\sum_{j=1}^{\lfloor p/24\rfloor}\frac{1}{j^2}\equiv 10\left(\frac{p}{3}\right)B_{p-2}(\tfrac{1}{3})+40(-1)^{\frac{p-1}{2}}E_{p-3}$$
$$+32(-1)^{\frac{(p-1)(p+5)}{8}}B_{\chi_{-8}}^{p-2}+36\left(\frac{p}{3}\right)(-1)^{\frac{p^2-1}{8}}B_{\chi_{-3}\chi_8}^{p-2}\pmod{p},\tag{5.1}$$
$$\sum_{j=1}^{\lfloor 11p/24\rfloor}\frac{1}{j^2}\equiv -10\left(\frac{p}{3}\right)B_{p-2}(\tfrac{1}{3})-40(-1)^{\frac{p-1}{2}}E_{p-3}$$
$$+32(-1)^{\frac{(p-1)(p+5)}{8}}B_{\chi_{-8}}^{p-2}+36\left(\frac{p}{3}\right)(-1)^{\frac{p^2-1}{8}}B_{\chi_{-3}\chi_8}^{p-2}\pmod{p}.\tag{5.2}$$

Furthermore, $\sum_{j=1}^{\lfloor 13p/24\rfloor}1/j^2$ *and* $\sum_{j=1}^{\lfloor 23p/24\rfloor}1/j^2$ *are congruent* $\pmod{p}$ *to negative the right-hand sides of* (5.2) *and* (5.1), *respectively.*

Proof. Similarly to earlier proofs, (5.1) is obtained by taking the second congruence of Theorem 2.1(vii) modulo p and applying (3.6) and (3.9). The congruence (5.2) follows from (4.5), (4.12) and (5.1), and the final statement follows from (2.16). □

The sums for $k=5$ and $k=7$ (and by extension $k=17$ and 19) require a somewhat different treatment. We use the generalized Bernoulli polynomials which, in analogy to (3.1), are defined by the generating function

$$\sum_{a=1}^{f}\frac{\chi(a)te^{(a+x)t}}{e^{ft}-1}=\sum_{n=0}^{\infty}B_{\chi}^{n}(x)\frac{t^n}{n!},\tag{5.3}$$

with χ and f as in Section 3. Among the most basic properties (see, e.g., [2] or [20] for more details) are the identities

$$B_{\chi}^{n}(x)=\sum_{j=0}^{n}\binom{n}{j}B_{\chi}^{j}x^{n-j},\tag{5.4}$$
$$B_{\chi}^{n}(-x)=(-1)^{n-\delta}B_{\chi}^{n}(x),\tag{5.5}$$
$$B_{\chi}^{n}(x+N)-B_{\chi}^{n}(x)=n\sum_{j=1}^{N}\chi(j)(x+j)^{n-1},\tag{5.6}$$

where N is a multiple of f, and $\delta=0$ or 1 for χ even, respectively odd. We note in passing that with $x=0$ and $N=f$, (5.6) and (5.4) immediately give (3.3).

Now we set $\chi=\chi_{-4}$, so that $f=4$, and we let $x=0$ and $n=p-2$. Then with (5.6), Fermat's Little Theorem, and with (3.6) we get

$$B_{\chi_{-4}}^{p-2}(N)-E_{p-3}\equiv -2\sum_{j=1}^{N}\frac{\chi_{-4}(j)}{j^2}\pmod{p}.\tag{5.7}$$

We are now ready to state and prove a result that complements Proposition 9. For the sake of simplicity we restrict ourselves to the case $p\equiv 1\pmod{24}$, which is of particular interest for the applications in [1].

Proposition 10 ($M = 24$ and $r = 2$, continued)**.** *For all primes $p \equiv 1 \pmod{24}$ we have*

$$\sum_{j=1}^{\lfloor 5p/24\rfloor} \frac{1}{j^2} \equiv -10B_{p-2}(\tfrac{1}{3}) + 40E_{p-3} + 4B_{\chi_{-4}}^{p-2}(\tfrac{1}{6}) \pmod{p}, \tag{5.8}$$

$$\sum_{j=1}^{\lfloor 5p/24\rfloor} \frac{1}{j^2} \equiv 10B_{p-2}(\tfrac{1}{3}) - 40E_{p-3} + 4B_{\chi_{-4}}^{p-2}(\tfrac{1}{6}) \pmod{p}. \tag{5.9}$$

Furthermore, $\sum_{j=1}^{\lfloor 17p/24\rfloor} 1/j^2$ and $\sum_{j=1}^{\lfloor 19p/24\rfloor} 1/j^2$ are congruent $\pmod{p}$ to negative the right-hand sides of (5.9) *and* (5.8)*, respectively.*

