GRAPH-THEORETIC MODELS FOR THE UNIVARIATE FIBONACCI FAMILY

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ABSTRACT. We construct graph-theoretic models for an extended univariate Fibonacci family, which includes Fibonacci, Lucas, Pell, and Pell-Lucas polynomials.

1. Introduction

It is well-known that Fibonacci, Lucas, Pell, and Pell-Lucas numbers can be studied combinatorially using tilings of linear and circular boards [1, 2, 7]. This combinatorial approach manifests the beauty in establishing some of their elegant properties. In this article, we employ different tools to explore the corresponding univariate polynomials, namely, graph-theoretic models.

The celebrated Fibonacci polynomials $f_n(x)$ were originally studied by Eugene Charles Catalan (1814–1894) in 1883, and the Lucas polynomials $l_n(x)$ by Marjorie Bicknell-Johnson in 1970. They both satisfy the same polynomial recurrence $g_n(x) = x g_{n-1}(x) + g_{n-2}(x)$, where $n \geq 2$. When $g_0(x) = 0$ and $g_1(x) = 1$, $g_n(x) = f_n(x)$; and when $g_0(x) = 2$ and $g_1(x) = x$, $g_n(x) = l_n(x)$. Clearly, $f_n(1) = F_n$ and $l_n(1) = L_n$.

Fibonacci and Lucas polynomials can also be defined by the Binet-like formulas

$$f_n(x) = \frac{\alpha^n - \beta^n}{\alpha - \beta} \text{ and } l_n(x) = \alpha^n + \beta^n,$$

where $\alpha = \alpha(x)$ and $\beta = \beta(x)$ are the solutions of the equation $t^2 - xt - 1 = 0$.

Pell polynomials $p_n(x)$ and Pell-Lucas polynomials $q_n(x)$ are defined by $p_n(x) = f_n(2x)$ and $q_n(x) = l_n(2x)$, respectively. The Pell numbers $P_n$ and Pell-Lucas numbers $Q_n$ are given by $P_n = p_n(1)$ and $Q_n = \frac{1}{2}q_n(1)$, respectively.

In the interest of brevity, we will delete the argument from the functional notation; so $g_n$ will mean $g_n(x)$.

2. Q-Matrix and Connected Graph

To construct graph-theoretic models for the univariate Fibonacci family, we introduce a $2 \times 2$ matrix, called the $Q$-matrix:

$$Q(x) = (q_{ij})_{2 \times 2} = \begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix}.$$ 

(In 1960, C. H. King studied the special case $Q(1)$ to extract some Fibonacci delights [4].) Interestingly, it can be translated into a connected graph $G$ with two vertices $v_1$ and $v_2$, and three edges. The edge from $v_i$ to $v_j$ is denoted by $v_i \rightarrow v_j$, or by the “word” $ij$ when there is no confusion. We define the weight $w_{ij}$ of edge $v_i \rightarrow v_j$ to be $q_{ij}$, where $1 \leq i \leq j \leq 2$; see Figure 1. Since a weight is assigned to each edge, $G$ is a weighted graph and $Q(x)$ is its weighted adjacency matrix.

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Next we make a few more graph-theoretic definitions for clarity.

3. Weighted Paths

A path from vertex $v_i$ to vertex $v_j$ in a connected graph is a sequence $v_i - e_i - v_{i+1} - \cdots - v_{j-1} - e_j - v_j$ of vertices $v_k$ and edges $e_k$, where edge $e_k$ is incident with vertices $v_k$ and $v_{k+1}$. The path is closed if its endpoints are the same; otherwise, it is open. The length $\ell$ of a path is the number of edges in the path; that is, it takes $\ell$ steps to reach one endpoint of the path from the other. The weight of a path is the product of the weights of the edges along the path. (Note that this definition is different from that of a path in graph theory.) For example, the weight of the path $1112$ is $x \cdot x = x^2$.

The $Q$-matrix has the property that

$$Q^n(x) = \begin{bmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{bmatrix},$$

where $n \geq 1$. Consequently, we can give a nice graph-theoretic interpretation of the recurrence $f_{n+1} = xf_n + f_{n-1}$:

$$\left( \text{sum of the weights of closed paths of length } n \text{ from } v_1 \text{ to } v_1 \right) = x \left( \text{sum of the weights of paths of length } n \text{ from } v_1 \text{ to } v_2 \right) + \left( \text{sum of the weights of closed paths of length } n \text{ from } v_2 \text{ to } v_2 \right).$$

The Lucas polynomial $l_n$ also can be interpreted using this model. The sum of the weights of closed paths of length $n$ originating at $v_1$ is $f_{n+1}$, and that of closed paths of the same length originating at $v_2$ is $f_{n-1}$; so the sum of the weights of closed paths of length $n$ is $f_{n+1} + f_{n-1} = l_n$.

