GRAPH-THEORETIC MODELS FOR THE UNIVARIATE FIBONACCI FAMILY

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Abstract. We construct graph-theoretic models for an extended univariate Fibonacci family, which includes Fibonacci, Lucas, Pell, and Pell-Lucas polynomials.

1. Introduction

It is well-known that Fibonacci, Lucas, Pell, and Pell-Lucas numbers can be studied combinatorially using tilings of linear and circular boards [1, 2, 7]. This combinatorial approach manifests the beauty in establishing some of their elegant properties. In this article, we employ different tools to explore the corresponding univariate polynomials, namely, graph-theoretic models.

The celebrated *Fibonacci polynomials* $f_n(x)$ were originally studied by Eugene Charles Catalan (1814–1894) in 1883, and the *Lucas polynomials* $l_n(x)$ by Marjorie Bicknell-Johnson in 1970. They both satisfy the same polynomial recurrence $g_n(x) = xg_{n-1}(x) + g_{n-2}(x)$, where $n \geq 2$. When $g_0(x) = 0$ and $g_1(x) = 1$, $g_n(x) = f_n(x)$; and when $g_0(x) = 2$ and $g_1(x) = x$, $g_n(x) = l_n(x)$. Clearly, $f_n(1) = F_n$ and $l_n(1) = L_n$.

Fibonacci and Lucas polynomials can also be defined by the Binet-like formulas

$$f_n(x) = \frac{\alpha^n - \beta^n}{\alpha - \beta} \text{ and } l_n(x) = \alpha^n + \beta^n,$$

where $\alpha = \alpha(x)$ and $\beta = \beta(x)$ are the solutions of the equation $t^2 - xt - 1 = 0$.

Pell polynomials $p_n(x)$ and *Pell-Lucas polynomials* $q_n(x)$ are defined by $p_n(x) = f_n(2x)$ and $q_n(x) = l_n(2x)$, respectively. The *Pell numbers* P_n and *Pell-Lucas numbers* Q_n are given by $P_n = p_n(1)$ and $Q_n = \frac{1}{2}q_n(1)$, respectively.

In the interest of brevity, we will delete the argument from the functional notation; so g_n will mean $g_n(x)$.

2. Q-Matrix and Connected Graph

To construct graph-theoretic models for the univariate Fibonacci family, we introduce a 2×2 matrix, called the *Q-matrix*:

$$Q(x) = (q_{ij})_{2\times 2} = \begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix}.$$

(In 1960, C. H. King studied the special case $Q(1)$ to extract some Fibonacci delights [4].) Interestingly, it can be translated into a connected graph G with two vertices v_1 and v_2, and three edges. The edge from v_i to v_j is denoted by v_i-v_j, i-j, or by the "word" ij when there is *no* confusion. We define the *weight* w_{ij} of edge v_i-v_j to be q_{ij}, where $1 \leq i \leq j \leq 2$; see Figure 1. Since a weight is assigned to each edge, G is a *weighted graph* and $Q(x)$ is its *weighted adjacency matrix*.

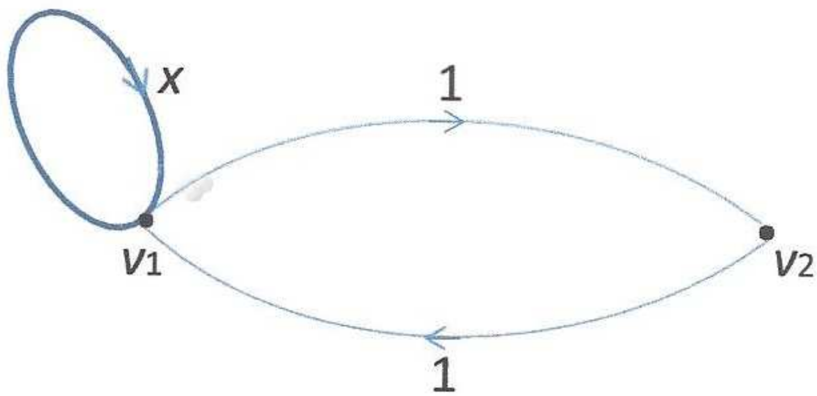

Figure 1.

Next we make a few more graph-theoretic definitions for clarity.

