QUASI-PERIODS FOR THE HOFSTADTER Q FUNCTION

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Abstract. The purpose of this paper is to answer, in the negative, an open question of Ruskey on whether all solutions of Hofstadter's Q function have quasi-period 3. We solve this problem by presenting for each positive integer e initial conditions for the Q function such that the resulting sequence satisfying the recursion and initial conditions has quasi-period e.

1. Background

Several authors have studied nested recursions, that is, recursions, where at least one of the arguments of at least one occurrence of the recursive function in the defining recursion, references that recursive function. Nested recursions are also called self-referencing functions; the sequences satisfying a nested recursion are also called meta-Fibonacci. The following examples, some of which omit initial conditions, illustrate this concept.

Example 1.1. *Conolly's recursion* [3] *is defined by*

$$F(n) = F(n - F(n-1)) + F(n-1-F(n-2)), \tag{1.1}$$

with initial conditions,

$$F(1) = 0, F(2) = 1. \tag{1.2}$$

Note that Conolly actually defined four functions, H, F, C, K.

Example 1.2. *Hostadter's Q-function* [7] *is defined by*

$$Q(n) = Q(n - Q(n-1)) + Q(n - Q(n-2)). \tag{1.3}$$

The Q function is sometimes called the U function.

Depending on initial conditions, the Q function may fail to be well-defined for all n. In fact, it is an open question whether it is well-defined for all positive n with initial conditions $Q(1) = Q(2) = 1$. It is an open question to describe the behavior of the Q function for large n (e.g. [16]). Researchers still actively study variants of the Q function (e.g. [1]).

In the next section, we will present the two known solutions where the Q function is defined for all n. Both of them have quasi-period 3. This led Ruskey to ask whether all known solutions have quasi-period 3. A main result of this paper is a negative answer to this question. More specifically, we produce for every positive integer e a solution of the Q function with quasi-period e.

Example 1.3. *The V function,* [1], *a variant of Hostadter's Q-function, is defined by*

$$V(n) = V(n - V(n-1)) + V(n - V(n-4)),$$

with initial conditions

$$V(1) = V(2) = V(3) = V(4) = 1.$$

Although the V function is too complicated to study in this paper, we completely solve a similar complex sequence, (1.3) with initial conditions $Q(1) = 1, Q(2) = 5, Q(3) = 3$. Under these initial conditions, the resulting infinite sequence $\{Q(i)\}_{i \geq 1}$, naturally divides into blocks, similar to the *generations* of the V sequence. The lengths of the blocks are growing and these lengths are governed by a recursion. The blocks themselves have specific periodicities which we identify.

The solutions to nested recursions are sometimes *quasi-periodic*. To explain this, we need to develop some compact notation and introduce terminology, which we do in the next section.

2. Notation and Terminology

Golumb [5] presented the first solved case of the Hofstadter Q function.

Assume initial conditions, $Q(1) = 3, Q(2) = 2, Q(3) = 1$;

then by (1.3), $Q(3k-2) = 3, Q(3k-1) = 3k-1, Q(3k) = 3k-2, \qquad k \geq 1. \qquad (2.1)$

Ruskey [17] provides a further solution to the Hostadter Q function.

Assume initial conditions, $Q(0) = Q(3) = 3, Q(1) = Q(4) = 6, Q(2) = 5, Q(5) = 8$;

then by (1.3), $Q(3k) = 3, Q(3k+1) = 6, Q(3k+2) = F_{k+5}, \qquad k \geq 0, \qquad (2.2)$

where F_k are the Fibonacci numbers.