For example, consider the closed paths of length 4:

Paths originating at $v_1$: 1111 1112 1121 11211 12111 12121
Sum of their weights: $x^4 + 3x^2 + 1 = f_5$

Paths originating at $v_2$: 21112 21212
Sum of their weights: $x^2 + 1 = f_3$
Cumulative Sum: $x^4 + 4x^2 + 2 = l_4$. 

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Also, since \( l_n = xf_n + 2f_{n-1} \), \( l_n \) can be interpreted differently:

\[
l_n = x \left( \text{sum of the weights of paths of length } n \text{ from } v_1 \text{ to } v_2 \right) + 2 \left( \text{sum of the weights of closed paths of length } n \text{ from } v_2 \text{ to } v_2 \right).
\]

For example, let \( n = 4 \). There are three paths of length 4 from \( v_1 \) to \( v_2 \): 11112, 11212, and 12112; the sum of their weights is \( x^3 + 2x = f_4 \). There are two paths of length 4 from \( v_2 \) to itself: 21112 and 21212; the sum of their weights is \( x^2 + 1 = f_3 \). Then \( xf_4 + 2f_3 = x(x^3 + 2x) + 2(x^2 + 1) = x^4 + 4x^2 + 2 = l_4 \), as expected.

**Interesting Special Cases.** Clearly, the graph-theoretic model provides one for Fibonacci and Lucas numbers by letting \( x = 1 \) [6]; one for Pell and Pell-Lucas polynomials by replacing \( x \) with \( 2x \) and hence for Pell and Pell-Lucas numbers. But we need to choose the initial conditions appropriately when the length of the path is zero.

4. **Matrix \( Q(x) \) Revisited**

The weighted adjacency matrix of a weighted graph can be employed to compute the sum of the weights of paths of a given length \( n \) between any two vertices, as the next theorem shows. The proof follows by induction [5].

**Theorem 1.** Let \( A \) be the weighted adjacency matrix of a connected graph with vertices \( v_1, v_2, \ldots, v_k \), and \( n \) a positive integer. Then the \( ij \)th entry of the matrix \( A^n \) records the weights of paths of length \( n \) from \( v_i \) to \( v_j \). □

The next result follows by this theorem.

**Corollary 1.** The \( ij \)th entry of \( Q^n(x) \) gives the sum of the weights of paths of length \( n \) from \( v_i \) to \( v_j \), where \( 1 \leq i, j \leq 2 \). □

For example, we have \( Q^4(x) = \begin{bmatrix} f_5 & f_4 \\ f_4 & f_3 \end{bmatrix} \). So the sum of the weights of paths of length 4 from \( v_1 \) to itself is \( f_5 \); the sum of such paths from \( v_1 \) to \( v_2 \) is \( f_4 \), and also from \( v_2 \) to \( v_1 \); and the sum of such paths from \( v_2 \) to itself is \( f_3 \); see Table 1.

**Table 1**

Paths of Length 4.

<table>
<thead>
<tr>
<th>Paths from ( v_1 ) to ( v_1 )</th>
<th>Weight</th>
<th>Paths from ( v_1 ) to ( v_2 )</th>
<th>Weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>11111</td>
<td>( x^4 )</td>
<td>11112</td>
<td>( x^3 )</td>
</tr>
<tr>
<td>11121</td>
<td>( x^2 )</td>
<td>11212</td>
<td>( x )</td>
</tr>
<tr>
<td>11211</td>
<td>( x^2 )</td>
<td>12112</td>
<td>( x )</td>
</tr>
<tr>
<td>12111</td>
<td>( x^2 )</td>
<td>12121</td>
<td>1</td>
</tr>
<tr>
<td>Sum of the Weights</td>
<td>( x^4 + 3x^2 + 1 )</td>
<td>( x^3 + 2x )</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Paths from ( v_2 ) to ( v_1 )</th>
<th>Weight</th>
<th>Paths from ( v_2 ) to ( v_2 )</th>
<th>Weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>21111</td>
<td>( x^3 )</td>
<td>21112</td>
<td>( x^2 )</td>
</tr>
<tr>
<td>21121</td>
<td>( x )</td>
<td>21212</td>
<td>1</td>
</tr>
<tr>
<td>21211</td>
<td>( x )</td>
<td>21212</td>
<td>1</td>
</tr>
<tr>
<td>Sum of the Weights</td>
<td>( x^3 + 2x )</td>
<td>( x^2 + 1 )</td>
<td></td>
</tr>
</tbody>
</table>

The next result follows from this corollary.
Corollary 2. The $ij$th entry of $Q^n(1)$ records the number of paths of length $n$ from $v_i$ to $v_j$, where $1 \leq i, j \leq 2$.

Since $f_{n+1} + f_{n-1} = l_n$, the following result also follows from Corollary 1.

Corollary 3. The sum of the weights of all closed paths of length $n$ is $l_n$.