3. Weighted Paths

A *path* from vertex v_i to vertex v_j in a connected graph is a sequence $v_i - e_i - v_{i+1} - \cdots - v_{j-1} - e_j - v_j$ of vertices v_k and edges e_k, where edge e_k is incident with vertices v_k and v_{k+1}. The path is *closed* if its endpoints are the same; otherwise, it is *open*. The *length* ℓ of a path is the number of edges in the path; that is, it takes ℓ steps to reach one endpoint of the path from the other. The *weight* of a path is the product of the weights of the edges along the path. (Note that this definition is *different* from that of a path in graph theory.) For example, the weight of the path 1112 is $x \cdot x \cdot 1 = x^2$.

The Q-matrix has the property that

$$Q^n(x) = \begin{bmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{bmatrix},$$

where $n \geq 1$. Consequently, we can give a nice graph-theoretic interpretation of the recurrence $f_{n+1} = xf_n + f_{n-1}$:

$$\begin{pmatrix} \text{sum of the weights of} \\ \text{closed paths of length} \\ n \text{ from } v_1 \text{ to } v_1 \end{pmatrix} = x \begin{pmatrix} \text{sum of the weights of} \\ \text{paths of length } n \\ \text{from } v_1 \text{ to } v_2 \end{pmatrix} + \begin{pmatrix} \text{sum of the weights of} \\ \text{closed paths of length} \\ n \text{ from } v_2 \text{ to } v_2 \end{pmatrix}.$$

The Lucas polynomial l_n also can be interpreted using this model. The sum of the weights of closed paths of length n originating at v_1 is f_{n+1}, and that of closed paths of the same length originating at v_2 is f_{n-1}; so the sum of the weights of closed paths of length n is $f_{n+1} + f_{n-1} = l_n$.

For example, consider the closed paths of length 4:

Paths originating at v_1: 11111 11121 11211 12111 12121
Sum of their weights: $x^4 + 3x^2 + 1 = f_5$
Paths originating at v_2: 21112 21212
Sum of their weights: $x^2 + 1 = f_3$
Cumulative Sum: $x^4 + 4x^2 + 2 = l_4$.

Also, since $l_n = xf_n + 2f_{n-1}$, l_n can be interpreted differently:

$$l_n = x\begin{pmatrix}\text{sum of the weights of paths}\\ \text{of length } n \text{ from } v_1 \text{ to } v_2\end{pmatrix} + 2\begin{pmatrix}\text{sum of the weights of closed}\\ \text{paths of length } n \text{ from } v_2 \text{ to } v_2\end{pmatrix}.$$

For example, let $n = 4$. There are three paths of length 4 from v_1 to v_2: 11112, 11212, and 12112; the sum of their weights is $x^3 + 2x = f_4$. There are two paths of length 4 from v_2 to itself: 21112 and 21212; the sum of their weights is $x^2 + 1 = f_3$. Then $xf_4 + 2f_3 = x(x^3 + 2x) + 2(x^2 + 1) = x^4 + 4x^2 + 2 = l_4$, as expected.

Interesting Special Cases. Clearly, the graph-theoretic model provides one for Fibonacci and Lucas numbers by letting $x = 1$ [6]; one for Pell and Pell-Lucas polynomials by replacing x with $2x$ and hence for Pell and Pell-Lucas numbers. But we need to choose the initial conditions appropriately when the length of the path is zero.

4. Matrix $Q(x)$ Revisited

The weighted adjacency matrix of a weighted graph can be employed to compute the sum of the weights of paths of a given length n between any two vertices, as the next theorem shows. The proof follows by induction [5].

Theorem 1. *Let A be the weighted adjacency matrix of a connected graph with vertices $v_1, v_2, \cdots, v_k$, and n a positive integer. Then the ijth entry of the matrix A^n records the weights of paths of length n from v_i to v_j.* □

The next result follows by this theorem.

Corollary 1. *The ijth entry of $Q^n(x)$ gives the sum of the weights of paths of length n from v_i to v_j, where $1 \le i, j \le 2$.* □

For example, we have $Q^4(x) = \begin{bmatrix} f_5 & f_4 \\ f_4 & f_3 \end{bmatrix}$. So the sum of the weights of paths of length 4 from v_1 to itself is f_5; the sum of such paths from v_1 to v_2 is f_4, and also from v_2 to v_1; and the sum of such paths from v_2 to itself is f_3; see Table 1.

Table 1
Paths of Length 4.

	Paths from v_1 to v_1	Weight	Paths from v_1 to v_2	Weight
	11111	x^4	11112	x^3
	11121	x^2	11212	x
	11211	x^2	12112	x
	12111	x^2		
	12121	1		
Sum of the Weights		$x^4 + 3x^2 + 1$		$x^3 + 2x$
	Paths from v_2 to v_1	Weight	Paths from v_2 to v_2	Weight
	21111	x^3	21112	x^2
	21121	x	21212	1
	21211	x		
Sum of the Weights		$x^3 + 2x$		$x^2 + 1$

The next result follows from this corollary.