In the sequel, it will be convenient to state equations like (2.1) using a more compact notation which we now introduce and explain.

$$Q_{1:3} = \langle 3, 2, 1 \rangle \longrightarrow Q_{1:\infty} = \langle 3, 3k-1, 3k-2 \rangle^{\infty}. \tag{2.3}$$

The correspondence between (2.1) and (2.3) should be clear. We make the following clarifying remarks:

- Sequences (in contrast to sets) are indicated by angle brackets.
- Vector notation $v_{i:j} = \langle w(i), \ldots, w(j) \rangle$ compactly summarizes the $j - i + 1$ equations $v(k) = w(k), i \leq k \leq j$.
- A right arrow indicates how assumed initial conditions stated on the left of the right arrow yield the results stated on the right side of the right arrow.
- The vector-function notation $\langle v(f_1(k)), \ldots, v(f_m(k)) \rangle_{k \in S}$ where the $f_i, 1 \leq i \leq m$ are functions and $S = \langle s_1, s_2, \ldots \rangle$ is a possibly infinite sequence of integers, is compact shorthand for the sequence
$$\langle v(f_1(s_1)), \ldots, v(f_m(s_1)), v(f_1(s_2)), \ldots, v(f_m(s_2)), \ldots \rangle.$$
- Similarly, the word exponent notation $\langle v(f_1(k)), \ldots, v(f_m(k)) \rangle^n$ indicates the sequence
$$\langle v(f_1(1)), \ldots, v(f_m(1)), v(f_1(2)), \ldots, v(f_m(2)), \ldots, v(f_1(n), \ldots, v(f_m(n) \rangle.$$
We allow n to be infinite.
- We will abuse notation and use unions of sequences.

For purposes of notational clarity, in the sequel, we may use the notation $Q(x)$ interchangeably with Q_x, especially when x is a long expression.

The following examples further clarify the notation.

Example 2.1. *Equation* (2.2) *can be restated as follows:*

$$Q_{0:5} = \langle 3, 6, 5, 3, 6, 8 \rangle \longrightarrow Q_{0:\infty} = \langle 3, 6, F_k \rangle_{k \geq 5}. \tag{2.4}$$

Example 2.2. *The expression*

$$\langle 4\rangle \cup \left(\langle 6k, 2\rangle^2 \cup \langle 6k, 6\rangle \right)^{\infty}, \tag{2.5}$$

indicates the sequence, $\langle 4, 6, 2, 6, 2, 6, 6, 12, 2, 12, 2, 12, 6, 18, 2, 18, 2, 18, 6, \ldots\rangle$. *Note how we use parentheses to indicate order. First we evaluate* $\langle 6k, 2\rangle^2$ *and* $\langle 6k, 6\rangle$ *at* $k = 1$*; then we evaluate them at* $k = 2$*, etc.*

Heuristically, we would like to say that (2.5) is *quasi-periodic,* a concept first introduced by Golumb. There are a variety of ways to define this heuristic. We find the following definition not too restrictive and not too broad.

Definition 2.3. *A sequence* $\{a_n\}_{n\geq 1}$ *is quasi-periodic if there exists some positive integer* $k \geq 1$*, the quasi-period, a non-negative integer* n_0 *and a collection of functions* $g_i, 1 \leq i \leq k$*, such that (i) for all non-negative integer* m *and for* $1 \leq i \leq k$*,* $a_{n_0+i+mk} = g_i(m)$*, and (ii) at least one of the* $g_i, 1 \leq i \leq k$*, is identically constant. By analogy with decimal representations of fractions, if* $n_0 = 0$ *the sequence is called* purely *quasi-periodic; otherwise the sequence is called* eventually *quasi-periodic.*

Example 2.4. *Equation* (2.5) *is eventually quasi-periodic with quasi-period 6. This can be seen by letting* $n_0 = 1$*,* $g_2(m) = g_4(m) = 2, g_6(m) = 6, g_1(m) = g_3(m) = g_5(m) = 6m + 6$*, and applying the definition just given.*

Example 2.5. *Equation* (2.3) *is purely quasi-periodic with quasi-period 3. This can be seen by letting* $n_0 = 0, g_1(m) = 3, g_2(m) = 3m + 2, g_3(m) = 3m + 1$.

Example 2.6. *Equation* (2.4) *is purely quasi-periodic with quasi-period 3. This can be seen by letting* $n_0 = -1, g_1(m) = 3, g_2(m) = 6, g_3(m) = F_{5+m}$*. Notice the minor adjustment in the definition to account for the fact that* (2.4) *starts at index 0.*

Example 2.7. *(Traditional periodicity.) A sequence* $\{a_n\}_{n\geq 1}$ *is periodic if it is quasi-periodic and all* $g_i, 1 \leq i \leq k$*, are the constant function.*

Certain subtleties about and problems with Definition 2.2 are as follows.

