A NOTE ON PRIME FIBONACCI SEQUENCES

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Abstract. In this paper, we define a variant of Fibonacci-like sequences that we call prime Fibonacci sequences, where one takes the sum of the previous two terms and returns the smallest odd prime divisor of that sum as the next term. We prove that these sequences always terminate in a power of two but can be extended infinitely to the left.

1. Introduction

In [3], Guy, Khovanova, and Salazar study a variant of Fibonacci-like sequences they call subprime Fibonacci sequences—a variant suggested by Conway. To compute a term of a subprime Fibonacci sequence, one takes the sum of the previous two terms and, if the sum is composite, divides by its smallest prime divisor. They study periodic subprime Fibonacci sequences and derive many interesting results; however, they are unable to prove that any such sequence "diverges" (i.e., does not eventually end in a cycle). Indeed, it is difficult to imagine how one might prove such a thing. The question, "Do all subprime Fibonacci sequences eventually end in a cycle?" may belong to the class of extremely difficult (possibly even formally unsolvable) problems discussed in [1]; one such example is the generalized Collatz problem, which was shown in [5] to be undecidable.

In this paper, we consider a different variant of the Fibonacci sequence that submits itself to more complete analysis: instead of adding two terms and *dividing* by the smallest prime divisor, we *return* the smallest *odd* prime divisor as the next term.

Definition 1.1. *Let p_1, p_2 be odd primes. Then the prime Fibonacci sequence generated by p_1, p_2 is (a_i), where a_{i+2} is the smallest odd prime divisor of $a_i + a_{i+1}$ if $a_i + a_{i+1}$ is not a power of 2; otherwise, the sequence terminates.*

For example, consider starting with $5, 7$:

$$5, 7, 3, 5.$$

The sequence terminates because the last two terms add up to a power of two. As we will see, these prime Fibonacci sequences always terminate, but can be made arbitrarily long. In fact, they can be infinite to the left, so we will take one and "turn it around" to get an infinite sequence.

2. Proofs of Main Results

Theorem 2.1. *Given distinct odd primes p_1, p_2, the prime Fibonacci sequence (a_i) with $a_1 = p_1, a_2 = p_2$ terminates in a power of 2.*

Proof. Let (a_i) be any prime Fibonacci sequence. Since a_i and a_{i+1} are odd, their sum is even. So, $a_{i+2} \leq \frac{a_i + a_{i+1}}{2}$. In particular, $a_{i+2} < \max[a_i, a_{i+1}]$ if $a_i \neq a_{i+1}$. (It is easy to show both that any eventually constant prime Fibonacci sequence must have been constant from the beginning and that no nontrivial periodic sequences exist.) The conclusion follows. □

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Theorem 2.2. *Given distinct odd primes p_1, p_2, we can always find an odd prime p_0 so that p_2 is the smallest odd prime dividing $p_0 + p_1$.*

Proof. Let p_1, p_2 be distinct odd primes. If $p_2 \mid p_0 + p_1$, then $p_2 \cdot m = p_0 + p_1$ for some $m \in \mathbb{Z}^+$, and $p_0 = p_2 \cdot m - p_1$. Let

$$2, 3, 5, \ldots, p', p_2$$

be the list of primes up to p_2. Let $Q = \prod_{2<p\leq p_2} p$, and consider the equivalences

$$x \equiv 1 - p_1 \pmod 3 \tag{2.1}$$

$$x \equiv 1 - p_1 \pmod 5 \tag{2.2}$$

$$x \equiv 1 - p_1 \pmod 7 \tag{2.3}$$

$$\vdots$$

$$x \equiv 1 - p_1 \pmod{p'} \tag{2.4}$$

$$x \equiv -p_1 \pmod{p_2} \tag{2.5}$$

By the Chinese Remainder Theorem, there is a unique solution to (2.1)-(2.5) modulo Q; let a be this solution. Note that $(a, Q) = 1$. Then Dirichlet's Theorem on primes in arithmetic progressions (see [7] for a proof) tells us that the sequence

$$a, a + Q, a + 2Q, a + 3Q, \ldots$$

contains infinitely many primes. Let p_0 be the smallest of these. We verify that p_0 has the desired properties.

Write $p_0 = a + nQ$ for some $n \in \mathbb{Z}$, so

$$\begin{aligned} p_0 + p_1 &= a + nQ + p_1 \\ &\equiv -p_1 + 0 + p_1 \pmod{p_2} \\ &= 0. \end{aligned}$$

Hence, $p_2 \mid p_0 + p_1$. Now let $q < p_2$ be an odd prime. Again,

$$\begin{aligned} p_0 + p_1 &= a + nQ + p_1 \\ &\equiv 1 - p_1 + 0 + p_1 \pmod q \\ &\equiv 1 \pmod q. \end{aligned}$$

Hence, $q \nmid p_0 + p_1$. □

Corollary 2.3. *For all positive integers k, there is a prime Fibonacci sequence of length at least k.*

Corollary 2.4. *Any prime Fibonacci sequence can be extended indefinitely to the left.*

We can give a second proof of Corollary 2.3 using the celebrated Green-Tao Theorem [2] that for all n one can find n primes in arithmetic progression.