Corollary 2. *The ijth entry of $Q^n(1)$ records the number of paths of length n from v_i to v_j, where $1 \le i,\ j \le 2$.* □

Since $f_{n+1} + f_{n-1} = l_n$, the following result also follows from Corollary 1.

Corollary 3. *The sum of the weights of all closed paths of length n is l_n.* □

Interesting Observations. Notice that the *eigenvalues* of the matrix $Q(x)$ are α and β, and hence those of $Q^n(x)$ are α^n and β^n. Since the sum of the weights of all closed paths of length n is l_n, it follows that this sum is indeed the sum of the eigenvalues of $Q^n(x)$. Since $l_n = f_{n+1} + f_{n-1}$, the sum also equals the *trace* of $Q^n(x)$.

5. Byproducts of the Model

To showcase the beauty of this approach, we will now confirm a few elegant properties of Fibonacci and Lucas polynomials. The essence of our technique lies in computing the sum of the weights of the same objects in two different ways, and then equating the two counts.

Example 1. *Prove that $f_{2n} = f_n l_n$.*

Proof. Consider the sum of the weights of paths of length $2n$ from v_1 to v_2. By Corollary 1, the sum is f_{2n}.

We will now count it in a different way. Such a path can land at v_1 or v_2 after n steps. Suppose it stops at v_1 after n steps: $v_1 \underbrace{\text{— . . . . —}}_{n \text{ steps}} v_1 \underbrace{\text{— . . . . —}}_{n \text{ steps}} v_2$. The sum of the weights of paths from v_1 to itself is f_{n+1}, and that from v_1 to v_2 is f_n. So, by the multiplication principle, the sum of the weights of paths from v_1 to v_2 that pass through v_1 after n steps is $f_{n+1}f_n$.

On the other hand, suppose the path lands at v_2 after n steps: $v_1 \underbrace{\text{— . . . . —}}_{n \text{ steps}} v_2 \underbrace{\text{— . . . . —}}_{n \text{ steps}} v_2$. The sum of the weights of paths from v_1 to v_2 is f_n, and that from v_2 to itself is f_{n-1}. So, again by the multiplication principle, the sum of the weights of paths from v_1 to v_2 that pass through v_2 after n steps is $f_n f_{n-1}$.

Thus, by the addition principle, the sum of the weights of paths of length $2n$ from v_1 to v_2 is $f_{n+1}f_n + f_n f_{n-1} = f_n(f_{n+1} + f_{n-1}) = f_n l_n$.

Equating the two sums, we get the desired result. □

For example, there are exactly $F_6 = 8$ paths of length 6 from v_1 to v_2:

111**1**112	121**1**112
111**1**212	121**2**112
111**2**112	112**1**212
112**1**112	121**1**212.

The sum of their weights is $x^5 + 4x^3 + 3x = f_6$.

Six of them land at v_1 after 3 steps (see the **1**'s in boldface); and two at v_2 after 3 steps (see the **2**'s in boldface).

$$\begin{aligned}
\text{Sum of their weights} &= (x^5 + 3x^3 + 2x) + (x^3 + x) \\
&= (x^3 + 2x)(x^2 + 1) + (x^2 + 1)x \\
&= (x^2 + 1)(x^3 + 3x) \\
&= f_3 l_3.
\end{aligned}$$

Next we establish the Fibonacci addition formula.

Example 2. *Prove that* $f_{m+n} = f_{m+1}f_n + f_m f_{n-1}$.

Proof. We will compute in two different ways the sum of the weights of paths of length $m+n$ from v_1 to v_2. By Corollary 1, the sum of the weights of such paths is f_{m+n}.

Such a path can take us to v_1 or v_2 after m steps. Suppose it lands at v_1 after m steps: $v_1 \underbrace{- \cdot \cdot \cdot -}_{m \text{ steps}} v_1 \underbrace{- \cdot \cdot \cdot -}_{n \text{ steps}} v_2$. The sum of the weights of paths from v_1 to itself after m steps is f_{m+1}, and that from v_1 to v_2 after n steps is f_n. Consequently, the sum of the weights of paths of length $m+n$ from v_1 to v_2 that land at v_1 after m steps is $f_{m+1}f_n$.