Proof. If $p \equiv 1 \pmod{24}$, then

$$\sum_{j=1}^{\lfloor\frac{5p}{24}\rfloor} \frac{1}{j^2} = \sum_{j=1}^{\frac{p-1}{4}} \frac{1}{j^2} - \sum_{j=\lfloor\frac{5p}{24}\rfloor+1}^{\frac{p-1}{2}} \frac{1}{j^2} \equiv 4E_{p-3} - \sum_{j=1}^{\frac{p-1}{24}} \frac{1}{(\frac{p-1}{4}+1-j)^2} \pmod{p}$$

$$\equiv 4E_{p-3} - 16\sum_{j=1}^{\frac{p-1}{24}} \frac{1}{(4j-3)^2} \pmod{p}, \tag{5.10}$$

having used (2.7). To determine the right-most sum in (5.10), we first note that

$$\sum_{j=1}^{\frac{p-1}{24}} \frac{1}{(4j-3)^2} + \sum_{j=1}^{\frac{p-1}{24}} \frac{1}{(4j-1)^2} = \sum_{j=1}^{\frac{p-1}{12}} \frac{1}{(2j-1)^2} = \sum_{j=1}^{\frac{p-1}{6}} \frac{1}{j^2} - \frac{1}{4}\sum_{j=1}^{\frac{p-1}{12}} \frac{1}{j^2}$$

$$\equiv \frac{5}{4}B_{p-2}(\tfrac{1}{3}) - 5E_{p-3} \pmod{p}, \tag{5.11}$$

where we have used (2.19) and (4.12), keeping in mind that $p \equiv 1 \pmod{24}$. On the other hand we have, by (5.7) with $N = (p-1)/6$,

$$\sum_{j=1}^{\frac{p-1}{24}} \frac{1}{(4j-3)^2} - \sum_{j=1}^{\frac{p-1}{24}} \frac{1}{(4j-1)^2} = \sum_{j=1}^{\frac{p-1}{6}} \frac{\chi_{-4}(j)}{j^2} \equiv \frac{1}{2}\Big(E_{p-3} - B_{\chi_{-4}}^{p-2}(\tfrac{p-1}{6})\Big) \pmod{p}. \tag{5.12}$$

With (5.4) and (5.5) we see that

$$B_{\chi_{-4}}^{p-2}(\tfrac{p-1}{6}) \equiv B_{\chi_{-4}}^{p-2}(\tfrac{-1}{6}) = B_{\chi_{-4}}^{p-2}(\tfrac{1}{6}) \pmod{p},$$

and upon adding (5.11) and (5.12) we therefore get

$$2\sum_{j=1}^{\frac{p-1}{24}} \frac{1}{(4j-3)^2} \equiv \frac{5}{4}B_{p-2}(\tfrac{1}{3}) - \tfrac{9}{2}E_{p-3} - \tfrac{1}{2}B_{\chi_{-4}}^{p-2}(\tfrac{1}{6}) \pmod{p}.$$

This, with (5.10), immediately gives (5.8).

The congruence (5.9) then follows from (4.5) with $M = 12$ and $k = 5$ and using (5.8) and (4.13). The final statement follows once again from (2.16). □

Proposition 11 ($M = 24$ and $r = 1$)**.** *For all primes $p \geq 5$ we have*

$$\sum_{j=1}^{\lfloor p/24 \rfloor} \frac{1}{j} \equiv -4q_p(2) - \frac{3}{2}q_p(3) + 3(-1)^{\frac{p-1}{2}} \left(\frac{p}{3}\right) \frac{B_{\chi_{-3}\chi_{-4}}^{p^2-p}}{p^2-p}$$
$$+ 4(-1)^{\frac{p^2-1}{8}} \frac{B_{\chi_8}^{p^2-p}}{p^2-p} + 3(-1)^{\frac{(p-1)(p+5)}{8}} \left(\frac{p}{3}\right) \frac{B_{\chi_{-3}\chi_{-8}}^{p^2-p}}{p^2-p}$$
$$+ p\Bigg(2q_p(2)^2 + \frac{3}{4}q_p(3)^2 - \frac{5}{12}\left(\frac{p}{3}\right) B_{p-2}(\tfrac{1}{3}) - \frac{5}{3}(-1)^{\frac{p-1}{2}} E_{p-3}$$
$$-\frac{3}{2}\left(\frac{p}{3}\right)(-1)^{\frac{p^2-1}{8}} B_{\chi_{-3}\chi_8}^{p-2} - \frac{4}{3}(-1)^{\frac{(p-1)(p+5)}{8}} B_{\chi_8}^{p-2}\Bigg) \pmod{p^2}. \tag{5.13}$$

This last congruence is obtained completely analogously to (4.16). Given the complexity of this result, we skip the other cases, namely the sums to $\lfloor kp/24 \rfloor$ for $k = 5$, 7, 11, 13, 17, 19 and 23.

6. Double Sums

In the applications of interest to us, in particular in [1] and [6], we also require certain double sums of reciprocals. These are very easy to derive; in fact, for any fixed integer $N \geq 1$ we have

$$\sum_{1 \leq j < k \leq N} \frac{1}{jk} = \frac{1}{2}\left(\sum_{j=1}^{N} \frac{1}{j}\right)^2 - \frac{1}{2}\sum_{j=1}^{N} \frac{1}{j^2}, \tag{6.1}$$

which is easy to verify. Using appropriate results in Section 2, we immediately obtain the following list of congruences.

