BINOMIAL IDENTITIES AND CONGRUENCES FOR EULER NUMBERS

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Abstract. We obtain two identities for binomial coefficients and apply them to prove congruences for Euler numbers.

1. Introduction

The Euler numbers $E_n$ are defined by

$$
\sum_{n \geq 0} E_n \frac{x^n}{n!} = \frac{1}{\cosh(x)}.
$$

Note that $E_n = 0$ if $n$ is odd.

N. Nielsen [1, pp. 258–261], proved that for $n \geq 1$,

$$
E_{4n} \equiv 5 \pmod{60} \text{ and } E_{4n+2} \equiv -1 \pmod{60}.
$$

We also will give proofs of these congruences. To do so, we will first show that if the \{a_n\} and \{b_n\} are given by

$$
a_n = \sum_{k=0}^{n} (-1)^k \binom{4n}{4k} \quad \text{and} \quad b_n = \sum_{k=0}^{n} (-1)^k \binom{4n+2}{4k},
$$

then

$$
\sum_{n \geq 0} a_n x^n = \frac{1 + 68x^2}{1 + 136x^2 + 16x^4} \quad \text{and} \quad \sum_{n \geq 0} b_n x^n = \frac{1 - 14x - 26x^2 + 8x^3}{1 + 136x^2 + 16x^4}.
$$

2. The Binomial Identities

We start with

$$
\frac{1}{4} ((x + 1)^{4n} + (x - 1)^{4n} + (x + i)^{4n} + (x - i)^{4n}) = \sum_{k=0}^{n} \binom{4n}{4k} x^{4n-4k}.
$$

Let $x = \frac{1 + i}{\sqrt{2}}$ and we obtain

$$
(-1)^n \sum_{k=0}^{n} (-1)^k \binom{4n}{4k}
$$

$$
= \frac{1}{4} \left( \left( \frac{1 + i}{\sqrt{2}} + 1 \right)^{4n} + \left( \frac{1 + i}{\sqrt{2}} - 1 \right)^{4n} + \left( \frac{1 + i}{\sqrt{2}} + i \right)^{4n} + \left( \frac{1 + i}{\sqrt{2}} - i \right)^{4n} \right)
$$

$$
= \frac{1}{4} \left( \left( \frac{\sqrt{2} + 1 + i}{\sqrt{2}} \right)^{4n} + \left( \frac{-\sqrt{2} - 1 - i}{\sqrt{2}} \right)^{4n} + \left( \frac{1 + (\sqrt{2} + 1)i}{\sqrt{2}} \right)^{4n} \right)
$$
\[ \left( \frac{1 - (\sqrt{2} - 1)i}{\sqrt{2}} \right)^{4n} \]
\[ + \left( \frac{1 - (\sqrt{2} - 1)i}{\sqrt{2}} \right)^{4n} = \frac{1}{4} \cdot \left( \left( (\sqrt{2} + 1) + i \right)^{4n} + \left( (\sqrt{2} - 1) + i \right)^{4n} + \left( 1 + (\sqrt{2} + 1)i \right)^{4n} + \left( 1 - (\sqrt{2} - 1)i \right)^{4n} \right) \]
\[ = \frac{1}{4} \cdot \left( (2\sqrt{2} + 2)^{2n} + (2\sqrt{2} - 2)^{2n} + (2\sqrt{2} + 2)^{2n} + (2\sqrt{2} - 2)^{2n} \right) \]
\[ = \frac{1}{4} \cdot \left( (12 + 8\sqrt{2})^{2n} + (12 - 8\sqrt{2})^{2n} + (12 + 8\sqrt{2})^{2n} + (12 - 8\sqrt{2})^{2n} \right) \]
\[ = \frac{1}{4} \cdot \left( (6 + 4\sqrt{2})^{2n} + (6 - 4\sqrt{2})^{2n} \right) (i^{n} + (-i)^{n}) . \]

So

\[ a_n = \frac{1}{4} \cdot \left( (6 + 4\sqrt{2})^{2n} + (6 - 4\sqrt{2})^{2n} \right) (i^{n} + (-i)^{n}) \]

and

\[ \sum_{n \geq 0} a_n x^n = \frac{1}{4} \left( \frac{1}{1 - (6 + 4\sqrt{2})ix} + \frac{1}{1 + (6 + 4\sqrt{2})ix} + \frac{1}{1 - (6 - 4\sqrt{2})ix} + \frac{1}{1 + (6 - 4\sqrt{2})ix} \right) \]
\[ = \frac{1}{2} \left( \frac{1}{1 + (6 + 4\sqrt{2})^2 x^2} + \frac{1}{1 + (6 - 4\sqrt{2})^2 x^2} \right) \]
\[ = \frac{1}{2} \left( \frac{1}{1 + (68 + 48\sqrt{2}) x^2} + \frac{1}{1 + (68 - 48\sqrt{2}) x^2} \right) \]
\[ = \frac{1 + 68x^2}{1 + 136x^2 + 16x^4} . \]

The proof of the result for \( \sum_{n \geq 0} b_n x^n \) is similar, so is omitted.