On the other hand, suppose the path takes us to v_2 after m steps: $v_1 \underbrace{- \cdot \cdot \cdot -}_{m \text{ steps}} v_2 \underbrace{- \cdot \cdot \cdot -}_{n \text{ steps}} v_2$. The sum of the weights of paths of length $m+n$ from v_1 to v_2 that land at v_2 after m steps is $f_m f_{n-1}$.

Combining the two cases, the sum of the weights of all such paths of length $m+n$ is $f_{m+1}f_n + f_m f_{n-1}$.

The addition formula follows by equating the two sums. □

We can employ the same technique to establish independently that $f_{n+1}^2 + f_n^2 = f_{2n+1}$ and $l_{m+n} = f_{m+1}l_n + f_m l_{n-1}$.

Example 3. *Prove the Lucas formula* $f_{n+1} = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} x^{n-2k}$.

Proof. This time, we will focus on the sum of the weights of closed paths of length n originating at v_1. By Corollary 1, the sum is f_{n+1}.

Suppose such a path contains k closed paths 121 of two edges, where $k \geq 0$; call them d-edges ("d" for "double") for convenience. The k d-edges account for $2k$ edges, so there are $n-2k$ edges remaining in the path. Consequently, the total number of *elements* (edges or d-edges) is $(n-2k)+k = n-k$. The $n-2k$ edges contribute x^{n-2k} and the k d-edges 1^k to the weight of the path; so the weight of such a path is $x^{n-2k} \cdot 1^k = x^{n-2k}$.

The k d-edges can be selected from the $n-k$ elements in $\binom{n-k}{k}$ ways, where $0 \leq 2k \leq n$. So the sum of the weights of all closed paths originating at v_1 equals $\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} x^{n-2k}$.

Equating the two sums yields the desired formula. □

For example, let $n = 5$. It follows from Table 2 that the sum of the weights of all closed paths originating at v_1 is $x^5 + 4x^3 + 3x = f_6$. (The d-edges are boldfaced or parenthesized in the table.)

Table 2
Closed Paths of Length 5 from v_1 to v_1.

Number of d-edges k	Closed Paths of Length 5 with k d-edges	Sum of the Weights of such Paths
0	111111	x^5
1	111**121** 11**121**1 1**121**11 **121**111	$4x^3$
2	1**12**(121) **121**(121) **12**(**1**21)1	$3x$

Similarly, there is one closed path of length 4 starting at v_1 with no d-edges: 11111; three with one d-edge: 11**121**, 1**121**1, and **121**11; and one with two d-edges: **12**(121). The sum of their weights is $x^4 + 3x^2 + 1 = f_5$.

The next identity expresses f_{2n} in terms of the first n Fibonacci polynomials.

Example 4. *Prove the identity* $f_{2n} = \sum_{k=1}^{n} \binom{n}{k} f_k x^k$.

Proof. Consider the closed paths of length $2n-1$ originating at v_1. The sum of the weights of such paths is f_{2n}.

We will now compute this sum in a different way. Since $2n-1$ is odd, each such path P must contain an odd number of edges (loops) 11. The remaining edges must be d-edges. Since there can be a maximum of $n-1$ d-edges, every path must contain at least $(n-1)+1 = n$ elements.

Suppose there are k loops among the first n elements of path P. The corresponding subpath A contains $n-k$ d-edges; its length is $k + 2(n-k) = 2n-k$. The remaining subpath B is of length $(2n-1)-(2n-k) = k-1$; so path P is of the form $\underbrace{\text{subpath A}}_{\text{lenth } 2n-k}\ \underbrace{\text{subpath } B}_{\text{lenth } k-1}$.

The k loops in subpath A can be placed among the n elements in $\binom{n}{k}$ distinct ways. The sum of the weights of subpaths B is f_k. Consequently, the sum of the weights of such paths P is $\binom{n}{k} f_k x^k$, where $1 \le k \le n$.

Thus the sum of the weights of all closed paths of length $2n-1$ is $\sum_{k=1}^{n} \binom{n}{k} f_k x^k$.

The given identity now follows by equating the two sums. □

We will now illustrate this combinatorial technique with $n = 3$. There are $8 = F_6$ closed paths of length 5 originating at v_1; see Table 3, where the first three edges of each path are boldfaced for convenience.

Table 3

k	Closed Paths	Sum of the Weights
1	**112121** **121211** **121121**	$3x$
2	**11121**1 **11211**1 **12111**1	$3x^3$
3	**111**11 **111**21	$x^5 + x^3$

It follows from the table that the cumulative sum of the weights of all closed paths is $x^5 + 4x^3 + 3x = f_6$, as expected.

