BINOMIAL IDENTITIES AND CONGRUENCES FOR EULER NUMBERS

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Abstract. We obtain two identities for binomial coefficients and apply them to prove congruences for Euler numbers.

1. Introduction

The Euler numbers E_n are defined by

$$\sum_{n\geq 0} E_n \frac{x^n}{n!} = \frac{1}{\cosh(x)}.$$

Note that $E_n = 0$ if n is odd.

N. Nielsen [1, pp. 258–261], proved that for $n \geq 1$,

$$E_{4n} \equiv 5 \pmod{60} \text{ and } E_{4n+2} \equiv -1 \pmod{60}.$$

We also will give proofs of these congruences. To do so, we will first show that if the $\{a_n\}$ and $\{b_n\}$ are given by

$$a_n = \sum_{k=0}^{n} (-1)^k \binom{4n}{4k} \quad \text{and} \quad b_n = \sum_{k=0}^{n} (-1)^k \binom{4n+2}{4k}$$

then

$$\sum_{n\geq 0} a_n x^n = \frac{1+68x^2}{1+136x^2+16x^4} \quad \text{and} \quad \sum_{n\geq 0} b_n x^n = \frac{1-14x-26x^2+8x^3}{1+136x^2+16x^4}.$$

2. The Binomial Identities

We start with

$$\frac{1}{4}\left((x+1)^{4n} + (x-1)^{4n} + (x+i)^{4n} + (x-i)^{4n}\right) = \sum_{k=0}^{n} \binom{4n}{4k} x^{4n-4k}.$$

Let $x = \dfrac{1+i}{\sqrt{2}}$ and we obtain

$$(-1)^n \sum_{k=0}^{n} (-1)^k \binom{4n}{4k}$$
$$= \frac{1}{4}\left(\left(\frac{1+i}{\sqrt{2}}+1\right)^{4n} + \left(\frac{1+i}{\sqrt{2}}-1\right)^{4n} + \left(\frac{1+i}{\sqrt{2}}+i\right)^{4n} + \left(\frac{1+i}{\sqrt{2}}-i\right)^{4n}\right)$$
$$= \frac{1}{4}\left(\left(\frac{(\sqrt{2}+1)+i}{\sqrt{2}}\right)^{4n} + \left(\frac{-(\sqrt{2}-1)-i}{\sqrt{2}}\right)^{4n} + \left(\frac{1+(\sqrt{2}+1)i}{\sqrt{2}}\right)^{4n}\right.$$

$$+\left(\frac{1-(\sqrt{2}-1)i}{\sqrt{2}}\right)^{4n}\Bigg)$$
$$=\frac{1}{4}\cdot\frac{1}{4^n}\left(\left((\sqrt{2}+1)+i\right)^{4n}+\left(-(\sqrt{2}-1)+i\right)^{4n}+\left(1+(\sqrt{2}+1)i\right)^{4n}\right.$$
$$\left.+\left(1-(\sqrt{2}-1)i\right)^{4n}\right)$$
$$=\frac{1}{4}\cdot\frac{1}{4^n}\left((2\sqrt{2}+2)^{2n}(1+i^{2n}+(2\sqrt{2}-2)^{2n}(1+i)^{2n}+(2\sqrt{2}+2)^{2n}(1-i)^{2n}\right.$$
$$\left.+(2\sqrt{2}-2)^{2n}(1-i)^{2n}\right)$$
$$=\frac{1}{4}\cdot\frac{1}{4^n}\left((12+8\sqrt{2})^n(2i)^n+(12-8\sqrt{2})^n(2i)^n+(12+8\sqrt{2})^n(-2i)^n\right.$$
$$\left.+(12-8\sqrt{2})^n(-2i)^n\right)$$
$$=\frac{1}{4}\left((6+4\sqrt{2})^n+(6-4\sqrt{2})^n\right)\left(i^n+(-i)^n\right).$$

So
$$a_n=\frac{1}{4}\left((6+4\sqrt{2})^n+(6-4\sqrt{2})^n\right)\left(i^n+(-i)^n\right)$$
and
$$\sum_{n\geq 0}a_nx^n=\frac{1}{4}\left(\frac{1}{1-(6+4\sqrt{2})ix}+\frac{1}{1+(6+4\sqrt{2})ix}+\frac{1}{1-(6-4\sqrt{2})ix}+\frac{1}{1+(6_-4\sqrt{2})ix}\right)$$
$$=\frac{1}{2}\left(\frac{1}{1+(6+4\sqrt{2})^2x^2}+\frac{1}{1+(6-4\sqrt{2})^2x^2}\right)$$
$$=\frac{1}{2}\left(\frac{1}{1+(68+48\sqrt{2})x^2}+\frac{1}{1+(68-48\sqrt{2})x^2}\right)$$
$$=\frac{1+68x^2}{1+136x^2+16x^4}.$$

The proof of the result for $\sum_{n\geq 0}b_nx^n$ is similar, so is omitted.

