A NOTE ON THE INVARIANCE OF THE GENERAL 2×2 MATRIX ANTI-DIAGONALS RATIO WITH INCREASING MATRIX POWER: FOUR PROOFS

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Abstract. An invariance matrix property, first observed empirically and seemingly absent from mainstream literature, is stated and established formally here. Four short, and different, proofs are given accordingly.

1. Introduction and Result

Occasionally an observation is made which is striking in its simplicity, and more than a little surprising. We present one such result here, giving what background there seems to be, together with four proofs that are distinct in flavor and which we hope the reader will enjoy. The starting point of Proof I is succinct, and it delivers the result routinely. Proof II, being inductive, is essentially procedural (with echoes of Proof I). Proof III uses a different line of logic, and is an elegant one in its own right. Proof IV argues inductively also, but is quite different from Proof II.

1.1. **Result.** Let $\mathbf{M}$ be the general 2×2 matrix

$$\mathbf{M} = \mathbf{M}(A, B, C, D) = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{1.1}$$

and, with $\alpha_1 = A$, $\beta_1 = B$, $\gamma_1 = C$, $\delta_1 = D$, suppose for $n \geq 1$ it has nth power

$$\mathbf{M}^n = \begin{pmatrix} \alpha_n & \beta_n \\ \gamma_n & \delta_n \end{pmatrix}, \tag{1.2}$$

where $\alpha_n = \alpha_n(A, B, C, D), \ldots, \delta_n = \delta_n(A, B, C, D)$. We assume that each of A, B, C, D is non-zero and further, for convenience, that the matrix $\mathbf{M}$ has non-zero trace and is non-singular.

We state an observation, made empirically originally, as a formal result and give four proofs. The paper has been motivated by anti-diagonals product invariance evident across powers of 2×2 matrix sets which characterize a particular class of polynomial families [1] associated with sequences having quadratic governing ordinary generating function equations (reflecting the number of such sequences, the polynomial class is large in quantity). The typical matrix therein is, however, less general than $\mathbf{M}$ here (A, B, C are drawn from $\mathbb{Z}[x]$ in [1], but $D = 0$), and the invariance result noted and proved is of a different type.

Theorem 1.1. *The ratio $\beta_n/\gamma_n = B/C$ is a quantity invariant with power $n \geq 1$ of the 2×2 matrix $\mathbf{M}(A, B, C, D)$.*

Before proofs are offered, we illustrate Theorem 1.1 with computations made algebraically in which we see, when forming the anti-diagonals ratios, the increasing scale of term cancellation in line with matrix power. Noting that $\beta_1/\gamma_1 = B/C$ trivially, and $\beta_2 = B(A+D)$, $\gamma_2 = C(A+D)$, we find

$$\begin{aligned}
\beta_3 &= B(A^2 + AD + BC + D^2),\\
\gamma_3 &= C(A^2 + AD + BC + D^2),\\
\\
\beta_4 &= B(A+D)(A^2 + 2BC + D^2),\\
\gamma_4 &= C(A+D)(A^2 + 2BC + D^2),\\
\\
\beta_5 &= B(A^4 + A^3D + A^2(3BC + D^2) + A(4BCD + D^3) + B^2C^2 + 3BCD^2 + D^4),\\
\gamma_5 &= C(A^4 + A^3D + A^2(3BC + D^2) + A(4BCD + D^3) + B^2C^2 + 3BCD^2 + D^4),\\
\\
\beta_6 &= B(A+D)(A^2 + AD + BC + D^2)(A^2 - AD + 3BC + D^2),\\
\gamma_6 &= C(A+D)(A^2 + AD + BC + D^2)(A^2 - AD + 3BC + D^2), \qquad (1.3)
\end{aligned}$$

and so on, with $\beta_2/\gamma_2 = \cdots = \beta_6/\gamma_6 = \cdots = B/C$.