Next we will confirm the identity $f_{2n+2} = \sum_{i,j\ge 0} \binom{n-i}{j}\binom{n-j}{i} x^{2n-2i-2j+1}$ using the graph-theoretic model. We will accomplish this job using closed paths of length $2n+1$ from v_1 to v_1, and d-edges. But, first we will make an important observation. Since every d-edge is of length 2, every such closed path must contain an odd number of loops 11. So there must be a special loop M with an equal number of loops on either side. For convenience, we call M the *median loop*.

Example 5. *Confirm the identity* $f_{2n+2} = \sum_{i,j\ge 0} \binom{n-i}{j}\binom{n-j}{i} x^{2n-2i-2j+1}$.

Proof. Consider the closed paths of length $2n+1$ from v_1 to v_1. The sum of the weights of such paths is f_{2n+2}.

Now consider such an arbitrary path P. Let M denote the median loop in it. Suppose there are i d-edges to the left of M and j d-edges to its right: $\underbrace{i \text{ d}-\text{edges}}_{\text{to left}} 11 \underbrace{j \text{ d}-\text{edges}}_{\text{to right}}$. Then P contains $2n+1-2i-2j$ loops. So there are $n-i-j$ loops on either side of M. Consequently, there are $(n-i-j)+i = n-j$ edges to the left of M, of which i are d-edges. The i d-edges can be placed among the $n-j$ edges in $\binom{n-j}{i}$ different ways. The weight of this subpath is $\binom{n-j}{i}x^{n-i-j}$.

Similarly, the weight of the subpath to the right of M is $\binom{n-i}{j}x^{n-i-j}$. So the weight of path P equals $\binom{n-j}{i}x^{n-i-j} \cdot x \cdot \binom{n-i}{j}x^{n-i-j} = \binom{n-i}{j}\binom{n-j}{i}x^{2n-2i-2j+1}$, where $0 \le i+j \le n$.

Thus the sum of the weights of all closed paths P is $\sum\limits_{i,j\ge 0}\binom{n-i}{j}\binom{n-j}{i}x^{2n-2i-2j+1}$. Equating the two sums yields the given identity. $\square$

For example, Table 4 gives the closed paths of length 5 from v_1 to itself, where we have identified the loops in boldface. The uparrows indicate the median loops, and the numbers below their locations.

Table 4

111111	**1111**21	**111**2**11**	**11**2**111**	12**1111**	**11**2121	12**11**21	1212**11**
$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$
3	2	2	4	4	1	3	5

Table 5 shows the possible locations m of the median loops, corresponding value(s) of i and j, and the weights of corresponding path(s). It follows from the table that the sum of the weights of all closed paths of length 5 from v_1 to v_1 is $x^5+4x^3+3x = f_6 = \sum\limits_{0\le i+j\le 2}\binom{2-i}{j}\binom{2-j}{i}x^{5-2i-2j}$.

Table 5

m	i	j	Sum(s) of the Weight(s) of Path(s)
1	0	2	x
2	0	1	$2x^3$
3	0	0	x^5
	1	1	x
4	1	0	$2x^3$
5	2	0	x

$\uparrow$

Cumulative sum $= f_6$

In the next example, we will prove Cassini's formula using the graph-theoretic model and induction.

Example 6. *Prove that* $f_{n+1}f_{n-1} - f_n^2 = (-1)^n$.

Proof. Clearly, the formula works for $n = 1$ and $n = 2$. Suppose it is true for an arbitrary integer $n \ge 2$.

Form two lists A and B of pairs of closed paths from v_1 to v_1. List A consists of such pairs (v, w) of paths of length $n-1$ and $n+1$, respectively. List B consists of pairs (x, y) of paths of the same length n:

List A	List B
$v:\ 1v_2\cdots v_{n-1}1$	$x:\ 1x_2x_3\cdots x_n1$
$w:\ 1w_2w_3w_4\cdots w_{n+1}1$	$y:\ 1y_2\,y_3\cdots\, y_n1$

Clearly, the sum of the weights of the pairs (v,w) is f_nf_{n+2} and that of the pairs (x,y) is f_{n+1}^2.

We will now establish a bijection between two suitable subsets of A and B.

Case 1. Suppose $w_2 = 1$. Then moving $w_1 = 1$ to the beginning of v produces a pair (x,y) of paths of length n each:

$$\begin{gathered} 11v_2\cdots v_{n-1}1 \\ 1w_3w_4\cdots w_{n+1}1. \end{gathered}$$

Case 2. On the other hand, suppose $w_2 = 2$. Then shifting w_1 to v does *not* generate a pair (x,y) in B. So such a pair (v,w) in A does *not* have a matching pair (x,y) in B.

