HIDDEN FORMULAS IN FIBONACCI TILINGS

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ABSTRACT. In 1972, Alfred Brousseau, a founding editor of *The Fibonacci Quarterly*, published an entertaining account of how geometric tilings can be used to generate identities involving Fibonacci numbers. In this article, we explain how there is a hidden formula in each of his tilings, discoverable by consideration of the tiling's centroid. To demonstrate the utility of this approach, we provide a simple derivation of a new formula for the sum of cubes of the first n Fibonacci numbers.

1. INTRODUCTION

The Fibonacci numbers are defined by the recurrence relation $F_n = F_{n-1} + F_{n-2}$ where $F_0 = 0$ and $F_1 = 1$. It is well-known that rectangular tiles whose side lengths are Fibonacci numbers can be carefully arranged to demonstrate relationships between Fibonacci numbers. Alfred Brousseau gives a range of examples in [3]. For instance, in Figure 1 we compare the total area to the sum of its parts to find that

$$F_1^2 + F_2^2 + \cdots + F_n^2 = F_n F_{n+1}.$$

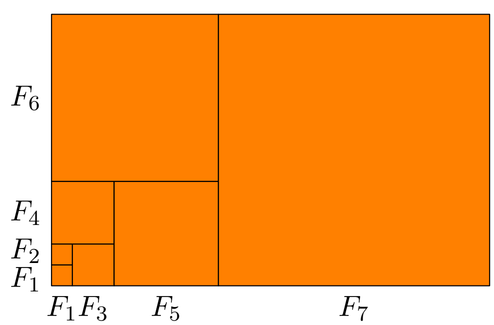

FIGURE 1

By considering the centroid of such figures we can obtain a second hidden identity. And by applying this general approach to various tilings we find a new formula for the sum of the first n cubes of Fibonacci numbers.

2. THE CENTROID

We provide a cursory summary of the required theory. The *centroid* of a planar figure $X \subset \mathbb{R}^2$ with area A is the mean position $(\overline{x}, \overline{y})$ of all the points in X, defined by the integrals

$$\overline{x} = \frac{\iint_X x \, dx dy}{A} \text{ and } \overline{y} = \frac{\iint_X y \, dx dy}{A}.$$

If $\{X_i\}$ is a partition of X then its centroid can be found by first finding the centroid $(\overline{x}_i, \overline{y}_i)$ and area A_i of each part and then computing

$$\overline{x} = \frac{\sum_i A_i \overline{x}_i}{A} \text{ and } \overline{y} = \frac{\sum_i A_i \overline{y}_i}{A}.$$

That is, the centroid of a figure is well behaved with respect to decomposition, in the sense that it can be found as a weighted average of its constituent parts. The proof follows at once from the definition of the centroid and the calculation,

$$\overline{x} = \frac{\iint_X x\,dxdy}{A} = \frac{\sum_i \iint_{X_i} x\,dxdy}{A} = \frac{\sum_i A_i \overline{x}_i}{A}.$$

In the work that follows, we will only be finding the x-component of the centroid. We will call this the *x-mean*. We outline how it will be employed in the subsequent section.

3. The Method

We have already seen how geometric diagrams can illustrate algebraic identities. Perhaps the simplest such example is a demonstration of the fact that for non-negative real numbers a, b and c,

$$c(a+b) = ca + cb. \tag{3.1}$$

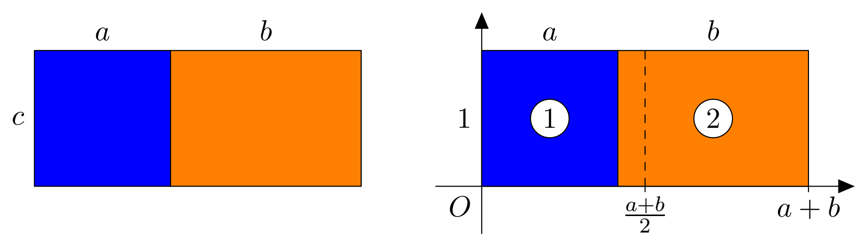

Figure 2. This figure can demonstrate identities (3.1) and (3.2).

It may be surprising that lurking in that same simple diagram one can *also* find that

$$(a+b)^2 = a^2 + 2ab + b^2. \tag{3.2}$$

To see this, we let $c = 1$ and then construct perpendicular axes along the left and lower edges of the rectangle. As the rectangle is symmetric, its x-mean is located at

$$\overline{x} = \frac{a+b}{2}. \tag{3.3}$$

Now decompose the figure into two parts, as indicated on Figure 2. Here, and later, we let $\overline{x}_i$ and A_i be the x-mean and area of the rectangle $\textcircled{i}$, respectively. We can also locate the x-mean of the outer rectangle by taking a weighted average of these two parts. Since $c = 1$,

$$\overline{x} = \frac{\overline{x}_1 A_1 + \overline{x}_2 A_2}{A_1 + A_2} = \frac{\frac{a}{2}a + (a + \frac{b}{2})b}{(a+b)}. \tag{3.4}$$

Equating (3.3) and (3.4) gives identity (3.1). Likewise, Figure 3 below is ordinarily used to demonstrate the truth of (3.2). However, it can also be leveraged to demonstrate that

