THE EXTENDIBILITY OF $D(4)$-PAIR $\{F_{2k}, 5F_{2k}\}$

ALAN FILIPIN

Abstract. Let $k \geq 1$ be an integer and let F_k be the kth Fibonacci number. In this paper we prove that if $\{F_{2k}, 5F_{2k}, c, d\}$ with $c < d$ is the set of four positive integers such that any product of its two distinct elements increased by 4 is a perfect square, then d is uniquely determined by k and c.

1. Introduction

Let $n \neq 0$ be an integer. A set $\{a_1, a_2, \ldots, a_m\}$ of m positive integers is called a $D(n)$-m-tuple if $a_ia_j + n$ is a perfect square for all i, j with $1 \leq i < j \leq m$. The problem of finding such sets has a long and rich history. All details about it, together with references and open problems, can be found on webpage `web.math.pmf.unizg.hr/\~{}duje/dtuples.html`. Here we will only consider the case $n = 4$. With the exception of that case, cases $n = 1$ and $n = -1$ have been studied extensively in the last few years.

In the case $n = 4$ there is a conjecture that does not exist, a $D(4)$-quintuple. Actually, there exists a stronger version of that conjecture [4, Conjecture 1] that if $\{a, b, c, d\}$ is a $D(4)$-quadruple such that $a < b < c < d$, then

$$d = d_+ = a + b + c + \frac{1}{2}\left(abc + rst\right),$$

where r, s and t are positive integers defined by $ab + 4 = r^2$, $ac + 4 = s^2$, and $bc + 4 = t^2$. The $D(4)$-quadruple $\{a, b, c, d\}$, where $d > \max\{a, b, c\}$ is called a regular quadruple if $d = d_+$. We also define $d_- = a + b + c + 1/2\left(abc - rst\right)$. The set $\{a, b, c, d_-\}$ is also a $D(4)$-quadruple if $d_- \neq 0$, but $d_- < c$. There are many results that support this Conjecture (see [11, 10, 4, 9, 7, 8, 1, 2]).

In this paper we will generalize the result obtained in [4] where Dujella and Ramasamy proved that if $\{F_{2k}, 5F_{2k}, 4F_{2k+2}, d\}$ is $D(4)$-quadruple, then $d = 4L_{2k}F_{4k+2}$, where $k \geq 1$ is an integer and F_k and L_k denote kth Fibonacci and kth Lucas number. Our main result is the following theorem.

Theorem 1.1. *Let $k \geq 2$, $\nu \geq 1$ be integers and let $c = c_\nu$ be an integer defined by*

$$c_\nu^{\pm} := \frac{4}{5F_{2k}^2}\Bigg\{\left(\frac{3F_{2k} \pm F_{2k}\sqrt{5}}{2}\right)\left(\frac{L_{2k} + F_{2k}\sqrt{5}}{2}\right)^{2\nu} + \left(\frac{3F_{2k} \mp F_{2k}}{2}\right)\left(\frac{L_{2k} - F_{2k}\sqrt{5}}{2}\right)^{2\nu} - 3F_{2k}\Bigg\}. \quad (1.1)$$

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Assume that $c \neq c_1^{\pm} = 4F_{2k\pm 2}$. If the set $\{F_{2k}, 5F_{2k}, c, d\}$ is $D(4)$-quadruple, then $d = c_{\nu-1}^{\pm}$ or $d = c_{\nu+1}^{\pm}$. Moreover, if $c = c_1^{\pm} = 4F_{2k\pm 2}$ and $\{F_{2k}, 5F_{2k}, c, d\}$ is $D(4)$-quadruple, then $d = c_2^{\pm}$.

Let us mention the case when $k = 1$ is solved completely in [2] where Baćić and the author considered the extensions of $D(4)$-triples $\{k-2, k+2, c\}$ so the triples $\{1, 5, c\}$ were covered there. The same authors [3, Lemma 3] proved this theorem for $k \leq 8$. From this point onward we will assume $k \geq 9$. Also, the case $c = c_1^{+}$ is exactly what is proved in [4]. Theorem 1.1 also implies the following corollary.