We will now count those non-matchable pairs in A. When $w_2 = 1, w_3 = 1$. So $w = 121w_4\cdots w_{n+1}1$. The sum of the weights of such paths is f_n. So the sum of the weights of such pairs (v,w) is f_n^2; *no* such pairs have matching elements (x,y) in B. Consequently, the sum of the weights of the pairs in A that have matching elements in B equals $f_nf_{n+2} - f_n^2$.

Let us now reverse the order. Shift $x_1 = 1$ from x to the beginning of y.

Case 1. Suppose $x_2 = 1$. Then $x = \underbrace{1x_3\cdots x_n1}_{\text{length } n-1}$ and $y = \underbrace{11y_2\cdots y_n1}_{\text{length } n+1}$. The corresponding pair (x,y) is a valid element in A.

Case 2. Suppose $x_2 = 2$. Then $x_3 = 1$; and $x = \underbrace{21\cdots x_n1}_{\text{length } n-1}$ and $y = \underbrace{1y_2\cdots y_n1}_{\text{length } n}$. The corresponding pair (x,y) does *not* generate a matching element in A. The sum of the weights of such unmatchable pairs equals $f_{n-1}f_{n+1}$; this equals $f_n^2+(-1)^n$, by the inductive hypothesis.

Consequently, the sum of the weights of the pairs (x,y) that have matchable counterparts in A equals $f_{n+1}^2 - [f_n^2 + (-1)^n]$.

Since the matching between the two sets of matchable pairs is bijective, the sums of their weights must be equal; that is, $f_nf_{n+2} - f_n^2 = f_{n+1}^2 - [f_n^2 + (-1)^n]$. This implies that $f_nf_{n+2} - f_{n+1}^2 = (-1)^{n+1}$. So Cassini's formula works for $n+1$ also. Thus, by induction, it works for every $n \geq 1$. $\square$

We will now illustrate the essence of the proof for the case $n = 3$. Table 6 lists the pairs (v,w) of paths $v = v_1v_2$ and $w = w_1w_2w_3w_4$ from v_1 to v_1. Six of them are numbered 1 through 6 for convenience; the sum of the weights of these pairs is $x^2(x^4+2x^2) + 1\cdot(x^4+2x^2) = (x^2+1)(x^4+2x^2) = f_3f_5 - f_3^2$. The others are labeled a through d; the sum of the weights of these four pairs equals $x^2(x^2+1) + 1\cdot(x^2+1) = (x^2+1)^2$. The grand total is $(x^2+1)(x^4+3x^2+1) = f_3f_5$.

Table 6

111	111	111	111	111	121	121	121	121	121
11111	11121	11211	12111	12121	11111	11121	11211	12111	12121
1	2	3	a	b	4	5	6	c	d

Table 7 shows the pairs (x,y) of closed paths $x = x_1x_2x_3$ and $y = y_1y_2y_3$ from v_1 to itself. Again, six of them are labeled 1 through 6; and the sum of their weights is $(x^3+x)(x^3+2x) =$

$f_4^2 - [f_3^2 + (-1)^3]$. The sum of the weights of the remaining three, labeled α, β, and γ, is $x(x^3 + 2x)$. Their grand total is $(x^3 + 2x)^2 = f_4^2$.

Table 7

1111	1111	1111	1121	1121	1121	1211	1211	1211
1111	1121	1211	1111	1121	1211	1111	1121	1211
1	2	3	4	5	6	α	β	γ

The matchable pairs from both lists are numbered 1 through 6, and the others by letters. Both sets have the same sum of weights: $(x^2 + 1)(x^4 + 2x^2) = (x^3 + x)(x^3 + 2x)$, as expected.

Next we will prove the identity $l_{n+1} + l_{n-1} = (x^2 + 4)f_n$. We will achieve this task by establishing a *one-to-five correspondence* from the set of paths of length n from v_1 to v_2 to that of closed paths of length $n + 1$ or $n - 1$.

Example 7. *Establish the identity* $l_{n+1} + l_{n-1} = (x^2 + 4)f_n$.

Proof. By Corollary 3, the sum of the weights of closed paths of length $n+1$ is l_{n+1}, and that of length $n-1$ is l_{n-1}. So the sum of the weights of paths of length $n+1$ or $n-1$ is $l_{n+1}+l_{n-1}$.