1.2. **Background.** A very familiar result for powers of the special case matrix $\mathbf{M}(1,1,1,0)$ involves Fibonacci numbers, and there is certainly evidence of powers of 2×2 matrices having been examined for instances in which initial matrix terms are not all retained symbolically. Our more general result is available from but very few sources and is not nearly as widely known as one might imagine, if so at all.

Surveying the literature we see that over a decade ago J. McLaughlin [2] defined a parameter

$$y_n = y_n(T, M) = \sum_{i=0}^{\lfloor \frac{1}{2}n \rfloor} \binom{n-i}{i} T^{n-2i}(-M)^i = y_n(A, B, C, D) \qquad (1.4)$$

(where $T = T(A, D) = A + D$ is the trace of $\mathbf{M}$ and $M = M(A, B, C, D) = |\mathbf{M}| = AD - BC$ its determinant), and proved inductively that, for $n \geq 1$,

$$\mathbf{M}^n(A, B, C, D) = \begin{pmatrix} y_n - Dy_{n-1} & By_{n-1} \\ Cy_{n-1} & y_n - Ay_{n-1} \end{pmatrix}, \qquad (1.5)$$

from which Theorem 1.1 is trivial [2, Theorem 1, p. 3]; the formulation was used as a basis to derive some combinatorial identities. Beyond this, a variety of expressions representing powers of general 2×2 matrices have been derived but, it seems, not in ways that relate to our result here since forms for entries of such matrix powers are neither tractable nor explicit (the same applies to those occasional studies of exponentiated matrices of arbitrary dimension). Note that McLaughlin does, though, allude to some overlapping work by Schwerdtfeger (occurring in a 1962 textbook section on the iteration of a Möbius transformation [3, pp. 104–105]) which itself was motivated by a much earlier paper by Jacobsthal from 1919–20, suggesting that powers of 2×2 matrices have a reasonably long history as an item of interest to some mathematicians. The author has recently developed a new formulation of McLaughlin's result (1.5) (which will be disseminated at some point in the future) based on so called Catalan polynomials referred to in Remark 2.1; interestingly, a minor variation of them are seen in Schwerdtfeger's text (and Jacobsthal's work), where they are termed Fibonacci polynomials.

We now present our four proofs. Readers are, of course, cordially invited to add to this number (see also the open question posed in the Summary).

2. The Proofs

Proof I. Our first proof (generalized slightly in the Appendix) is a direct one.

Proof. We write the self-satisfying identity $\mathbf{M}^{n+1} = \mathbf{M}^{n+1}$ as

$$\mathbf{M}\mathbf{M}^n = \mathbf{M}^n\mathbf{M}, \tag{I.1}$$

which yields the four equations

$$\begin{aligned} 0 &= B\gamma_n - C\beta_n, \\ 0 &= A\beta_n + B\delta_n - (B\alpha_n + D\beta_n), \\ 0 &= C\alpha_n + D\gamma_n - (A\gamma_n + C\delta_n), \\ 0 &= C\beta_n - B\gamma_n, \end{aligned} \tag{I.2}$$

using (1.1) and (1.2). The result follows trivially from either the first/last equation, or (as an alternative) by combining the middle two equations, of (I.2). □

Proof II. Our second proof proceeds inductively.

Proof. Clearly, Theorem 1.1 holds for $n = 1, 2$. These being the first odd and even values of n then, assuming the result is true for some $n = k \geq 2$, it suffices to show that the case $n = k+2$ is valid in consequence.

By hypothesis, therefore, we have $B\gamma_k - C\beta_k = 0$. Consider now

$$\begin{pmatrix} \alpha_{k+1} & \beta_{k+1} \\ \gamma_{k+1} & \delta_{k+1} \end{pmatrix} = \mathbf{M}^{k+1} = \mathbf{M}^k\mathbf{M} = \begin{pmatrix} \alpha_k & \beta_k \\ \gamma_k & \delta_k \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{II.1}$$

with $\alpha_{k+1} = A\alpha_k + C\beta_k$, $\beta_{k+1} = B\alpha_k + D\beta_k$, $\gamma_{k+1} = A\gamma_k + C\delta_k$, $\delta_{k+1} = B\gamma_k + D\delta_k$ as a result. Thus, writing