We have the following Corollary.

Corollary 2.1. *Modulo 3,*
$$a_n\equiv 1\quad \textit{if } n \textit{ is even},\quad 0\quad \textit{if } n \textit{ is odd},$$
$$b_n\equiv 1.$$

Proof. Modulo 3, we have
$$\sum_{n\geq 0}a_nx^n=\frac{1+68x^2}{1+136x^2+16x^4}\equiv\frac{1-x^2}{1+x^2+x^4}=\frac{(1-x^2)^2}{1-x^6}\equiv\frac{1+x^2+x^4}{1-x^6}=\frac{1}{1-x^2},$$
$$\sum_{n\geq 0}b_nx^n=\frac{1-14x-28x^2+8x^3}{1+136x^2+16x^4}\equiv\frac{1+x-x^2-x^3}{1+x^2+x^4}=\frac{(1+x)(1-x^2)}{1+x^2+x^4}$$
$$=\frac{(1+x)(1-x^2)^2}{1-x^6}\equiv\frac{(1+x)(1+x^2+x^4)}{1-x^6}=\frac{1+x}{1-x^2}=\frac{1}{1-x}.$$

□

The results follow.

3. The Euler Numbers

The Euler numbers are given by

$$\begin{aligned}\sum_{n\geq 0} E_n x^n &= \frac{1}{\cosh(x)}\\ &= \frac{1}{2}\left(\frac{1}{\cosh(x)}+\frac{1}{\cos(x)}\right)-\frac{1}{2}\left(\frac{1}{\cos(x)}-\frac{1}{\cosh(x)}\right)\\ &= \frac{\frac{1}{2}\left(\cosh(x)+\cos(x)\right)}{\cosh(x)\cos(x)}-\frac{\frac{1}{2}\left(\cosh(x)-\cos(x)\right)}{\cosh(x)\cos(x)}\\ &= \sum_{n\geq 0}\frac{x^{4n}}{(4n)!}\Big/\sum_{n\geq 0}\frac{(-4)^n x^n}{(4n)!}-\sum_{n\geq 0}\frac{x^{4n+2}}{(4n+2)!}\Big/\sum_{n\geq 0}\frac{(-4)^n x^n}{(4n)!}.\end{aligned}$$

It follows that

$$E_{4n} = -\sum_{k=1}^{n}(-4)^k\binom{4n}{4k}E_{4n-4k}+1$$

and

$$E_{4n+2} = -\sum_{k=1}^{n}(-4)^k\binom{4n+2}{4k}E_{4n+2-4k}-1.$$

Modulo 4, these equations yield

$$E_{4n}\equiv 1 \quad\text{and}\quad E_{4n+2}\equiv -1.$$

Modulo 5, we have

$$E_{4n}\equiv -\sum_{k=1}^{n-1}\binom{4n}{4k}E_{4n-4k}$$

since $-(-4)^n E_0 + 1 \equiv -(1)^n\times 1 + 1 = 0$ and

$$E_{4n+2}\equiv -\sum_{k=1}^{n}\binom{4n+2}{4k}E_{4n+2-4k}-1.$$

It follows by induction that

$$E_{4n}\equiv 0 \quad\text{for}\quad n\geq 1 \quad\text{and}\quad E_{4n+2}\equiv -1.$$

All that is required for the second of these is that

$$\sum_{k=0}^{n}\binom{4n+2}{4k} = \frac{1}{4}\times 2^{4n+2} = 4^{2n}\equiv(-1)^{2n}=1.$$

Modulo 3, we have

$$E_{4n}\equiv -\sum_{k=1}^{n-1}(-1)^k\binom{4n}{4k}E_{4n-4k}-(-1)^n+1$$

and

$$E_{4n+2}\equiv -\sum_{k=1}^{n}(-1)^k\binom{4n+2}{4k}E_{4n+2-4k}-1.$$

It follows by induction that

$$E_{4n} \equiv -1 \quad \text{for} \quad n \geq 1 \quad \text{and} \quad E_{4n+2} \equiv -1.$$

For these we need the Corollary in Section 2.

It follows that for $n \geq 1$,

$$E_{4n} \equiv 5 \pmod{60} \quad \text{and} \quad E_{4n+2} \equiv -1 \pmod{60}.$$

References

[1] N. Nielsen, Traité Élementaire des Nombres de Bernoulli, Gauthier–Villars, Paris, 1923.

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