- The definition does not require that the g_i be linear functions.
- For *any* sequence $\{a_n\}_{n\geq 1}$, we can define $g_1(m) = a_m$. So without the requirement that at least one of the g_i be constant, all sequences would be quasi-periodic of quasi-period 1. This is clearly not desirable. Consequently, the requirement that at least one of the g_i be constant is needed.
- Definition 2.2 excludes the sequence $1, 2, 1, 2, 1, 1, 2, 1, 1, 1 \ldots$ from being classified as quasi periodic. The point here is that *quasi-periodic* is not a synonym for *pattern.* For this reason, we require quasi-periods to be constants.
- For the reals, *periodicity* implies rationality; similarly, with continued fractions, *periodicity* implies quadratic irrationality; thus for the reals and continued fractions there is a need to be precise in the use of the word *periodic* since it implies other properties. However, with nested recursions, quasi-periodicity does not predict any other property. For this reason, we have restricted quasi-periodicity to situations where some constant is repeating periodically.

In the sequel, it will suffice to state that a sequence is (eventually or purely) quasi-periodic without going into details about application of the definition.

In Example 1.2 we have already noted that, depending on the initial conditions, the Q sequence may not be well defined for all positive n.

Ruskey [17] notes that "One way to make the Q sequence well-defined is to simply specify the initial values of Q for all $n < 1$." Consequently, throughout this paper, we assume the following.

$$Q(n) = 0, \qquad \text{if } n \leq 0. \tag{2.6}$$

Ruskey himself did not require $Q(0) = 0$. There however is no loss of generality in so assuming.

3. Ruskey's Open Problem

In the last section of his paper, Ruskey lists open problems including the following: "Both of the known quasi-periodic solutions to the Hofstadter recurrence ((2.3) and (2.4)) have quasi-period 3. Are other quasi-periods possible?"

We answer this open problem in the negative with the following theorem.

Theorem 3.1.

(a) *For positive odd integer o and positive even integer e with $o > e$ we have*

$$Q_{1:3} = \langle e, o, 2\rangle \longrightarrow Q_{1:\infty} = \langle e\rangle \cup \langle o, 2\rangle^{\frac{o-1}{2}} \cup \left(\langle o + ek, 2\rangle^{\frac{e}{2}}\right)^{\infty},$$

a sequence with eventual quasi-period e.

(b) *For two positive even integers e_2 and e_1 with $e_2 > e_1$ we have*

$$Q_{1:3} = \langle e_1, e_2, 2\rangle \longrightarrow Q_{1:\infty} = \langle e_1\rangle \cup \left(\langle ke_2, 2\rangle^{\frac{e_2-2}{2}} \cup \langle ke_2, e_1 + 2\rangle\right)^{\infty},$$

a sequence with eventual quasi-period e_2.

Graphical illustrations of Theorem 3.1(a) and (b) for specific values of e and o are presented in Figures 1 and 2, respectively.

Corollary 3.2. *For every positive even integer e there is a Q-sequence with eventual quasi-period e.*

4. Proof of Theorem 3.1

In the introductory section, we presented several nested recursions. Some of these recursions have been solved or partially solved while others remain open problems.

When solutions exist, the proof methods vary. Some popular techniques are use of ceiling functions [5, 9], treatment of slow growing functions [4, 12, 13, 14], or a combinatoric approach using labeled tree methods [1, 4, 8, 10, 11, 13, 14, 15, 18].

A classical approach is proof by nested inductions. In a nested inductive proof, the induction step or the base step of an induction argument requires an inductive proof. Some authors refer to these as *multi-statement induction proofs*. For examples, see [6, 2, 19].

In this paper, we exclusively focus on proofs by nested inductions. We note that very often nested inductive arguments naturally combine with alternative methods such as labeled-tree methods. Thus the methods of this paper have wider applicability.

In this section, we prove Theorem 3.1. We only prove Theorem 3.1(a), the proof of Theorem 3.1(b) being similar and hence omitted. We first prove three propositions.