Interesting Observations. Notice that the eigenvalues of the matrix $Q(x)$ are $\alpha$ and $\beta$, and hence those of $Q^n(x)$ are $\alpha^n$ and $\beta^n$. Since the sum of the weights of all closed paths of length $n$ is $l_n$, it follows that this sum is indeed the sum of the eigenvalues of $Q^n(x)$. Since $l_n = f_{n+1} + f_{n-1}$, the sum also equals the trace of $Q^n(x)$.

5. Byproducts of the Model

To showcase the beauty of this approach, we will now confirm a few elegant properties of Fibonacci and Lucas polynomials. The essence of our technique lies in computing the sum of the weights of the same objects in two different ways, and then equating the two counts.

Example 1. Prove that $f_{2n} = f_n l_n$.

Proof. Consider the sum of the weights of paths of length $2n$ from $v_1$ to $v_2$. By Corollary 1, the sum is $f_{2n}$.

We will now count it in a different way. Such a path can land at $v_1$ or $v_2$ after $n$ steps. Suppose it stops at $v_1$ after $n$ steps: $v_1 \ldots \ldots v_1 \ldots \ldots v_2$. The sum of the weights of paths from $v_1$ to itself is $f_{n+1}$, and that from $v_1$ to $v_2$ is $f_n$. So, by the multiplication principle, the sum of the weights of paths from $v_1$ to $v_2$ that pass through $v_1$ after $n$ steps is $f_{n+1}f_n$.

On the other hand, suppose the path lands at $v_2$ after $n$ steps: $v_1 \ldots \ldots v_2 \ldots \ldots v_2$. The sum of the weights of paths from $v_1$ to $v_2$ is $f_n$, and that from $v_2$ to itself is $f_{n-1}$. So, again by the multiplication principle, the sum of the weights of paths from $v_1$ to $v_2$ that pass through $v_2$ after $n$ steps is $f_n f_{n-1}$.

Thus, by the addition principle, the sum of the weights of paths of length $2n$ from $v_1$ to $v_2$ is $f_{n+1}f_n + f_n f_{n-1} = f_n(f_{n+1} + f_{n-1}) = f_n l_n$.

Equating the two sums, we get the desired result.

For example, there are exactly $F_6 = 8$ paths of length 6 from $v_1$ to $v_2$:

1111112 1211112
11112112 1212112
11121112 11212112
11211112 12111212.

The sum of their weights is $x^5 + 4x^3 + 3x = f_6$.

Six of them land at $v_1$ after 3 steps (see the 1’s in boldface); and two at $v_2$ after 3 steps (see the 2’s in boldface).

Sum of their weights = $(x^5 + 3x^3 + 2x) + (x^3 + x)$

$= (x^3 + 2x)(x^2 + 1) + (x^2 + 1)x$

$= (x^2 + 1)(x^3 + 3x)$

$= f_3 l_3$.

Next we establish the Fibonacci addition formula.
Example 2. Prove that $f_{m+n} = f_{m+1}f_n + f_mf_{n-1}$.

Proof. We will compute in two different ways the sum of the weights of paths of length $m + n$ from $v_1$ to $v_2$. By Corollary 1, the sum of the weights of such paths is $f_{m+n}$.

Such a path can take us to $v_1$ or $v_2$ after $m$ steps. Suppose it lands at $v_1$ after $m$ steps: $v_1 \cdots v_1 \cdots v_2$. The sum of the weights of paths from $v_1$ to itself after $m$ steps is $f_{m+1}$, and that from $v_1$ to $v_2$ after $n$ steps is $f_n$. Consequently, the sum of the weights of paths of length $m + n$ from $v_1$ to $v_2$ that land at $v_1$ after $m$ steps is $f_{m+1}f_n$.

On the other hand, suppose the path takes us to $v_2$ after $n$ steps: $v_1 \cdots v_2 \cdots v_2$. The sum of the weights of paths of length $m + n$ from $v_1$ to $v_2$ that land at $v_2$ after $m$ steps is $f_{m+n-1}$.

The sum of the weights of paths of length $m + n$ from $v_1$ to $v_2$ that land at $v_2$ after $m$ steps is $f_{m}f_{n-1}$.

Combining the two cases, the sum of the weights of all such paths of length $m + n$ is $f_{m+1}f_n + f_mf_{n-1}$.

The addition formula follows by equating the two sums. \qed

We can employ the same technique to establish independently that $f_{n+1}^2 + f_n^2 = f_{2n+1}$ and $l_{m+n} = f_{m+1}l_n + f_ml_{n-1}$.

Example 3. Prove the Lucas formula $f_{n+1} = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} x^{n-2k}$.

Proof. This time, we will focus on the sum of the weights of closed paths of length $n$ originating at $v_1$. By Corollary 1, the sum is $f_{n+1}$.

