PERFECT BALANCING NUMBERS

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Abstract. Perfect numbers are scarce, only 48 are known, and when the search is restricted to a specified sequence, the possibility of their adequate presence still reduces. The objective of this paper is to show that the only perfect number in the balancing sequence is 6.

1. Introduction

A perfect number is a natural number which is equal to the sum of its proper positive divisors. These numbers are very scarce and to date only 48 numbers have been found, the largest one contains 34850340 digits. The infinitude of these numbers is yet to be established. Surprisingly, all the known perfect numbers are even although many properties and conjectures about odd perfect numbers are available in the literature. All the even perfect numbers are of the form $2^{n-1}(2^n - 1)$ where $2^n - 1$ is a prime (popularly known as Mersenne prime). It is well-known that any odd perfect number, if exists, is very large. A recent result by Ochem and Rao [7] ascertains that odd perfect numbers must be greater than $10^{1500}$. Euler proved that every odd perfect number is of the form $p^{4\alpha+1} x^2$, where $p$ is a prime of the form $4n + 1$. Neilson [8] verified that every odd perfect number has at least 9 prime factors.

The search of perfect numbers in various number sequences has been a motivating job for mathematicians. In 1994, McDaniel [5] proved that the only triangular number in the Pell sequence is 1, which is a clear indication that there is no even perfect number in this sequence. In [4], Luca established the absence of perfect numbers in the Fibonacci and Lucas sequences. The absence of even perfect numbers in the associated Pell sequence follows from the paper [12] by Prasad and Rao. The objective of this work is to explore all perfect numbers in the balancing sequence.

As defined by Behera and Panda [1], a natural number $n$ is a balancing number if $1 + 2 + \cdots + (n - 1) = (n + 1) + (n + 2) + \cdots + (n + r)$ for some natural number $r$, which is the balancer corresponding to $n$. As a consequence of the definition, a natural number $n > 1$ is a balancing number if and only if $n^2$ is a triangular number, or equivalently, $8n^2 + 1$ is a perfect square. The $n$th balancing number is denoted by $B_n$, and $C_n = \sqrt{8B_n^2 + 1}$ is called the $n$th Lucas-balancing number [13, p. 25]. Customarily, 1 is accepted as the first balancing number, that is, $B_1 = 1$. Panda in [9] proved that the Lucas-balancing numbers are associated with balancing numbers in the way Lucas numbers are associated with Fibonacci numbers. The identities $B_{2n} = 2B_nC_n$ and $C_{2n} = C_n^2 + 8B_n^2$ (see [9]) resemble respectively $F_{2n} = F_nL_n$ and $L_{2n} = \frac{1}{2}(F_n^2 + 5L_n^2)$ will prove their usefulness in the next section.

2. Even Perfect Balancing Numbers

This section is devoted to the search of even perfect numbers in the balancing sequence. Since the balancing numbers are alternately odd and even and all the known perfect numbers are even, we first focus our attention on even balancing numbers. The second balancing number $B_2 = 6$ is perfect and using Microsoft Mathematics, we verified that there is no
The importance of Pell and associated Pell sequences lies in the fact that the associated Pell sequence is defined as

$$Q_n = P_{n+1} + P_{n-1} = 2P_n + P_{n-1}$$

for each positive integer $n$. We will prove that there is only one balancing number of the form $2^rB_n$ in connection with odd perfect numbers. Robbins [14, 15] explored terms of the form $x^2$ in Fibonacci and Pell sequences while McDaniel [6] studied terms of the form $kx^2$ in these sequences. We will prove that there is only one balancing number of the form $px^2$ which is not a perfect number.

Identification of terms of the form $kx^2$ in well-known sequences have been considered by many authors. In Section 1, we have already discussed the importance of these type of numbers in connection with odd perfect numbers. Robbins [14, 15] explored terms of the form $px^2$ in Fibonacci and Pell sequences while McDaniel [6] studied terms of the form $kx^2$ in these sequences. We will prove that there is only one balancing number of the form $px^2$ which is not a perfect number.