$$\begin{pmatrix} \alpha_{k+2} & \beta_{k+2} \\ \gamma_{k+2} & \delta_{k+2} \end{pmatrix} = \mathbf{M}^{k+2} = \mathbf{M}\mathbf{M}^{k+1} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \alpha_{k+1} & \beta_{k+1} \\ \gamma_{k+1} & \delta_{k+1} \end{pmatrix}, \tag{II.2}$$

we see that

$$\begin{aligned} &B\gamma_{k+2} - C\beta_{k+2} \\ &\quad = B(C\alpha_{k+1} + D\gamma_{k+1}) - C(A\beta_{k+1} + B\delta_{k+1}) \\ &\quad = BC(A\alpha_k + C\beta_k) + BD(A\gamma_k + C\delta_k) - [CA(B\alpha_k + D\beta_k) + CB(B\gamma_k + D\delta_k)] \\ &\quad = M(B\gamma_k - C\beta_k) \\ &\quad = 0 \end{aligned} \tag{II.3}$$

by assumption, completing the proof. □

A trivial corollary to Theorem 1.1 is that for $n \geq 1$ the anti-diagonals ratio $\beta_n/\gamma_n = 1$ if $C = B$, while setting further (or indeed independently) $D = A$ gives a diagonals ratio $\alpha_n/\delta_n = 1$ also.

Remark 2.1. Note that setting $D = 0$ in $\mathbf{M}$ leaves Theorem 1.1 unchanged, and under such a condition it is immediate from (I.1) of [1, p. 176] which employs so called Catalan polynomials that are closely related to McLaughlin's parameters; the general $(n+1)$th Catalan polynomial is defined for $n \geq 0$ as $P_n(x) = \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n-i}{i}(-x)^i$, with $y_n(A,B,C,D) = T^n P_n(M/T^2)$ and in particular $y_n(A,B,C,0) = A^n P_n(-BC/A^2)$.

Remark 2.2. From the computations in (1.3) it would appear that should $T = 0$ then Theorem 1.1 would hold only for n odd, since $\beta_2 = \beta_4 = \beta_6 = \cdots = 0 = \gamma_2 = \gamma_4 = \gamma_6 = \cdots$, with the anti-diagonals ratios $\beta_2/\gamma_2, \beta_4/\gamma_4, \ldots$, undefined. As an aside, then in this instance the ratio of diagonal terms α_n/δ_n exists for both parity cases, being 1 (n even) or -1 (n odd).

Remark 2.3. We can force $\mathbf{M}$ to be algebraically singular by choosing $D = BC/A$, in which case Theorem 1.1 remains valid still (with the diagonals ratio $\alpha_n/\delta_n = A^2/BC$ equally invariant for matrix power $n \geq 1$).

Proof III. Our third proof is elegant.

Proof. Define a diagonal matrix

$$\mathbf{D}(x) = \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix}, \tag{III.1}$$

with inverse

$$\mathbf{D}^{-1}(x) = \begin{pmatrix} 1 & 0 \\ 0 & 1/x \end{pmatrix}. \tag{III.2}$$

Then the matrix

$$\mathbf{X} = \mathbf{X}(A,B,C,D,x) = \mathbf{D}(x)\mathbf{M}(A,B,C,D)\mathbf{D}^{-1}(x) = \begin{pmatrix} A & B/x \\ Cx & D \end{pmatrix} \tag{III.3}$$

will be symmetric so long as $x = \pm\sqrt{B/C}$. Under this condition any nth power of the matrix $\mathbf{X}(A,B,C,D) = \mathbf{X}(A,B,C,D,x(B,C))$ will also be a symmetric matrix ($n > 1$). Thus, if we write (where $r_1 = A$, $s_1 = \pm\sqrt{BC}$, $t_1 = D$)