Corollary 1.2. *Let $k \geq 2$ be an integer. If the set $\{F_{2k}, 5F_{2k}, c, d\}$ is $D(4)$-quadruple with $c < d$ then d is uniquely determined. Moreover, $\{F_{2k}, 5F_{2k}, c, d\}$ is a regular $D(4)$-quadruple and $D(4)$-pair $\{F_{2k}, 5F_{2k}\}$ cannot be extended to a $D(4)$-quintuple.*

Proof. The statement of the Corollary follows immediately from Theorem 1.1 if we know that all possible c's that extend $D(4)$-pair $\{F_{2k}, 5F_{2k}\}$ are given by (1.1) and noticing that $c_{\nu-1}^{\pm}c_{\nu+1}^{\pm} + 4$ cannot be a perfect square. To prove this let us remember that if we want to extend $D(4)$-pair $\{a, b\}$ with element c, there should exist positive integers s and t such that $ac + 4 = s^2$ and $bc + 4 = t^2$. Eliminating c, we get a pellian equation

$$bs^2 - at^2 = 4(b - a).$$

However, in our case we have $a = F_{2k}$ and $b = 5a = 5F_{2k}$ which implies $t^2 - 5s^2 = -16$. It is not hard to see that all fundamental solutions which generate all the solutions of this equation are given by $(t_0, s_0) = (\pm 2, 2), (\pm 8, 4)$. The fundamental solution $(t_0, s_0) = (\pm 2, 2)$ will generate the extensions with $c = c_{\nu}^{\pm}$, while the fundamental solution $(\pm 8, 4)$ will not give any extension to a triple because in that case $c = (s^2 - 4)/a$ and we can prove by induction on parameter ν that $s^2 \equiv s_0^2 \equiv 16 \pmod{a}$. So, c can be an integer only if a is divisor of 12, which is obviously possible only for $a = F_2 = 1$ and $a = F_4 = 3$. □

In the proof of the main theorem we will use the already well-known standard methods in solving those kind of problems. The main purpose of the paper is to illustrate the use of Theorem 1 in [3] which will make our proof much faster and more elegant, because we will have to consider the extendibility of $D(4)$-triples $\{F_{2k}, 5F_{2k}, c\}$ only for few values of c. This will help us give a lower bound for the solutions of the system of simultaneous pellian equations using congruence methods which have some difficulties considering general c, because then it is not so obvious which congruences to consider. We will show the application of some other results from [4] and [3]. Both papers are easily accessible online. In the end we will combine the lower bounds for the solution with the upper bound which was already known in [4] where the authors used the hypergeometric method. Another interesting part here is that we do not need to use linear forms in logarithms nor any reduction method which will also save us a lot of time. However, those two tools were used in Lemma 3 [3] which allows us to assume that $k \geq 9$.

2. Preliminaries

Let $a = F_{2k}, b = 5F_{2k}$ and $\{a, b, c, d\}$ be a $D(4)$-quadruple with $c = c_{\nu}$, which is given by (1.1), and $k \geq 9$. Notice that in the proof of Corollary 1.2 we showed that all c's which extend the pair $\{F_{2k}, 5F_{2k}\}$ are given by (1.1). Moreover, let r, s and t be positive integers defined by $ab + 4 = r^2$, which implies $r = L_{2k}$, and $F_{2k}c + 4 = s^2$, $5F_{2k}c + 4 = t^2$. Then there exist integers x, y and z such that

$$F_{2k}d+4=x^2, \qquad 5F_{2k}d+4=y^2, \qquad cd+4=z^2. \tag{2.1}$$

Eliminating d, we obtain the following system of simultaneous pellian equations

$$F_{2k}z^2-cx^2=4(F_{2k}-c) \tag{2.2}$$

$$5F_{2k}z^2-cy^2=4(5F_{2k}-c). \tag{2.3}$$

From [4, Lemma 2] we know that if (z,x) and (z,y) are positive solutions of (2.2) and (2.3), respectively, then there exist indices i and m such that $z=v_m^{(i)}$, where

$$v_0^{(i)}=z_0^{(i)}, \quad v_1^{(i)}=\frac{1}{2}\Big(sz_0^{(i)}+cx_0^{(i)}\Big), \quad v_{m+2}^{(i)}=sv_{m+1}^{(i)}-v_m^{(i)}, \tag{2.4}$$

and there exist indices j and n such that $z=w_n^{(j)}$, where

$$w_0^{(j)}=z_1^{(j)}, \quad w_1^{(j)}=\frac{1}{2}\Big(tz_1^{(j)}+cy_1^{(j)}\Big), \quad w_{n+2}^{(j)}=tw_{n+1}^{(j)}-w_n^{(j)}. \tag{2.5}$$