Alternate proof of Corollary 2.3. We construct a prime Fibonacci sequence of length k. By [2], there is an arithmetic progression $p_0, p_1, \ldots, p_n$ of primes of length $2^{k-2} + 1$ (so $n = 2^{k-2}$). We let $a_1 = p_0$ and $a_2 = p_n$. Notice that, because the p_i's are in arithmetic progression, $p_0 + p_n = 2p_{n/2}$, so $a_3 = p_{n/2}$. Similarly, $a_2 + a_3 = p_n + p_{n/2} = 2p_{3n/4}$, so $a_4 = p_{3n/4}$. In each case, once $a_i = p_\alpha$ and $a_{i+1} = p_\beta$ have been found, $a_{i+2} = p_{(\alpha+\beta)/2}$. The claim that $a_1, \ldots, a_k$ is a prime Fibonacci sequence is equivalent to the claim that the sequence

$$b_1 = 0, b_2 = 2^{k-2}, \text{ and } b_{i+2} = \frac{b_i + b_{i+1}}{2}$$

is an integer for $i \leq k$, which is easily verified by induction. □

3. Reversed Prime Fibonacci Sequences

In light of Corollary 2.4, we make the following definition.

Definition 3.1. *Let p, q be distinct odd primes. Then (a_i) is the reversed prime Fibonacci sequence generated by p, q if $a_1 = p, a_2 = q$, and for all $i \geq 1$, a_{i+2} is the smallest odd prime with the following property: a_i is the smallest odd prime divisor of $a_{i+1} + a_{i+2}$.*

Then we have the following result.

Theorem 3.2. *Let (a_i) be an eventually monotonic reversed prime Fibonacci sequence. Then (a_i) has asymptotic density zero in the primes.*

Proof. Let (a_i) be any (eventually) monotonic reversed prime Fibonacci sequence, and let (p_i) be the sequence of primes. Since $a_i \mid (a_{i+1} + a_{i+2})$ and (a_i) is monotonic, we have $2a_i < a_{i+1} + a_{i+2}$. Since a_i is a divisor of $a_{i+1} + a_{i+2}$, we get the stronger inequality $4a_i \leq a_{i+1} + a_{i+2}$. Then we have $a_{i+2} \geq 4a_i - a_{i+1}$. Consider a sequence (b_i) satisfying the recurrence $b_{i+2} = 4b_i - b_{i+1}$. This sequence has characteristic equation $r^2 + r - 4 = 0$, with positive root $\alpha = \frac{1}{2}(\sqrt{17} - 1) \approx 1.56$. Certainly, we have $a_n \gg b_n$, no matter the initial conditions on either sequence, so $a_n \gg \alpha^n$. Note also that from [4] we have $p_n \ll (1.2)^n$. Combining these, we get

$$\frac{p_n}{a_n} \ll \frac{1.2^n}{\alpha^n} < \frac{1.2^n}{1.5^n} \to 0.$$

□

We do not know whether any reversed prime Fibonacci sequence is eventually monotonic. Getting good computational data is difficult, because although reversed prime Fibonacci sequences may not be monotonic, they still grow quite quickly. For instance, consider the sequence below beginning with 3, 5, for which we can compute 23 terms:

$$3, 5, 7, 3, 11, 7, 37, 19, 277, 331, 223, 439, 7, 406507, 67, 330515394367, 967,$$
$$10576492618777, 116041, 22372439224849182406250739 7, 3691561,$$
$$1001052073739140571449182973141607102075256301115093 17, 423951181 \ldots.$$

(This is sequence number A255562 in OEIS [6].) The authors were able to compute the first 15 terms; the next eight were added in OEIS by Giovanni Resta.

Reversed prime Fibonacci sequences seem to exhibit oscillating behavior, as in the sequence below, which begins with 11, 19, and which alternates between increasing and decreasing:

$$11, 19, 3, 73, 5, 1163, 17, 2309, 3, 147773, 7, 2364361, 43, 75659509, 109, 605275963, 601.$$

Because $a_{i+2} < \max[a_i, a_{i+1}]$ in any prime Fibonacci sequence, if (b_i) is a reversed prime Fibonacci sequence that oscillates, then either (b_{2k}) or (b_{2k+1}) is increasing.

Conjecture 3.3. *Every reversed prime Fibonacci sequence eventually oscillates.*

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