Suppose such a path contains $k$ closed paths 121 of two edges, where $k \geq 0$; call them d-edges (“d” for “double”) for convenience. The $k$ d-edges account for $2k$ edges, so there are $n - 2k$ edges remaining in the path. Consequently, the total number of elements (edges or d-edges) is $(n - 2k) + k = n - k$. The $n - 2k$ edges contribute $x^{n-2k}$ and the $k$ d-edges $1^k$ to the weight of the path; so the weight of such a path is $x^{n-2k} \cdot 1^k = x^{n-2k}$.

The $k$ d-edges can be selected from the $n - k$ elements in $\binom{n-k}{k}$ ways, where $0 \leq 2k \leq n$. So the sum of the weights of all closed paths originating at $v_1$ equals $\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} x^{n-2k}$.

Equating the two sums yields the desired formula. \qed

For example, let $n = 5$. It follows from Table 2 that the sum of the weights of all closed paths originating at $v_1$ is $x^5 + 4x^3 + 3x = f_6$. (The d-edges are boldfaced or parenthesized in the table.)

<table>
<thead>
<tr>
<th>Number of d-edges $k$</th>
<th>Closed Paths of Length 5 with $k$ d-edges</th>
<th>Sum of the Weights of such Paths</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1111111</td>
<td>$x^5$</td>
</tr>
<tr>
<td>1</td>
<td>111121 111211 112111 121111 121111</td>
<td>$4x^3$</td>
</tr>
<tr>
<td>2</td>
<td>112(121) 121(121) 12(121)1</td>
<td>$3x$</td>
</tr>
</tbody>
</table>

Similarly, there is one closed path of length 4 starting at $v_1$ with no d-edges: 11111; three with one d-edge: 11121, 11211, and 12111; and one with two d-edges: 12(121). The sum of their weights is $x^4 + 3x^2 + 1 = f_5$. 

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The next identity expresses $f_{2n}$ in terms of the first $n$ Fibonacci polynomials.

**Example 4.** Prove the identity $f_{2n} = \sum_{k=1}^{n} \binom{n}{k} f_k x^k$.

*Proof.* Consider the closed paths of length $2n - 1$ originating at $v_1$. The sum of the weights of such paths is $f_{2n}$.

We will now compute this sum in a different way. Since $2n - 1$ is odd, each such path $P$ must contain an odd number of edges (loops) 11. The remaining edges must be d-edges. Since there can be a maximum of $n - 1$ d-edges, every path must contain at least $(n - 1) + 1 = n$ elements.

Suppose there are $k$ loops among the first $n$ elements of path $P$. The corresponding subpath $A$ contains $n - k$ d-edges; its length is $k + 2(n - k) = 2n - k$. The remaining subpath $B$ is of length $(2n - 1) - (2n - k) = k - 1$; so path $P$ is of the form subpath $A$ \( \begin{array} {l} \text{length} \ 2n-k \ \text{length} \ k-1 \end{array} \) subpath $B$.

The $k$ loops in subpath $A$ can be placed among the $n$ elements in \( \binom{n}{k} \) distinct ways. The sum of the weights of subpaths $B$ is $f_k$. Consequently, the sum of the weights of such paths $P$ is \( \binom{n}{k} f_k x^k \), where $1 \leq k \leq n$.

Thus the sum of the weights of all closed paths of length $2n - 1$ is $\sum_{k=1}^{n} \binom{n}{k} f_k x^k$.

The given identity now follows by equating the two sums. \( \square \)

We will now illustrate this combinatorial technique with $n = 3$. There are $8 = F_6$ closed paths of length 5 originating at $v_1$; see Table 3, where the first three edges of each path are boldfaced for convenience.

<table>
<thead>
<tr>
<th>$k$</th>
<th>Closed Paths</th>
<th>Sum of the Weights</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>111211 121211 121111</td>
<td>3x</td>
</tr>
<tr>
<td>2</td>
<td>111211 112111 121111</td>
<td>3x^3</td>
</tr>
<tr>
<td>3</td>
<td>111111 111111</td>
<td>$x^5 + x^3$</td>
</tr>
</tbody>
</table>

It follows from the table that the cumulative sum of the weights of all closed paths is $x^5 + 4x^3 + 3x = f_6$, as expected.

Next we will confirm the identity $f_{2n+2} = \sum_{i,j \geq 0} \binom{n-i}{j} \binom{n-j}{i} x^{2n-2i-2j+1}$ using the graph-theoretic model. We will accomplish this job using closed paths of length $2n + 1$ from $v_1$ to $v_1$, and d-edges. But, first we will make an important observation. Since every d-edge is of length 2, every such closed path must contain an odd number of loops 11. So there must be a special loop $M$ with an equal number of loops on either side. For convenience, we call $M$ the median loop.

**Example 5.** Confirm the identity $f_{2n+2} = \sum_{i,j \geq 0} \binom{n-i}{j} \binom{n-j}{i} x^{2n-2i-2j+1}$.

*Proof.* Consider the closed paths of length $2n + 1$ from $v_1$ to $v_1$. The sum of the weights of such paths is $f_{2n+2}$.