Here $(z_0^{(i)},x_0^{(i)})$ and $(z_1^{(j)},y_1^{(j)})$ are fundamental solutions of (2.2) and (2.3), respectively. So now we have transformed the problem of solving the system of simultaneous pellian equations to solving finitely many Diophantine equations $z=v_m^{(i)}=w_n^{(j)}$. For simplicity's sake, from now on, we will omit the superscripts (i) and (j). Initial terms of the sequences (v_m) and (w_n) are almost completely determined in the following lemma.

Lemma 2.1. *Let $a<b<c$.*

- (i) *If the equation $v_{2m}=w_{2n}$ has a solution, then $z_0=z_1$. Moreover, $|z_0|=2$ or $|z_0|=\frac{1}{2}(cr-st)$ or $|z_0|<1.608a^{\frac{-5}{14}}c^{\frac{9}{14}}$.*
- (ii) *If the equation $v_{2m+1}=w_{2n}$ has a solution, then $|z_0|=t$, $|z_1|=\frac{1}{2}(cr-st)$, $z_0z_1<0$.*
- (iii) *If the equation $v_{2m}=w_{2n+1}$ has a solution, then $|z_1|=s$, $|z_0|=\frac{1}{2}(cr-st)$, $z_0z_1<0$.*
- (iv) *If the equation $v_{2m+1}=w_{2n+1}$ has a solution, then $|z_0|=t$, $|z_1|=s$, $z_0z_1>0$.*

Moreover, if $c=a+b-2r<b$, then the equations $v_{2m+1}=w_{2n}$, $v_{2m}=w_{2n+1}$, and $v_{2m+1}=w_{2n+1}$ do not have a solution, while if the equation $v_{2m}=w_{2n}$ has a solution, then $z_0=z_1=2$.

Proof. Statements (i) to (iv) are exactly [5, Lemma 9] and the last statement can be proven the same way we found the initial terms in the proof after Lemma 1 in [8]. Notice only that we had $a<b<c=a+b+2r$ while here $a<c<b=a+c+2s$. □

From now on we will assume that c is "minimal" in some sense, which will help us narrow the possibilities for the fundamental solutions (z_0,x_0) and (z_1,y_1).

Assumption 2.2. *Assume that $\{F_{2k},5F_{2k},c',c\}$ is not a $D(4)$-quadruple for any c' with $0<c'<c_{\nu-1}$.*

Remark 2.3. *Notice that this assumption is not restrictive in any sense because we know all possible values of c. If we prove that under Assumption 2.2 $D(4)$-triple $\{F_{2k},5F_{2k},c\}$ has only two extensions to a quadruple (with $d=d_-=c_{\nu-1}$ and $d=d_+=c_{\nu+1}$), which implies the statement of Theorem 1.1.*

Now remember in [3, Theorem 1], Baćić and the author proved that if $\{a,b,c\}$ with $b\geq 5a$ is $D(4)$-triple whose extension satisfies the Assumption 2.2, then $c<6b^5$ or it can be extended to a quadruple only with $d=d_{\pm}$. It implies that we have to consider the extensions of our triples $\{F_{2k},5F_{2k},c\}$ only with $c=c_1^{\pm},c_2^{\pm},c_3^{\pm}$ and that is what we will do now. As we mentioned, the case $c=c_1^+$ is solved in [4].

Lemma 2.4. *Let $c = c_2^{\pm}, c_3^{\pm}$. Then solving the equations $v_m = w_n$, it is enough to consider*

(i) $v_{2m} = w_{2n}$ *if* $z_0 = z_1 = \pm 2$ *and*

(ii) $v_{2m+1} = w_{2n+1}$ *if* $z_0 = \pm t, z_1 = \pm s$ *and* $z_0 z_1 > 0$.