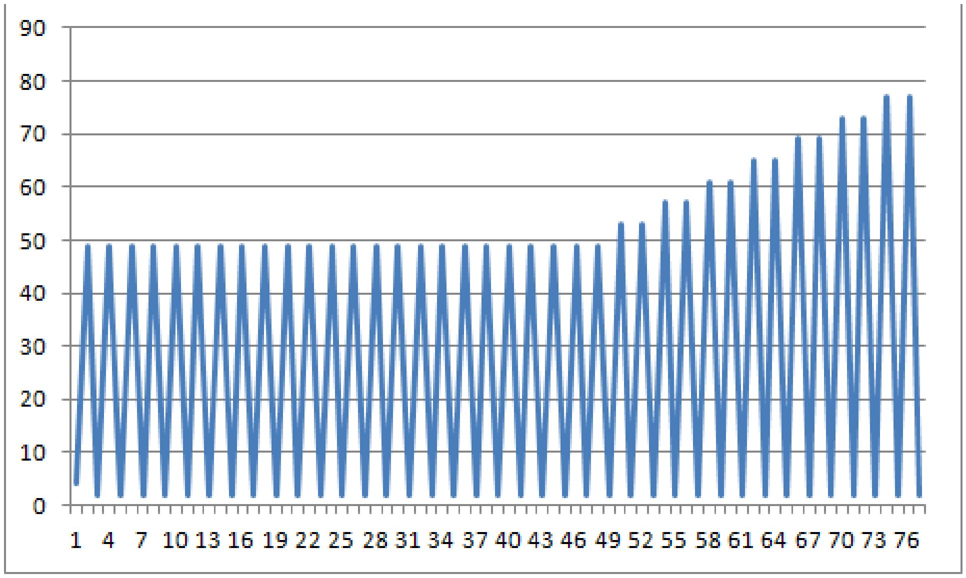

FIGURE 1. Graph of $Q(n)$ with $Q_{1:3} = \langle 4, 49, 2\rangle$. The graph illustrates Theorem 3.1(a). After the initial 4, there is an initial flat segment of a repeating $\langle 49, 2\rangle$ followed by linearly ascending segments of period 4, $\langle 53, 2, 53, 2\rangle, \langle 57, 2, 57, 2\rangle, \ldots$.

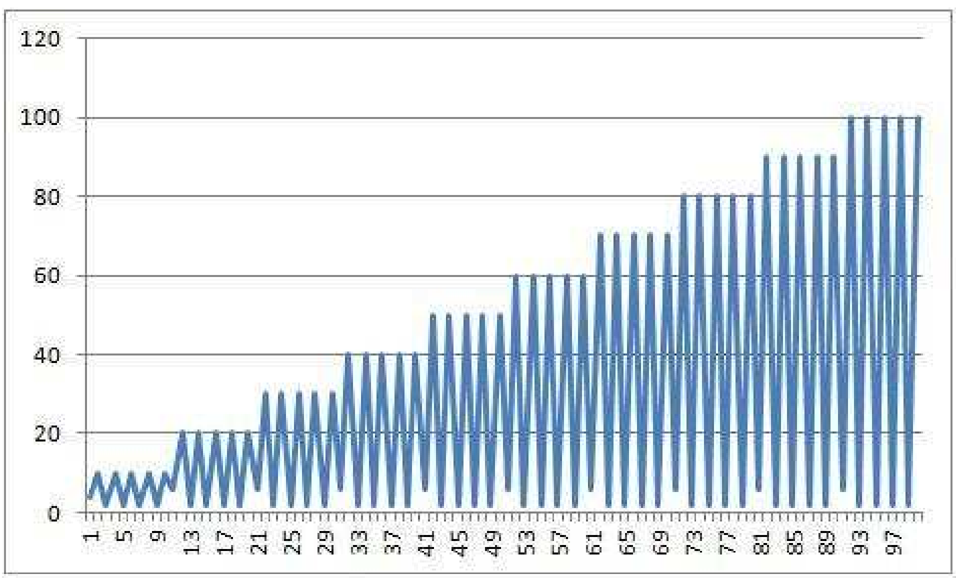

FIGURE 2. Graph of $Q(n)$ with $Q_{1:3} = \langle 4, 10, 2\rangle$. The graph illustrates Theorem 3.1(b). After the initial 4, there are segments with period 10 consisting of 4 repeating pairs $\langle 10k, 2\rangle$ followed by a terminal pair $\langle 10k, 6\rangle$.