By Corollary 1, the sum of the weights of paths of length n from v_1 to v_2 is f_n. Let $w = w_1w_2 \cdots w_nw_{n+1}$ be such a path. Clearly, $w_1 = 1$ and $w_{n+1} = 2$. In addition, every v_2 must be preceded by v_1; so $w_n = 1$. Thus $w = \underbrace{1w_2 \cdots w_{n-1}12}_{\text{length } n}$.

We will now devise an algorithm in five steps to establish the aforementioned correspondence.

Step 1. Append a 1 at the end of w. This generates a closed path of length $n + 1$ from v_1 to v_1: $\underbrace{1w_2 \cdots w_{n-1}121}_{\text{length } n+1}$. The sum of the weights of such paths equals f_n.

Step 2. Delete $w_{n+1} = 2$. This results in a closed path of length $n - 1$ from v_1 to itself: $\underbrace{1w_2 \cdots w_{n-1}1}_{\text{length } n-1}$. The sum of the weights of such paths also equals f_n.

Step 3. Place a 2 at the beginning of w. This creates a closed path of length $n + 1$ from v_2 to v_2: $\underbrace{21w_2 \cdots w_{n-1}12}_{\text{length } n+1}$. The sum of the weights of such paths is again f_n.

Step 4. Replace $w_{n+1} = 2$ with 11. This operation produces a closed path of length $n + 1$ from v_1 to itself: $\underbrace{1w_2 \cdots w_{n-1}111}_{\text{length } n+1}$. The sum of the weights of such paths is $x^2 f_n$.

These four steps do *not* account for all closed paths of length $n + 1$ or $n - 1$, namely, the ones that begin with $w_1w_2 = 11$ or 12. This takes us to Step 5, which has therefore two parts.

Step 5A. Suppose $w_2 = 1$. Then delete w_1 and insert 11 at the end of w. This generates a closed path of length $n + 1$ from v_1 to itself: $\underbrace{w_2 \cdots w_{n-1}1211}_{\text{length } n+1}$. The sum of the weights of such paths is xf_{n-1}.

Step 5B. Suppose $w_2 = 2$. Then delete w_1. This gives a closed path of length $n - 1$ from v_2

to itself: $\underbrace{w_2 \cdots w_{n+1}}_{\text{length } n-1}$. Such paths contribute f_{n-2} to the sum of the weights.

The sum of the weights of closed paths generated by Step 5 equals $xf_{n-1} + f_{n-2} = f_n$.

Steps 1–5 do *not* produce duplicate paths. So the sum of the weights of closed paths they create is $(x^2 + 4)f_n$; some are of length $n + 1$ and the rest of length $n - 1$. Thus the two sums of weights must be equal; that is, $l_{n+1} + l_{n-1} = (x^2 + 4)f_n$, as desired. □

We will now illustrate the five steps of the algorithm for the case $n = 4$. There are 11 closed paths of length 5 and 4 of length three, a total of 15 closed paths of length 5 or 3:

Length 5: 111111 111121 111211 112111 121111 112121 121121 121211
Length 4: 211112 211212 212112
Length 3: 1111 1121 1211
2112

The sum of the weights of paths of length 5 is $l_5 = x^5 + 5x^3 + 5x$, and that of paths of length 3 is $l_3 = x^3 + 3x$. So the sum for paths of length 5 or 3 is $x^5 + 6x^3 + 8x$.

Now consider paths of length 4 from v_1 to v_2. There are 3 such paths $w = w_1w_2w_3w_4w_5$: 11112, 11212, 12112. The sum of their weights is $f_4 = x^3 + 2x$.

1) Appending a 1 at the end of w yields 3 closed paths length 5 from v_1 to v_1: 111121, 112121, 121121. The sum of their weights is f_4.
2) Deleting $w_5 = 2$ produces 3 closed paths length three from v_1 to v_1: 1111, 1121, 1211. The sum of their weights is also f_4.
3) Placing a 2 at the beginning of w gives three closed paths length 5 from v_2 to v_2: 211112, 211212, 212112. The sum of their weights is once again f_4.
4) Replacing w_5 with 11 generates three more closed paths of length 5 from v_1 to v_1: 111111, 112111, 121111. The sum of their weights is x^2f_4.

5a) When $w_2 = 1$, delete w_1 and append 11 at the end. This yields two closed paths of length 5: 111211, 121211. The sum of their weights equals $x^3 + x$.

5b) When $w_2 = 2$, we delete w_1. This creates exactly one closed path of length 3: 2112. Its weight is x.

So the sum of the weights of paths generated by Step 5 equals $(x^3 + x) + x = f_4$.