Proof. The proof for this follows immediately from the proof for [5, Lemma 9] and Assumption 2.2. The Assumption will help us to remove the third case of (i) from Lemma 2.1 because in that case we must have an irregular $D(4)$-quadruple $\{a, b, d_0, c\}$ with $0 < d_0 < c$ which contradicts the Assumption. Furthermore, it is easy to see that other cases ($v_{2m+1} = w_{2n}$ and $v_{2m} = w_{2n+1}$) give the exact same intersections of the sequences (v_m) and (w_n) as (i) and (ii) in this Lemma. □

3. Lower Bounds for the Solutions

In this section we will give the lower bounds of the indices m and n in the equation $v_m = w_n$ for $m > n \geq 2$. It is not hard to check that all solutions of $v_m = w_n$ with smaller indices will give the extension of $D(4)$-triple $\{a, b, c\}$ to a quadruple with $d = d_- = c_{\nu-1}^{\pm}$ or $d = d_+ = c_{\nu+1}^{\pm}$. So to prove Theorem 1.1 we must show that $v_m = w_n$ for $m > n \geq 2$ does not have a solution for $c = c_1^-, c_2^{\pm}, c_3^{\pm}$.

Lemma 3.1.

(i) *Let $c = c_1^- = 4F_{2k-2}$. If $v_{2m} = w_{2n}$ has a solution with $n > 1$, then*

$$m > n \geq \sqrt{\frac{F_{2k-2}}{6}}.$$

(ii) *Let $c = c_2^{\pm} = 4L_{2k}F_{4k\pm 2}$. If $v_{2m} = w_{2n}$ has a solution with $n > 1$, then $m > F_{2k}/4$. Moreover, if $v_{2m+1} = w_{2n+1}$ has a solution with $n \geq 1$, then*

$$m > \frac{\sqrt[4]{L_{2k}}}{3}.$$

(iii) *Let $c = c_3^{\pm} = 4(L_{4k}+1)F_{6k\pm 2}$. If $v_{2m} = w_{2n}$ has a solution with $n > 1$, then $m > F_{2k}/4$. Moreover, if $v_{2m+1} = w_{2n+1}$ has a solution with $n \geq 1$, then*

$$m > \frac{\sqrt[4]{F_{2k}^3}}{2}.$$

Proof. (i) Let us assume the opposite, i.e. $n < \sqrt{F_{2k-2}/6}$. Let $c = c_1^- = 4F_{2k-2}$. Then $s = 2F_{2k-1}$ and $t = 2L_{2k-1}$. Considering congruences modulo c^2 (see [5, Lemma 12]) we have

$$am^2 \pm sm \equiv bn^2 \pm tn \pmod{c}.$$

Now using $F_{2k-1} = F_{2k} - F_{2k-2}$ and $L_{2k-1} = F_{2k} + F_{2k-2}$ we get

$$F_{2k}m^2 \pm 2F_{2k}m \equiv 5F_{2k} \pm 2F_{2k}n \pmod{F_{2k-2}}.$$

Moreover, F_{2k} and F_{2k-2} are relatively prime, so it implies

$$m^2 \pm 2m \equiv 5n^2 \pm 2n \pmod{F_{2k-2}}.$$

Now from assumption $n < \sqrt{F_{2k-2}/6}$ and the fact that we always have $n < m < 2n$ we see that both sides of the congruence are positive and less than F_{2k-2} which yields that we actually have an equation

$$m^2 \pm 2m = 5n^2 \pm 2n$$

which obviously does not have any solution in positive integers because the right-hand side is larger. So we obtain a contradiction which implies (i).

(ii) Let us first consider the case with even indices $v_{2m} = w_{2n}$. Again assume the opposite, $n < m \leq F_{2k}/4$. Also let us consider the case $c = c_2^+ = 4L_{2k}F_{4k+2}$. From [5, Lemma 12] we conclude

$$\pm am^2 + sm \equiv \pm bn^2 + tn \pmod{c}.$$

Using that $s = L_{2k}(L_{2k} + F_{2k}) - 2$, $t = L_{2k}(L_{2k} + 5F_{2k}) - 2$ and $n < m \leq F_{2k}/4$ we conclude that the absolute values of both sides of the congruence are less than $c/2$. It implies that we actually have an equation

$$\pm am^2 + sm = \pm bn^2 + tn.$$

Now, considering congruences modulo F_{2k} together with $L_{2k}^2 = 5F_{2k}^2 + 4$, we have

$$2m \equiv 2n \pmod{F_{2k}}.$$

Now using the assumption $n < m \leq F_{2k}/4$ we get an equation $m = n$ which does not have a solution, so again we have a contradiction. The case $c = c_2^-$ can be solved in the exactly same way. There we have $s = L_{2k}(L_{2k} - F_{2k}) - 2$ and $t = L_{2k}(5F_{2k} - L_{2k}) + 2$ which would imply $m = -n$ and gives a contradiction.