Proposition 4.1. $Q_{2:o} = \langle o, 2\rangle^{\frac{o-1}{2}}$.

Proof. To prove Proposition 4.1, it suffices to show that for

$$1 \leq K \leq \frac{o-1}{2},$$

that

$$Q_{2:2K+1} = \langle o, 2\rangle^{K}. \tag{4.1}$$

The initial conditions for Theorem 3.1(a), $Q_{1:3} = \langle e, o, 2\rangle$, imply that (4.1) is satisfied for $K = 1$. This forms the base step for an induction argument.

For an induction step, we assume that for some K satisfying

$$1 \leq K \leq \frac{o-3}{2} \tag{4.2}$$

that (4.1) holds. We proceed to show that (4.1) holds with K replaced by $K+1$. To do so, we must show that $Q(2K+2) = o$ and $Q(2K+3) = 2$. We suffice with showing that $Q(2K+1) = o$, the proof that $Q(2K+3) = 2$ being similar. Note that our induction assumption that (4.1) holds for K satisfying (4.2) implies

$$Q(2K) = o, \qquad Q(2K+1) = 2. \tag{4.3}$$

$$\begin{aligned} Q(2K+2) &= Q(2K+2-Q(2K+1)) + Q(2K+2-Q(2K)), \text{ by (1.3)}, \\ &= Q(2K) + Q(2K+2-o), \text{ by (4.3)}, \\ &= o, \text{ by (4.3), (4.2) and (2.6)}. \end{aligned}$$

□

Proposition 4.2. *For* $K \geq 0$,

$$Q_{o+eK-1:o+eK} = \langle o+eK, 2\rangle \longrightarrow Q_{o+eK+1:o+eK+2} = \langle o+eK+e, 2\rangle.$$

Proof. By the hypothesis of Theorem 3.1(a), we have

$$Q(1) = e. \tag{4.4}$$

Assume for some $K \geq 0$, that

$$Q_{o+eK-1:o+eK} = \langle o+eK, 2\rangle. \tag{4.5}$$

We proceed to prove $Q_{o+eK+1:o+eK+2} = \langle o+eK+e, 2\rangle$. To do so requires proving $Q(o+eK+1) = o+eK+e$ and $Q(o+eK+2) = 2$. We suffice with proving $Q(o+eK+1) = o+eK+e$, the proof of the other equality being similar.

$$\begin{aligned} Q(o+eK+1) &= Q(o+eK+1-Q(o+eK)) + Q(o+eK+1-Q(o+eK-1)), \text{ by (1.3)}, \\ &= Q(o+eK-1) + Q(1), \text{ by (4.5)}, \\ &= o+eK+e, \text{ by (4.5) and (4.4)}. \end{aligned}$$

□

Proposition 4.3. *For* $e \geq 4$, *if*

$$Q_{o+ek+1:o+ek+2} = \langle o+ek+e, 2\rangle, \tag{4.6}$$

then

$$Q_{o+ek+1:o+ek+e} = \langle o+ek+e, 2\rangle^{\frac{e}{2}}. \tag{4.7}$$

Proof. To prove Proposition 4.3, it suffices to show that if (4.6) holds, then for

$$1 \le l \le \frac{e}{2},$$

that

$$Q_{o+ek+1:o+ek+2l} = \langle o+ek+e, 2\rangle^{l}. \tag{4.8}$$

Equation (4.6) implies that (4.8) holds for $l = 1$. This forms the base step for an induction argument.