Now consider such an arbitrary path $P$. Let $M$ denote the median loop in it. Suppose there are $i$ d-edges to the left of $M$ and $j$ d-edges to its right: \( \begin{array} {c} \text{i \text{- edges}} \ \text{11} \ j \ \text{d \text{- edges}} \end{array} \) \( \begin{array} {c} \text{to left} \ \text{to right} \end{array} \). Then $P$ contains $2n + 1 - 2i - 2j$ loops. So there are $n - i - j$ loops on either side of $M$. Consequently,
there are \((n - i - j) + i = n - j\) edges to the left of \(M\), of which \(i\) are d-edges. The \(i\) d-edges can be placed among the \(n - j\) edges in \(\binom{n-j}{i}\) different ways. The weight of this subpath is \(\binom{n-j}{i}x^{n-i-j}\).

Similarly, the weight of the subpath to the right of \(M\) is \(\binom{n-i}{j}x^{n-i-j} \cdot x \cdot \binom{n-i}{j}x^{n-i-j} = \binom{n-i}{j} \binom{n-i}{j} x^{2n-2i-2j+1}\), where \(0 \leq i + j \leq n\).

Thus the sum of the weights of all closed paths \(P\) is \(\sum_{i,j \geq 0} \binom{n-i}{j} \binom{n-i}{j} x^{2n-2i-2j+1}\). Equating the two sums yields the given identity. \(\square\)

For example, Table 4 gives the closed paths of length 5 from \(v_1\) to itself, where we have identified the loops in boldface. The uparrows indicate the median loops, and the numbers below their locations.

Table 4

<table>
<thead>
<tr>
<th>111111</th>
<th>111121</th>
<th>111211</th>
<th>112111</th>
<th>121111</th>
<th>112121</th>
<th>121121</th>
<th>121211</th>
</tr>
</thead>
<tbody>
<tr>
<td>↑</td>
<td>↑</td>
<td>↑</td>
<td>↑</td>
<td>↑</td>
<td>↑</td>
<td>↑</td>
<td>↑</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>2</td>
<td>4</td>
<td>4</td>
<td>1</td>
<td>3</td>
<td>5</td>
</tr>
</tbody>
</table>

Table 5 shows the possible locations \(m\) of the median loops, corresponding value(s) of \(i\) and \(j\), and the weights of corresponding path(s). It follows from the table that the sum of the weights of all closed paths of length 5 from \(v_1\) to \(v_1\) is \(x^5 + 4x^3 + 3x = f_6 = \sum_{0 \leq i+j \leq 2} \binom{2-i}{j} \binom{2-j}{i} x^{5-2i-2j}\).

Table 5

<table>
<thead>
<tr>
<th>(m)</th>
<th>(i)</th>
<th>(j)</th>
<th>Sum(s) of the Weight(s) of Path(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>2</td>
<td>(x)</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>1</td>
<td>(2x^3)</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
<td>(x^5)</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>1</td>
<td>(x)</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>0</td>
<td>(2x^3)</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>0</td>
<td>(x)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Cumulative sum = (f_6)</td>
</tr>
</tbody>
</table>

In the next example, we will prove Cassini’s formula using the graph-theoretic model and induction.

**Example 6.** Prove that \(f_{n+1}f_{n-1} - f_n^2 = (-1)^n\).

**Proof.** Clearly, the formula works for \(n = 1\) and \(n = 2\). Suppose it is true for an arbitrary integer \(n \geq 2\).

Form two lists \(A\) and \(B\) of pairs of closed paths from \(v_1\) to \(v_1\). List \(A\) consists of such pairs \((v, w)\) of paths of length \(n - 1\) and \(n + 1\), respectively. List \(B\) consists of pairs \((x, y)\) of paths of the same length \(n\):
Clearly, the sum of the weights of the pairs \((v, w)\) is \(f_n f_{n+2}\) and that of the pairs \((x, y)\) is \(f_{n+1}^2\).

We will now establish a bijection between two suitable subsets of \(A\) and \(B\).

**Case 1.** Suppose \(w_2 = 1\). Then moving \(w_1 = 1\) to the beginning of \(v\) produces a pair \((x, y)\) of paths of length \(n\) each:

\[
\begin{align*}
1v_2 & \cdots v_{n-1} \\
1w_2w_3w_4 & \cdots w_{n+1}.
\end{align*}
\]

**Case 2.** On the other hand, suppose \(w_2 = 2\). Then shifting \(w_1\) to \(v\) does not generate a pair \((x, y)\) in \(B\). So such a pair \((v, w)\) in \(A\) does not have a matching pair \((x, y)\) in \(B\).