The Pell sequence is defined as $P_1 = 1, P_2 = 2$ and $P_{n+1} = 2P_n + P_{n-1}$ for $n \geq 2$, while the associated Pell sequence is defined as $Q_1 = 1, Q_2 = 3$, and $Q_{n+1} = 2Q_n + Q_{n-1}$ for $n \geq 2$. The importance of Pell and associated Pell sequences lies in the fact that the $n$th convergent of $\sqrt{2}$ expressed as a continued fraction is $\frac{Q_n}{P_n}$. Further, a crucial relationship of these two numbers is

$$B_n = B_{2^r} = 2B_{2^r-1}C_{2^r-1} = \cdots = 2^rB_kC_kC_{2k}C_{4k} \cdots C_{2^r-1}k.$$
sequences with the balancing sequence needed to prove an important result of this section is $B_n = P_nQ_n$ [10, p.46].

To prove that there is only one balancing number of the form $px^2$, we need the following two lemmas. The first lemma ascertains the presence of only one square term in the associated Pell sequence.

**Lemma 3.1.** The only square term in the associated Pell sequence is $Q_1 = 1$.

**Proof.** It is known that for any odd order associated Pell number $k, 2(k^2 + 1)$ is a perfect square. Assume that $k$ is a square say $k = x^2$ and let $2(k^2 + 1) = z^2$. We thus have the Diophantine equation

$$2(x^4 + 1) = z^2. \tag{3.1}$$

Since the left-hand side is even, $z$ is also even, say $z = 2y$ and (3.1) reduces to

$$x^4 + 1 = 2y^2. \tag{3.2}$$

But (3.2) has no solution other than $x = y = 1$ [11, p. 133]. Thus, the only square term in the associated Pell sequence is $Q_1 = 1$ corresponding to $k = 1$. □

The following lemma due to Ljunggren [3] caters to all square terms in the Pell sequence.

**Lemma 3.2.** The only square terms in Pell sequence are $P_1 = 1$ and $P_7 = 169$.

Now, we are in a position to prove the main result of this section.

**Theorem 3.3.** The only balancing number of the form $px^2$ is $B_7 = 40391$.

**Proof.** Let $N$ be a balancing number of the form $px^2$. So it must have the prime factorization $N = p_1^{2a_1}p_2^{2a_2} \cdots p_m^{2a_m}$ for some natural number $m$. Since $N$ is a balancing number, $N = B_n$ for some $n$. Further, $B_n = P_nQ_n$ and it is well-known that $(P_n, Q_n) = 1$ for each $n$. Thus, $p$ is a factor of either $P_n$ or $Q_n$. If $p|P_n$, then $Q_n$ is a perfect square and by virtue of Lemma 3.1, this happens if $n = 1$ and consequently $N = 1$ which is not of the form $px^2$. On the other hand, if $p|Q_n$, then $P_n$ is a perfect square and by Lemma 3.2, this is possible only if $n = 1$ or 7. In the former case $N = 1$, which is not of the form $px^2$ and in the latter case $N = 40391 = 239 \cdot 13^2$. □

As mentioned earlier, Ochem and Rao [7] proved that an odd perfect number, if exists, must be greater than $10^{1500}$. Many great number theorists believe that odd perfect numbers do not exist. So how can one expect such a number in the balancing sequence? The following theorem ascertains their absence.

**Theorem 3.4.** There is no odd perfect balancing number.

**Proof.** Let $N$ be an odd perfect number. Hence, it has the canonical decomposition $N = p^{4s+1}p_1^{2a_1}p_2^{2a_2} \cdots p_m^{2a_m}$ where $p \equiv 1 \pmod{4}$ and $m \geq 9$ [8]. By virtue of Theorem 3.3, $B_7 = 40391 = 239 \cdot 13^2$ is the only balancing numbers of the form $px^2$. Since the prime $239 \not\equiv 1 \pmod{4}, 40391$ is not a perfect number. Hence, no odd balancing number is perfect. □

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