For an induction step, we assume that for some l satisfying

$$1 \le l \le \frac{e-2}{2}, \tag{4.9}$$

that (4.8) holds. We proceed to show that (4.8) holds with l replaced by $l+1$. To do so, we must show that $Q(o+ek+2l+1) = o+ek+e$ and $Q(o+ek+2l+2) = 2$. We suffice with showing that $Q(o+ek+2l+1) = o+ek+e$, the proof of the other equality being similar. Note that our induction assumption that (4.8) holds for l satisfying (4.9) implies

$$Q(o+ek+2l-1) = o+ek+e, \qquad Q(o+ek+2l) = 2. \tag{4.10}$$

$$\begin{aligned} Q_{o+ek+2l+1} &= Q(o+ek+2l+1-Q(o+ek+2l)) + Q(o+ek+2l \\ &\quad +1-Q(o+ek+2l-1)), \text{ by (1.3)}, \\ &= Q(o+ek+2l-1) + Q(2l+1-e), \text{ by (4.10)}, \\ &= o+ek+e, \text{ by (4.10), (4.9) and (2.6)}. \end{aligned}$$

□

We can now complete the proof of Theorem 3.1(a).

Proof. For a base step of an induction argument, assume that for some non-negative integer n that

$$Q_{2:o+en} = \langle o, 2\rangle^{\frac{o-1}{2}} \cup \left(\langle o+ek, 2\rangle^{\frac{e}{2}}\right)^{n}, \tag{4.11}$$

the case $n = 0$, justified by Proposition 4.1. We proceed to prove (4.11) with n replaced by $n+1$.

First, by (4.11) and Proposition 4.2, we have

$$Q_{2:o+en+2} = \langle o, 2\rangle^{\frac{o-1}{2}} \cup \left(\langle o+ek, 2\rangle^{\frac{e}{2}}\right)^{n} \cup \langle o+en+e, 2\rangle. \tag{4.12}$$

If $e = 2$, then (4.12) completes the proof that (4.11) holds with n replaced by $n+1$.

If $e \ge 4$, then Proposition 4.3 shows

$$Q_{o+en+1:o+en+e} = \langle o+en+e, 2\rangle^{\frac{e}{2}},$$

completing the proof that (4.11) holds with n replaced by $n+1$. The proof of Theorem 3.1(a) is completed by letting n go to infinity in (4.11). □

5. An Advanced Example

The example presented in this section is a nested recursion with several unusual features resembling the V sequence.

We use the Q function, (1.3), with initial conditions

$$Q_{1:3} = \langle 1, 5, 3 \rangle. \tag{5.1}$$

We easily compute that $Q_{4:14}$ is simply a sequence of numbers without a pattern and that

$$\begin{aligned} Q_{15:373} &= S_0 S_1 S_2 S_3 S_4, \qquad \text{with} \\ S_0 &= Q_{15:25} = \langle 23, 2, \ldots, 23, 2, 23, 3, 7 \rangle, \\ S_1 &= Q_{26:48} = \langle 46, 2, \ldots, 46, 2, 46, 3, 7 \rangle, \\ S_2 &= Q_{49:95} = \langle 92, 2, \ldots, 92, 2, 93, 3, 7 \rangle, \\ S_3 &= Q_{96:188} = \langle 185, 2, \ldots, 185, 2, 186, 3, 7 \rangle, \\ S_4 &= Q_{189:373} = \langle 371, 2, \ldots, 371, 2, 371, 3, 7 \rangle. \end{aligned} \tag{5.2}$$

We observe the following similarities with the V sequence:

- The blocks of S_i resemble the *generations* of the V sequence.
- The blocks have varying length. In fact, their length seems to be growing roughly as a power of 2.
- The blocks are not *quasi-periodic* in any sense of the word; however, although their length is going to infinity they can be described with a *bounded* number of symbols.
- As we will see in the main theorem of this section, the patterns in the blocks are governed by period 4 just as the patterns in the generations of the V sequence are governed by periods.

The following theorem precisely formulates the above observations.