We will now count those non-matchable pairs in \(A\). When \(w_2 = 1, w_3 = 1\). So \(w = 121w_4 \cdots w_{n+1}\). The sum of the weights of such pairs is \(f_n\). So the sum of the weights of such pairs \((v, w)\) is \(f_{n+1}^2\); no such pairs have matching elements \((x, y)\) in \(B\). Consequently, the sum of the weights of the pairs in \(A\) that have matching elements in \(B\) equals \(f_n f_{n+2} - f_n^2\).

Let us now reverse the order. Shift \(x_1 = 1\) from \(x\) to the beginning of \(y\).

**Case 1.** Suppose \(x_2 = 1\). Then \(x = 1x_3 \cdots x_{n-1}\) and \(y = 1y_2 \cdots y_{n+1}\). The corresponding pair \((x, y)\) is a valid element in \(A\).

**Case 2.** Suppose \(x_2 = 2\). Then \(x_3 = 1\); and \(x = 21 \cdots x_{n-1}\) and \(y = 1y_2 \cdots y_{n+1}\). The corresponding pair \((x, y)\) does not generate a matching element in \(A\). The sum of the weights of such unmatchable pairs equals \(f_n f_{n+1}\); this equals \(f_{n+1}^2 + (-1)^n\), by the inductive hypothesis.

Consequently, the sum of the weights of the pairs \((x, y)\) that have matchable counterparts in \(A\) equals \(f_{n+1}^2 - [f_n^2 + (-1)^n]\).

Since the matching between the two sets of matchable pairs is bijective, the sums of their weights must be equal; that is, \(f_n f_{n+2} - f_n^2 = f_{n+1}^2 - [f_n^2 + (-1)^n]\). This implies that \(f_n f_{n+2} - f_{n+1}^2 = (-1)^n\). So Cassini’s formula works for \(n + 1\) also. Thus, by induction, it works for every \(n \geq 1\).

We will now illustrate the essence of the proof for the case \(n = 3\). Table 6 lists the pairs \((v, w)\) of paths \(v = v_1v_2\) and \(w = w_1w_2w_3w_4\) from \(v_1\) to \(v_1\). Six of them are numbered 1 through 6 for convenience; the sum of the weights of these pairs is \(x^2(x^4 + 2x^2) + 1 \cdot (x^4 + 2x^2) = (x^2 + 1)(x^4 + 2x^2) = f_3 f_5 - f_3^2\). The others are labeled \(a\) through \(d\); the sum of the weights of these four pairs equals \(x^2(x^2 + 1) + 1 \cdot (x^2 + 1) = (x^2 + 1)^2\). The grand total is \((x^2 + 1)(x^4 + 3x^2 + 1) = f_3 f_5\).

<table>
<thead>
<tr>
<th>v</th>
<th>w</th>
</tr>
</thead>
<tbody>
<tr>
<td>111</td>
<td>111</td>
</tr>
<tr>
<td>11111</td>
<td>11121</td>
</tr>
<tr>
<td>121</td>
<td>121</td>
</tr>
<tr>
<td>12121</td>
<td>12111</td>
</tr>
<tr>
<td>121</td>
<td>121</td>
</tr>
<tr>
<td>12111</td>
<td>1211</td>
</tr>
<tr>
<td>12121</td>
<td>12121</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
</tr>
</tbody>
</table>

Table 6

Table 7 shows the pairs \((x, y)\) of closed paths \(x = x_1x_2x_3\) and \(y = y_1y_2y_3\) from \(v_1\) to itself. Again, six of them are labeled 1 through 6; and the sum of their weights is \((x^3 + x)(x^3 + 2x) =

\[
142
\]
Example 7. Establish the identity 

\[ f_n^2 = [f_n^2 + (-1)^2] \]. The sum of the weights of the remaining three, labeled α, β, and γ, is \( x(x^3 + 2x) \). Their grand total is \( (x^3 + 2x)^2 = f_n^2 \).

<table>
<thead>
<tr>
<th>Table 7</th>
</tr>
</thead>
<tbody>
<tr>
<td>1111 1111 1111 1211 1121 1121 1211 1211 1211 1211</td>
</tr>
<tr>
<td>1 2 3 4 5 6 α β γ</td>
</tr>
</tbody>
</table>

The matchable pairs from both lists are numbered 1 through 6, and the others by letters. Both sets have the same sum of weights: \( (x^2 + 1)(x^4 + 2x^2) = (x^3 + x)(x^3 + 2x) \), as expected.

Next we will prove the identity \( l_{n+1} + l_{n-1} = (x^2 + 4)f_n \). We will achieve this task by establishing a one-to-five correspondence from the set of paths of length \( n \) from \( v_1 \) to \( v_2 \) to that of closed paths of length \( n + 1 \) or \( n - 1 \).

**Example 7.** Establish the identity \( l_{n+1} + l_{n-1} = (x^2 + 4)f_n \).

**Proof.** By Corollary 3, the sum of the weights of closed paths of length \( n + 1 \) is \( l_{n+1} \), and that of length \( n - 1 \) is \( l_{n-1} \). So the sum of the weights of paths of length \( n + 1 \) or \( n - 1 \) is \( l_{n+1} + l_{n-1} \).