Thus, by Steps 1–5, the sum of the weights of all closed paths of length 5 or 3 equals $(x^2+4)f_4 = x^5 + 6x^3 + 8x = l_5 + l_3$, as expected.

Interestingly, there is a bijection between the set of closed paths of length n from v_1 to v_1, and that of Fibonacci tilings of a $1 \times n$ board. We will let the d-edges do the work for us.

6. A Bijection Algorithm

In the Fibonacci tilings of a $1 \times n$ board with squares and dominoes, we assign a weight x to each square and 1 to each domino [8]. Now consider a closed path of length n from v_1 to itself. Replacing the edge 11 with a square and a d-edge with a domino results in a tiling of the board. Clearly, this process is reversible. Consequently, this algorithm establishes the desired bijection.

To illustrate the bijection algorithm, consider the eight closed paths of length 5 starting at v_1 in Table 2. The algorithm produces the corresponding Fibonacci tilings of a 1×5 board with squares and dominoes; they are showcased in Table 8. As expected, the sum of the weights of the paths or tilings is $x^5 + 4x^3 + 3x = f_6$.

Table 8
Closed Paths from v_1 to v_1 and the Corresponding Fibonacci Tilings.

Closed Paths	Fibonacci Tilings	Weights of Tilings
111111	$\boxed{x}\boxed{x}\boxed{x}\boxed{x}\ \boxed{x}$	x^5
111121	$\boxed{x}\boxed{x}\boxed{x}\boxed{1}$	x^3
111211	$\boxed{x}\boxed{x}\boxed{1}\boxed{x}$	x^3
112111	$\boxed{x}\boxed{1}\boxed{x}\boxed{x}$	x^3
121111	$\boxed{1}\boxed{x}\boxed{x}\boxed{x}$	x^3
112121	$\boxed{x}\boxed{1}\boxed{1}$	x
121121	$\boxed{1}\boxed{x}\boxed{1}$	x
121211	$\boxed{1}\boxed{1}\boxed{x}$	x

7. Fibonacci and Lucas Sums

We can elicit the power of matrices and the graph-theoretic model to interpret combinatorially the following summation formulas:

1) $x\sum_{k=1}^{n} f_k = f_{n+1} + f_n - 1$
2) $x\sum_{k=1}^{n} l_k = l_{n+1} + l_n + x - 2$
3) $x\sum_{k=1}^{n} f_{2k-1} = f_{2n}$
4) $x\sum_{k=1}^{n} l_{2k-1} = l_{2n} - 2$
5) $x\sum_{k=1}^{n} f_{2k} = f_{2n+1}$
6) $x\sum_{k=1}^{n} l_{2k} = l_{2n+1} - x.$

In the interest of brevity, we will interpret formula (1) and leave the others for Fibonacci enthusiasts.

To this end, first we will make an important useful observation about the Q-matrix:

$$x\sum_{k=1}^{n} Q^k(x) = \begin{bmatrix} x\sum_{k=1}^{n} f_{k+1} & x\sum_{k=1}^{n} f_k \\ x\sum_{k=1}^{n} f_k & x\sum_{k=1}^{n} f_{k-1} \end{bmatrix}. \tag{1}$$

We will need the following result also [5].

Theorem 2. *Let A be the weighted adjacency matrix of a weighted graph with vertices $v_1, v_2, \cdots, v_k$, and n a positive integer. Then the ijth entry of the matrix $A + A^2 + \cdots + A^n$ gives the sum of the weights of the paths of length $\le n$ from vertex v_i to v_j.* □

With these two results at our finger tips, we are ready for the interpretation.

1) It follows from equation (1) that $x\sum_{k=1}^{n} f_k$ represents x times the sum of the weights of paths of length $\le n$ from v_1 to v_2. It equals $f_{n+1} + f_n - 1$.

For example, consider the paths of length ≤ 5 from v_1 to v_2; see Table 9. We then have

$$\begin{aligned} x\sum_{k=1}^{5} f_k &= x^5 + x^4 + 4x^3 + 3x^2 + 3x \\ &= (x^5 + 4x^3 + 3x) + (x^4 + 3x^2 + 1) - 1 \\ &= f_6 + f_5 - 1. \end{aligned}$$

Table 9
Paths of Length ≤ 5 from v_1 to v_2.

n	Paths of Length n from v_1 to v_2	Sum of the Weights
1	12	1
2	112	x
3	1112 1212	$x^2 + 1$
4	11112 11212 12112	$x^3 + 2x$
5	111112 111212 112112 121112 121212	$x^4 + 3x^2 + 1$

$\uparrow$
f_n

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