We have the following Corollary.

**Corollary 2.1.** Modulo 3,

\[ a_n \equiv 1 \text{ if } n \text{ is even, } 0 \text{ if } n \text{ is odd}, \]
\[ b_n \equiv 1. \]

**Proof.** Modulo 3, we have

\[ \sum_{n \geq 0} a_n x^n = \frac{1 + 68x^2}{1 + 136x^2 + 16x^4} \equiv \frac{1 - x^2}{1 + x^2 + x^4} = \frac{(1 - x^2)^2}{1 - x^6} \equiv \frac{1 + x^2 + x^4}{1 - x^6} = \frac{1}{1 - x^2} , \]
\[ \sum_{n \geq 0} b_n x^n = \frac{1 - 14x - 28x^2 + 8x^3}{1 + 136x^2 + 16x^4} \equiv \frac{1 + x - x^2 - x^3}{1 + x^2 + x^4} = \frac{(1 + x)(1 - x^2)}{1 + x^2 + x^4} \]
\[ = \frac{(1 + x)(1 - x^2)^2}{1 - x^6} \equiv \frac{(1 + x)(1 + x^2 + x^4)}{1 - x^6} = \frac{1 + x}{1 - x} = \frac{1}{1 - x} . \]
The results follow.

3. The Euler Numbers

The Euler numbers are given by

\[
\sum_{n \geq 0} E_n x^n = \frac{1}{\cosh(x)}
\]

\[
= \frac{1}{2} \left( \frac{1}{\cosh(x)} + \frac{1}{\cos(x)} \right) - \frac{1}{2} \left( \frac{1}{\cos(x)} - \frac{1}{\cosh(x)} \right)
\]

\[
= \frac{1}{2} \left( \frac{1}{\cosh(x) \cos(x)} \right) - \frac{1}{2} \left( \frac{1}{\cosh(x) \cos(x)} \right)
\]

\[
= \sum_{n \geq 0} \frac{x^{4n}}{(4n)!} \sum_{n \geq 0} \frac{(-4)^n x^n}{(4n)!} - \sum_{n \geq 0} \frac{x^{4n+2}}{(4n+2)!} \sum_{n \geq 0} \frac{(-4)^n x^n}{(4n)!}.
\]

It follows that

\[
E_{4n} = - \sum_{k=1}^{n} \left( -4 \right)^k \binom{4n}{4k} E_{4n-4k} + 1
\]

and

\[
E_{4n+2} = - \sum_{k=1}^{n} \left( -4 \right)^k \binom{4n+2}{4k} E_{4n+2-4k} - 1.
\]

Modulo 4, these equations yield

\[
E_{4n} \equiv 1 \quad \text{and} \quad E_{4n+2} \equiv -1.
\]

Modulo 5, we have

\[
E_{4n} \equiv - \sum_{k=1}^{n-1} \binom{4n}{4k} E_{4n-4k}
\]

since \(-4)^n E_0 + 1 \equiv -(1)^n \times 1 + 1 = 0\) and

\[
E_{4n+2} \equiv - \sum_{k=1}^{n} \binom{4n+2}{4k} E_{4n+2-4k} - 1.
\]

It follows by induction that

\[
E_{4n} \equiv 0 \quad \text{for} \quad n \geq 1 \quad \text{and} \quad E_{4n+2} \equiv -1.
\]

All that is required for the second of these is that

\[
\sum_{k=0}^{n} \binom{4n+2}{4k} = \frac{1}{4} \times 2^{4n+2} = 2^{2n+1} = (-1)^{2n+1} = 1.
\]

Modulo 3, we have

\[
E_{4n} \equiv - \sum_{k=1}^{n-1} (-1)^k \binom{4n}{4k} E_{4n-4k} - (-1)^n + 1
\]

and

\[
E_{4n+2} \equiv - \sum_{k=1}^{n} (-1)^k \binom{4n+2}{4k} E_{4n+2-4k} - 1.
\]
It follows by induction that
\[ E_{4n} \equiv -1 \text{ for } n \geq 1 \text{ and } E_{4n+2} \equiv -1. \]
For these we need the Corollary in Section 2.
It follows that for \( n \geq 1 \),
\[ E_{4n} \equiv 5 \pmod{60} \text{ and } E_{4n+2} \equiv -1 \pmod{60}. \]

REFERENCES


MSC2010: 11B68, 11B65, 05A30

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