Theorem 5.1. *We have*

$$Q_{15:\infty} = S_0 S_1 \ldots, \tag{5.3}$$

with

$$S_i = Q_{b_i : b_{i+1}-1} = \langle x_i, 2 \rangle^{\frac{l_i - 3}{2}} \langle x_i + \epsilon'_i, 3, 7 \rangle, \tag{5.4}$$

with

$$b_0 = 15, \qquad b_{i+1} = b_i + l_i,$$

with l_i the length of S_i and

$$l_0 = 11, l_{i+1} = 2l_i + \epsilon_i, \tag{5.5}$$

$$x_0 = 23, x_{i+1} = 2x_i + \epsilon'_i, \tag{5.6}$$

with

$$\epsilon_{0:3} = \langle 1, 1, -1, -1 \rangle, \epsilon_{i+4} = \epsilon_i, \tag{5.7}$$

and

$$\epsilon'_{0,3} = \langle 0, 0, 1, 1 \rangle, \epsilon'_{i+4} = \epsilon'_i. \tag{5.8}$$

Comment 5.2. *Equation* (5.2) *shows* (5.3)–(5.8) *satisfied for* $0 \le i \le 4$.

To prove the theorem, we will need the following preliminary proposition.

Proposition 5.3. *We have*

$$2\epsilon'_i + \epsilon_i = \epsilon'_{i-1} + \epsilon'_{i+1}. \tag{5.9}$$

$$Q(\epsilon'_i) = \epsilon'_i. \tag{5.10}$$

$$l_i = x_{i-1} + \epsilon'_i. \tag{5.11}$$

$$b_{i+1} = x_i + \epsilon'_i + 3. \tag{5.12}$$

Proof. Equations (5.9) and (5.10) are proven by direct verification using (5.7) and (5.8) (there are only 4 cases to check).

Equation (5.11) is proven by induction. The base case, when $i = 1$, is verified by (5.5)–(5.7). Then, assuming by induction that (5.11) holds for some i we can prove (5.11) with i replaced by $i + 1$.

$$\begin{aligned} l_{i+1} &= 2l_i + \epsilon_i, \text{ by (5.5)}, \\ &= 2(x_{i-1} + \epsilon'_i) + \epsilon_i, \text{ by an induction assumption}, \\ &= 2x_{i-1} + 2\epsilon'_i + \epsilon_i, \\ &= 2x_{i-1} + \epsilon'_{i-1} + \epsilon'_{i+1}, \text{ by (5.9)}, \\ &= x_i + \epsilon'_{i+1}, \text{ by (5.6)}. \end{aligned}$$

Equation (5.12) is similarly proven by a routine induction argument and is omitted. □

The following corollary is repeatedly used in the proof of Theorem 5.1.

Corollary 5.4. *We have*

$$b_{i+1} - x_{i+1} = 3 - x_i, \qquad b_{i+1} - x_i = 3 + \epsilon'_i. \tag{5.13}$$

Proof. By (5.6) and (5.12). □

To prove Theorem 5.1 we first prove two propositions.

Proposition 5.5. *If*

$$Q_{b_{i+1}-7:b_{i+1}-4} = \langle x_i, 2\rangle^2 \tag{5.14}$$

then

$$Q_{b_{i+1}-3:b_{i+1}-1} = \langle x_i + \epsilon'_i, 3, 7\rangle, \tag{5.15}$$

and

$$Q_{b_{i+1}:b_{i+1}+1} = \langle x_{i+1}, 2\rangle, \tag{5.16}$$

Proof. To prove (5.15) requires proving three assertions. We suffice with proving

$$Q(b_{i+1} - 3) = x_i + \epsilon'_i, \tag{5.17}$$

the proof of the other two assertions being similar and hence omitted.

$$\begin{aligned} Q(b_{i+1} - 3) &= Q(b_{i+1} - 3 - Q(b_{i+1} - 4)) + Q(b_{i+1} - 3 - Q(b_{i+1} - 5)), \text{ by (1.3)}, \\ &= Q(b_{i+1} - 5)) + Q(\epsilon'_i)), \text{ by (5.14) and (5.13)}, \\ &= x_i + \epsilon'_i, \text{ by (5.14) and (5.10)}. \end{aligned}$$

This completes the proof of (5.15).