By Corollary 1, the sum of the weights of paths of length \( n \) from \( v_1 \) to \( v_2 \) is \( f_n \). Let \( w = w_1w_2 \cdots w_{n+1} \) be such a path. Clearly, \( w_1 = 1 \) and \( w_{n+1} = 2 \). In addition, every \( v_2 \) must be preceded by \( v_1 \); so \( w_n = 1 \). Thus \( w = \overbrace{1w_2 \cdots w_{n-1}12}^{\text{length } n} \).

We will now devise an algorithm in five steps to establish the aforementioned correspondence.

**Step 1.** Append a 1 at the end of \( w \). This generates a closed path of length \( n + 1 \) from \( v_1 \) to \( v_1: \overbrace{1w_2 \cdots w_{n-1}121}^{\text{length } n+1} \). The sum of the weights of such paths equals \( f_n \).

**Step 2.** Delete \( w_{n+1} = 2 \). This results in a closed path of length \( n - 1 \) from \( v_1 \) to itself: \( \overbrace{1w_2 \cdots w_{n-1}11}^{\text{length } n-1} \). The sum of the weights of such paths also equals \( f_n \).

**Step 3.** Place a 2 at the beginning of \( w \). This creates a closed path of length \( n + 1 \) from \( v_2 \) to \( v_2: \overbrace{21w_2 \cdots w_{n-1}12}^{\text{length } n+1} \). The sum of the weights of such paths is again \( f_n \).

**Step 4.** Replace \( w_{n+1} = 2 \) with 11. This operation produces a closed path of length \( n + 1 \) from \( v_1 \) to itself: \( \overbrace{1w_2 \cdots w_{n-1}111}^{\text{length } n+1} \). The sum of the weights of such paths is \( x^2f_n \).

These four steps do not account for all closed paths of length \( n + 1 \) or \( n - 1 \), namely, the ones that begin with \( w_1w_2 = 11 \) or 12. This takes us to Step 5, which has therefore two parts.

**Step 5A.** Suppose \( w_2 = 1 \). Then delete \( w_1 \) and insert 11 at the end of \( w \). This generates a closed path of length \( n + 1 \) from \( v_1 \) to itself: \( \overbrace{w_2 \cdots w_{n-1}1211}^{\text{length } n+1} \). The sum of the weights of such paths is \( xf_{n-1} \).

**Step 5B.** Suppose \( w_2 = 2 \). Then delete \( w_1 \). This gives a closed path of length \( n - 1 \) from \( v_2 \).
to itself: \( w_2 \cdots w_{n+1} \). Such paths contribute \( f_{n-2} \) to the sum of the weights.

The sum of the weights of closed paths generated by Step 5 equals \( xf_{n-1} + f_{n-2} = f_n \).

Steps 1–5 do not produce duplicate paths. So the sum of the weights of closed paths they create is \((x^2 + 4)f_n\); some are of length \( n + 1 \) and the rest of length \( n - 1 \). Thus the two sums of weights must be equal; that is, \( l_{n+1} + l_{n-1} = (x^2 + 4)f_n \), as desired.

We will now illustrate the five steps of the algorithm for the case \( n = 4 \). There are 11 closed paths of length 5 and 4 of length three, a total of 15 closed paths of length 5 or 3:

Length 5:
- 111111
- 111211
- 112111
- 121111
- 112121
- 121121
- 121211

Length 4:
- 211112
- 211212
- 212112

Length 3:
- 1111
- 1121
- 1211
- 2112

The sum of the weights of paths of length 5 is \( l_5 = x^5 + 5x^3 + 5x \), and that of paths of length 3 is \( l_3 = x^3 + 3x \). So the sum for paths of length 5 or 3 is \( x^5 + 6x^3 + 8x \).

Now consider paths of length 4 from \( v_1 \) to \( v_2 \). There are 3 such paths \( w = w_1w_2w_3w_4w_5 \):
- 11112
- 11212
- 12112

The sum of their weights is \( f_4 \).

1) Appending a 1 at the end of \( w \) yields 3 closed paths length 5 from \( v_1 \) to \( v_1 \): 111121, 111211, 121121. The sum of their weights is \( f_4 \).

2) Deleting \( w_5 = 2 \) produces 3 closed paths length three from \( v_1 \) to \( v_1 \): 1111, 1121, 1211. The sum of their weights is also \( f_4 \).

3) Placing a 2 at the beginning of \( w \) gives three closed paths length 5 from \( v_2 \) to \( v_2 \): 211112, 211212, 212112. The sum of their weights is once again \( f_4 \).

4) Replacing \( w_5 \) with 11 generates three more closed paths of length 5 from \( v_1 \) to \( v_1 \):
- 111211
- 121121
- 211121

The sum of their weights is \( x^2f_4 \).