Let us now consider the case with odd indices. Here we will use the fact that from $s^2t^2 \equiv 16 \pmod{c}$ we know that $st \equiv \pm 4 \pmod{c'}$ for some c' which is a divisor of c and $c' \geq \sqrt{c}$. Again from [5, Lemma 12] we then conclude

$$am(m+1) \pm rm \equiv bn(n+1) \pm rn \pmod{\frac{c'}{2}}.$$

Notice that c' can be chosen to be divisible by 2, because c is always divisible by 4 which also implies that both $st + 4$ and $st - 4$ are even. Now we want both sides of the congruence are less that $c'/2$ (they are already positive), i.e. less than $\sqrt{L_{2k}F_{4k\pm2}}$. To get that it is sufficient to assume $n < m \leq \sqrt[4]{L_{2k}}/3$. So we have the equation

$$am(m+1) \pm rm = bn(n+1) \pm tn$$

instead of a congruence. Now considering congruences modulo F_{2k} we get

$$L_{2k}m \equiv L_{2k}n \pmod{F_{2k}}.$$

Since the greatest common divisor of F_{2k} and L_{2k} is 1 or 2, we get a contradiction $m = n$ using $n < m \leq \sqrt[4]{L_{2k}}/3 < F_{2k}/2$ and in the case with odd indices we must also have $n < m < 2n$. So we arrive at a contradiction again which finishes the proof of (ii).

(iii) This part can be proven in the exact same way as (ii). □

4. Upper Bounds for the Solutions and the Proof of the Main Theorem

Here we will use Lemma 5 and Lemma 6 from [4], which implies that if $z = v_m = w_n$, then

$$z^\lambda < 14445b^2c^2,$$

where

$$\lambda = \frac{\log\left(\frac{51bc'}{17600000}\right)}{\log(0.00425b^2c'^2)}$$

and $c' = c/4$. To prove this the authors have used the important facts that $b = 5a$ and that c is divisible by 4. Also, it was proved under the assumption $bc' > 5^9$ which is valid in all our cases of c remembering that we have $k \geq 9$.

So we have

$$\log z < \frac{\log(14445b^2c^2)\log(0.00425b^2c'^2)}{\log\left(\frac{51bc'}{17600000}\right)}.$$

We combine it with the lower bound for $z = v_m > (\sqrt{ac})^{m-1}$. It implies that if $v_m = w_n$, then

$$m - 1 < \frac{\log(14445b^2c^2)\log(0.00425b^2c'^2)}{\log(ac)\log\left(\frac{51bc'}{17600000}\right)}. \tag{4.1}$$

In the end we combine (4.1) with the lower bounds for m that we obtain in the previous section together (when it is needed for a small k) with the fact that if $v_m = w_n$, then $m > n > 7$ (see [6, Lemma 5]). It will give us the upper bound for k. Using that the right-hand side of (4.1) is decreasing in k for $k \geq 9$, we easily get that in all cases $k < 9$ which is a contradiction. So we can have the solution of the equation $v_m = w_n$ only with small indices and it is easy to check (and it was done many times in the papers on the topic of Diophantine m-tuples) that it will imply the statement of Theorem 1.1. In the case of $c = c_1^-$, we only get solution $v_0 = w_0 = \pm 2$ which implies $d = 0$ which is no real extension to a quadruple. And $v_2 = w_2 = (cr + st)/2$, which implies $d = d_+$. In the case of $c \geq c_2^-$, we get the solutions: $v_0 = w_0 = \pm 2$ which gives us $d = 0$, $v_1 = w_1 = (cr \pm st)/2$ which gives us $d = d_\pm$.

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Faculty of Civil Engineering, University of Zagreb, Zagreb, Croatia
E-mail address: filipin@grad.hr