To prove (5.16) requires proving two assertions. We suffice with proving

$$Q(b_{i+1}) = x_{i+1}, \tag{5.18}$$

the proof of the other assertion being similar and hence omitted.

$$\begin{aligned} Q(b_{i+1}) &= Q(b_{i+1} - Q(b_{i+1} - 1)) + Q(b_{i+1} - Q(b_{i+1} - 2)), \text{ by (1.3)}, \\ &= Q(b_{i+1} - 7) + Q(b_{i+1} - 3), \text{ by (5.15)}, \\ &= x_i + x_i + \epsilon_i', \text{ by (5.14) and (5.15)}, \\ &= x_{i+1}, \text{ by (5.6)}. \end{aligned}$$

This completes the proof of (5.16) and Proposition 5.5. □

Proposition 5.6. *If*

$$Q_{b_{i+1}:b_{i+1}+1} = \langle x_{i+1}, 2\rangle \tag{5.19}$$

then

$$Q_{b_{i+1}:b_{i+2}-4} = \langle x_{i+1}, 2\rangle^{\frac{l_{i+1}-3}{2}}. \tag{5.20}$$

Proof. To prove (5.20), it suffices to show that for

$$1 \le K \le \frac{l_{i+1} - 3}{2},$$

that

$$Q_{b_{i+1}:b_{i+1}+2K-1} = \langle x_{i+1}, 2\rangle^{K}. \tag{5.21}$$

Equation (5.19) shows that (5.21) is satisfied for $K = 1$. This forms the base step for an induction argument.

For an induction step, we assume that for some K satisfying

$$1 \le K \le \frac{l_{i+1} - 5}{2} \tag{5.22}$$

that (5.21) holds. We proceed to show that (5.21) holds with K replaced by $K + 1$. To do so, we must show that $Q(b_{i+1} + 2K) = x_{i+1}$ and $Q(b_{i+1} + 2K + 1) = 2$. We suffice with showing that $Q(b_{i+1} + 2K) = x_{i+1}$, the proof that $Q(b_{i+1} + 2K + 1) = 2$ being similar and hence omitted. Note that our induction assumption that (5.21) holds for K satisfying (5.22) implies

$$Q(b_{i+1} + 2K - 2) = x_{i+1}, \qquad Q(b_{i+1} + 2K - 1) = 2. \tag{5.23}$$

$$\begin{aligned} Q(b_{i+1} + 2K) &= Q(b_{i+1} + 2K - Q(b_{i+1} + 2K - 1)) + Q(b_{i+1} \\ &\quad + 2K - Q(b_{i+1} + 2K - 2)), \text{ by (1.3)}, \\ &= Q(b_{i+1} + 2K - 2) + Q(3 + 2K - x_i), \text{ by (5.23) and (5.13)}, \\ &= x_{i+1}, \text{ by (5.23) and (5.22), (5.11), (5.8) and (2.6)}. \end{aligned}$$

□

We now can complete the proof of (5.3).

Proof. The proof of (5.3) is equivalent to proving that

$$Q_{b_0:b_{i+1}-1} = S_0 S_1 \dots S_i, \tag{5.24}$$

for all non-negative i.

The proof is by induction. Equations (5.2) shows (5.24) true for $i \le 4$. This forms the base step for an induction argument.

For an induction step, we assume (5.24) holds for some $i \ge 4$ and proceed to prove (5.24) with i replaced by $i + 1$.

First, the induction assumption and (5.4) shows that (5.14) and (5.15) are satisfied, implying that (5.16) holds, and that therefore,

$$Q_{b_0:b_{i+1}+1} = S_0S_1 \ldots S_i \cup \langle x_{i+1}, 2\rangle.$$

But (5.16) is identical with (5.19), implying that (5.20) holds, and consequently

$$Q_{b_0:b_{i+2}-3} = S_0S_1 \ldots S_i \cup \langle x_{i+1}, 2\rangle^{\frac{l_{i+1}-3}{2}}.$$

But (5.20) is (5.14) with i replaced by $i+1$, and therefore by (5.15) and (5.4),

$$Q_{b_0:b_{i+2}-1} = S_0S_1 \ldots S_{i+1}.$$

We conclude that (5.24) holds with i replaced by $i+1$. This completes the proof of Theorem 5.1. □

6. Conclusion

This paper answers an open problem in [17] by presenting Q sequences which have exact quasi-periods, e, for any positive even integer.

We also introduced new notation methods by using vectors and word notation.

We believe this will prove useful in future studies of the Q function and other nested recursions.

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