5a) When \( w_2 = 1 \), delete \( w_1 \) and append 11 at the end. This yields two closed paths of length 5:
- 111211
- 121121

The sum of their weights equals \( x^3 + x \).

5b) When \( w_2 = 2 \), we delete \( w_1 \). This creates exactly one closed path of length 3: 2112. Its weight is \( x \).

So the sum of the weights of paths generated by Step 5 equals \( (x^3 + x) + x = f_4 \).

Thus, by Steps 1–5, the sum of the weights of all closed paths of length 5 or 3 equals \((x^2 + 4)f_4 = x^5 + 6x^3 + 8x = l_5 + l_3\), as expected.

Interestingly, there is a bijection between the set of closed paths of length \( n \) from \( v_1 \) to \( v_1 \), and that of Fibonacci tilings of a \( 1 \times n \) board. We will let the d-edges do the work for us.

6. A Bijection Algorithm

In the Fibonacci tilings of a \( 1 \times n \) board with squares and dominoes, we assign a weight \( x \) to each square and 1 to each domino [8]. Now consider a closed path of length \( n \) from \( v_1 \) to itself. Replacing the edge 11 with a square and a d-edge with a domino results in a tiling of the board. Clearly, this process is reversible. Consequently, this algorithm establishes the desired bijection.

To illustrate the bijection algorithm, consider the eight closed paths of length 5 starting at \( v_1 \) in Table 2. The algorithm produces the corresponding Fibonacci tilings of a \( 1 \times 5 \) board with squares and dominoes; they are showcased in Table 8. As expected, the sum of the weights of the paths or tilings is \( x^5 + 4x^3 + 3x = f_6 \).
7. Fibonacci and Lucas Sums

We can elicit the power of matrices and the graph-theoretic model to interpret combinatorially the following summation formulas:

1) \( x \sum_{k=1}^{n} f_k = f_{n+1} + f_n - 1 \)

2) \( x \sum_{k=1}^{n} l_k = l_{n+1} + l_n + x - 2 \)

3) \( x \sum_{k=1}^{n} f_{2k-1} = f_{2n} \)

4) \( x \sum_{k=1}^{n} l_{2k-1} = l_{2n} - 2 \)

5) \( x \sum_{k=1}^{n} f_{2k} = f_{2n+1} \)

6) \( x \sum_{k=1}^{n} l_{2k} = l_{2n+1} - x \).

In the interest of brevity, we will interpret formula (1) and leave the others for Fibonacci enthusiasts.

To this end, first we will make an important useful observation about the \( Q \)-matrix:

\[
\sum_{k=1}^{n} x Q^k(x) = \begin{bmatrix} \sum_{k=1}^{n} f_{k+1} & \sum_{k=1}^{n} f_k \\ \sum_{k=1}^{n} f_{k} & \sum_{k=1}^{n} f_{k-1} \end{bmatrix}. \tag{1}
\]

We will need the following result also [5].

**Theorem 2.** Let \( A \) be the weighted adjacency matrix of a weighted graph with vertices \( v_1, v_2, \ldots, v_k \), and \( n \) a positive integer. Then the \( ij \)-th entry of the matrix \( A + A^2 + \cdots + A^n \) gives the sum of the weights of the paths of length \( \leq n \) from vertex \( v_i \) to \( v_j \). \( \square \)

With these two results at our finger tips, we are ready for the interpretation.

1) It follows from equation (1) that \( x \sum_{k=1}^{n} f_k \) represents \( x \) times the sum of the weights of paths of length \( \leq n \) from \( v_1 \) to \( v_2 \). It equals \( f_{n+1} + f_n - 1 \).
For example, consider the paths of length $\leq 5$ from $v_1$ to $v_2$; see Table 9. We then have

$$x \sum_{k=1}^{5} f_k = x^5 + x^4 + 4x^3 + 3x^2 + 3x$$

$$= (x^5 + 4x^3 + 3x) + (x^4 + 3x^2 + 1) - 1$$

$$= f_6 + f_5 - 1.$$

Table 9

<table>
<thead>
<tr>
<th>$n$</th>
<th>Paths of Length $n$ from $v_1$ to $v_2$</th>
<th>Sum of the Weights</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>12</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>112</td>
<td>$x$</td>
</tr>
<tr>
<td>3</td>
<td>1112 1212</td>
<td>$x^2 + 1$</td>
</tr>
<tr>
<td>4</td>
<td>111112 11212 12112</td>
<td>$x^3 + 2x$</td>
</tr>
<tr>
<td>5</td>
<td>1111112 111212 11212 12112 121212</td>
<td>$x^4 + 3x^2 + 1$</td>
</tr>
</tbody>
</table>

Acknowledgment

The author would like to thank the reviewer for his/her thoughtful comments and suggestions for improving the original version.

References


MSC2010: 05A19, 11B39, 11Cxx

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