Proposition 12. *For all primes $p \geq 5$ we have*

$$\sum_{1 \leq j < k \leq p-1} \frac{1}{jk} \equiv 0 \pmod{p}, \tag{6.2}$$

$$\sum_{1 \leq j < k \leq \frac{p-1}{2}} \frac{1}{jk} \equiv 2q_p(2)^2 \pmod{p}, \tag{6.3}$$

$$\sum_{1 \leq j < k \leq \lfloor \frac{ap}{3} \rfloor} \frac{1}{jk} \equiv \frac{9}{8}q_p(3)^2 + \varepsilon_3 \frac{1}{4}\left(\frac{p}{3}\right) B_{p-2}(\tfrac{1}{3}) \pmod{p}, \tag{6.4}$$

where $\varepsilon_3 = -1$ when $a = 1$ and $\varepsilon_3 = 1$ when $a = 2$,

$$\sum_{1 \leq j < k \leq \lfloor \frac{ap}{4} \rfloor} \frac{1}{jk} \equiv \frac{9}{2}q_p(2)^2 + \varepsilon_4 2(-1)^{\frac{p-1}{2}} E_{p-3} \pmod{p}, \tag{6.5}$$

where $\varepsilon_4 = -1$ when $a = 1$ and $\varepsilon_4 = 1$ when $a = 3$,

$$\sum_{1 \leq j < k \leq \lfloor \frac{ap}{6} \rfloor} \frac{1}{jk} \equiv \frac{1}{2}\left(2q_p(2) + \frac{3}{2}q_p(3)\right)^2 + \varepsilon_6 \frac{5}{4}\left(\frac{p}{3}\right) B_{p-2}(\tfrac{1}{3}) \pmod{p}, \tag{6.6}$$

where $\varepsilon_6 = -1$ when $a = 1$ and $\varepsilon_6 = 1$ when $a = 5$.

For $M = 8$ and $M = 12$ we only give the first of the four possible congruences in each case, leaving the remaining ones to the reader. In deriving these identities we note that by the Kummer congruence (3.4) we have

$$\frac{B_\chi^{p^2-p}}{p^2-p} \equiv \frac{B_\chi^{p-1}}{p-1} \equiv -B_\chi^{p-1} \pmod{p}, \tag{6.7}$$

for $\chi = \chi_8$ and $\chi = \chi_{-3}\chi_{-4}$, respectively.

Proposition 13. *For all primes $p \geq 5$ we have*

$$\sum_{1\leq j<k\leq\lfloor\frac{p}{8}\rfloor} \frac{1}{jk} \equiv 2\left(2q_p(2) + (-1)^{\frac{p^2-1}{8}} B_{\chi_8}^{p-1}\right)^2 - 4(-1)^{\frac{p-1}{2}} E_{p-3}$$
$$- 4(-1)^{\frac{(p-1)(p+5)}{8}} B_{\chi_{-8}}^{p-2} \pmod{p}, \tag{6.8}$$

$$\sum_{1\leq j<k\leq\lfloor\frac{p}{12}\rfloor} \frac{1}{jk} \equiv \frac{1}{2}\left(3q_p(2) + \frac{3}{2}q_p(3) + 3(-1)^{\frac{p-1}{2}}\left(\frac{p}{3}\right) B_{\chi_{-3}\chi_{-4}}^{p-1}\right)^2$$
$$- \frac{5}{2}\left(\frac{p}{3}\right) B_{p-2}(\tfrac{1}{3}) - 10(-1)^{\frac{p-1}{2}} E_{p-3} \pmod{p}. \tag{6.9}$$

As far as $M = 24$ is concerned, we refrain from giving even the first case explicitly. It is now clear that by using (6.1), the desired congruence can be obtained from (5.1) and the modulo p specialization of (5.3), where (6.7) should be used for three different characters.

7. Further Remarks

We mentioned in the Introduction that the congruence (1.2) has been generalized in various directions; these are, in particular,

(1) different intervals over which the sum is taken,
(2) higher powers of the prime modulus,
(3) higher powers of j in the denominator,
(4) double or multiple sums,
(5) sums over certain arithmetical progressions, and
(6) composite moduli other than prime powers.

In this paper we have only dealt with (1)–(4), and for the sake of clarity and brevity not in the greatest possible generality. As far as (2) and (3) are concerned, numerous results modulo higher powers of p, or with higher powers of j in the denominator, were obtained by Z.-H. Sun [18], [19]. Similarly, the results of Kuzumaki and Urbanowicz [15] are, in the notation of (1.4), for $r = 1, 2, 3$ and $s = 4-r$, but only for $k = 1$. Numerous congruences for sums of the type (4) and (5) can be found in [19]. A large number of congruences of type (6) were recently obtained in [15], [12] and [14]; for some special cases related to (2.7), see [3], [7] and [13]. Finally, for congruences for sums of type (5) and (6) combined, see [14] and also [3], [4] and [8]. As most of these authors have acknowledged, much of this work goes back to Emma Lehmer's important 1938 paper [16].

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