$$\mathbf{X}^n = \begin{pmatrix} r_n(A,B,C,D) & s_n(A,B,C,D) \\ s_n(A,B,C,D) & t_n(A,B,C,D) \end{pmatrix}, \qquad n \geq 1, \tag{III.4}$$

we see that $\mathbf{M} = \mathbf{D}^{-1}\mathbf{X}\mathbf{D}$ by (III.3), and in turn (using (III.1), (III.2), and (III.4))

$$\mathbf{M}^n(A,B,C,D) = \mathbf{D}^{-1}(x(B,C))\mathbf{X}^n(r_n,s_n,t_n)\mathbf{D}(x(B,C)) = \begin{pmatrix} r_n & xs_n \\ s_n/x & t_n \end{pmatrix}, \tag{III.5}$$

from which, by comparison with (1.2),

$$\frac{\beta_n}{\gamma_n} = \frac{xs_n}{s_n/x} = x^2 = B/C, \tag{III.6}$$

as required. □

Proof IV. This proof is another inductive one.

Proof. Theorem 1.1 is valid for $n = 1, 2$, and we assume the same is true for some $n = k, k-1$ ($k \geq 2$). The key element of the proof is to see that (denoting the 2×2 identity matrix as $\mathbf{I}_2$) the matrix $\mathbf{M}^2$ is expressible as

$$\mathbf{M}^2 = T\mathbf{M} - M\mathbf{I}_2, \tag{IV.1}$$

hence, pre- or post-multiplying throughout by $\mathbf{M}^{k-1}$,

$$\mathbf{M}^{k+1} = T\mathbf{M}^k - M\mathbf{M}^{k-1}, \tag{IV.2}$$

of which (IV.1) is the $k = 1$ instance. By (1.2) this reads

$$\begin{pmatrix} \alpha_{k+1} & \beta_{k+1} \\ \gamma_{k+1} & \delta_{k+1} \end{pmatrix} = T\begin{pmatrix} \alpha_k & \beta_k \\ \gamma_k & \delta_k \end{pmatrix} - M\begin{pmatrix} \alpha_{k-1} & \beta_{k-1} \\ \gamma_{k-1} & \delta_{k-1} \end{pmatrix}, \tag{IV.3}$$

with the ratio

$$\frac{\beta_{k+1}}{\gamma_{k+1}} = \frac{T\beta_k - M\beta_{k-1}}{T\gamma_k - M\gamma_{k-1}} = B/C \tag{IV.4}$$

as required, since by assumption $\beta_k = (B/C)\gamma_k$ and $\beta_{k-1} = (B/C)\gamma_{k-1}$. □

3. Summary

In this note we have presented and proved an interesting 2×2 matrix invariance property, the origin of which lies with previous related work on matrices that characterize a class of polynomial families associated with sequences of certain type. In addition to the observation being promulgated, the mechanics of Proofs I-IV undoubtedly help to reveal its fundamental nature. With this in mind, the result might lend itself to an extension applicable to a general $n \times n$ matrix case of which our result would merely describe a particular instance, but we leave such a question to be discussed elsewhere.

Appendix

Here we generalize Proof I slightly, which we note proceeds using the instance $\mathbf{K} = \mathbf{M}^n$.

Proof. Let

$$\mathbf{K} = \begin{pmatrix} k_1 & k_2 \\ k_3 & k_4 \end{pmatrix} \tag{A.1}$$

be a general matrix that commutes with $\mathbf{M}$. Then $\mathbf{MK} = \mathbf{KM}$ and, matching entries across the equation, yields (I.2) with $\alpha_n, \beta_n, \gamma_n, \delta_n$ replaced by k_1, k_2, k_3, k_4, resp., from which we conclude that any matrix commuting with $\mathbf{M}$ has an anti-diagonals ratio $k_2/k_3 = B/C$. Our result is established since each of $\mathbf{M}, \mathbf{M}^2, \mathbf{M}^3, \ldots,$ commutes with $\mathbf{M}$, which is to say that $\mathbf{M}^n$ is such a commuting matrix for any $n \geq 1$, and will itself have an off-diagonals ratio B/C. □

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References

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