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3. \tag{3.5}$$

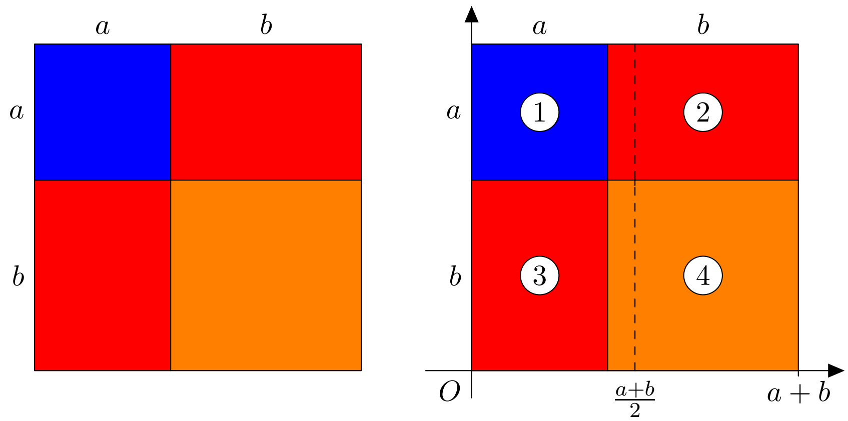

FIGURE 3. The above picture demonstrates both identities (3.2) and (3.5).

Once again, the x-mean is located at

$$\overline{x} = \frac{a+b}{2} \tag{3.6}$$

and alternatively,

$$\overline{x} = \frac{\overline{x}_1 A_1 + \overline{x}_2 A_2 + \overline{x}_3 A_3 + \overline{x}_4 A_4}{A_1 + A_2 + A_3 + A_4} = \frac{\frac{a}{2}a^2 + (a+\frac{b}{2})ab + \frac{a}{2}ab + (a+\frac{b}{2})b^2}{(a+b)^2}. \tag{3.7}$$

Equating (3.6) and (3.7) clearly gives identity (3.5).

Of course, we don't give these examples because they are new or profound. However, they do indicate a novel way of generating identities from figures with vertical symmetry. The procedure is simple. Take any such figure and then

(1) locate the x-mean along the line of symmetry then,
(2) locate the x-mean by dividing the figure into parts then,
(3) equate the two results.

As rectangles whose side lengths are Fibonacci numbers can be carefully arranged into larger vertically symmetric rectangles, this procedure is particularly fruitful when applied to such figures. With scarce effort we will be able to establish geometrically, that

$$\sum_{j=1}^{n} F_j F_{j+1} F_{j+2} = \frac{F_n^3 + F_{n+1}^3 + F_{n+2}^3 + F_{n-1}F_nF_{n+1} - 2}{4}. \tag{3.8}$$

We also obtain a new formula for the sum of cubes of the first n Fibonacci numbers. Notably, we make no appeal to the Binet formula, which features prominently in most works of this nature. In the work that follows, we will state results true for Fibonacci numbers. Some of our results generalize in obvious directions to other Fibonacci-like numbers. However our purpose is to illustrate a new technique, not exhaustively capture the full breadth of its consequences.

4. A Simple Fibonacci Tiling

To illustrate the ease and utility of this technique we begin with the simple Fibonacci tiling depicted in Figure 4 below.

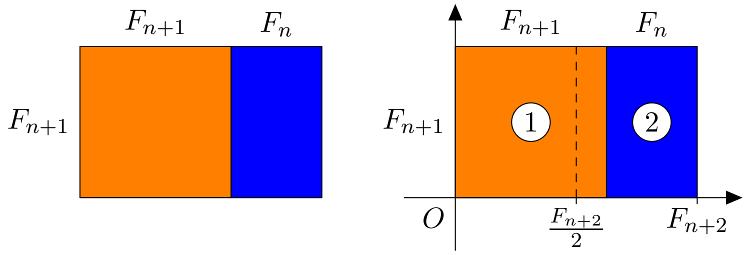

FIGURE 4. A simple Fibonacci tiling.

Superficially, the tiling demonstrates that $F_{n+2}F_{n+1} = F_{n+1}^2 + F_nF_{n+1}$. Consideration of the rectangle's x-mean reveals a further identity. First, the x-mean is located at

$$\overline{x} = \frac{F_{n+1} + F_n}{2} = \frac{F_{n+2}}{2}. \tag{4.1}$$

Now by considering a weighted average of its parts we also see that

$$\begin{aligned}\overline{x} &= \frac{\overline{x}_1 A_1 + \overline{x}_2 A_2}{A_1 + A_2} \\ &= \frac{\frac{F_{n+1}}{2} \cdot F_{n+1}^2 + (F_{n+1} + \frac{F_n}{2}) \cdot F_{n+1}F_n}{F_{n+1}(F_{n+1} + F_n)} \\ &= \frac{F_{n+1}^2 + (2F_{n+1} + F_n) \cdot F_n}{2F_{n+2}} \\ &= \frac{F_{n+1}^2 + F_{n+3}F_n}{2F_{n+2}}. \end{aligned} \tag{4.2}$$

Equating (4.1) and (4.2) gives the identity,

$$F_{n+2}^2 - F_{n+1}^2 = F_nF_{n+3}. \tag{4.3}$$

No doubt equation (4.3) is a trivial result, and algebraically obvious from the factorization of $F_{n+2}^2 - F_{n+1}^2$. Nonetheless, this example serves to demonstrate the technique that we will deploy in our subsequent work, where more complicated configurations of rectangles lead to more interesting results.

5. A Preliminary Result

However, before considering further tilings, we remind the reader of an elementary but remarkable result first observed by Block in [2]. We will call upon this formula in the following section. We provide the proof as it warrants a few subsequent remarks.

Proposition 5.1.

$$\sum_{j=1}^{n} F_j^2 F_{j+1} = \frac{1}{2} F_n F_{n+1} F_{n+2}. \tag{5.1}$$

Proof. This is a simple proof by induction. The base case is obvious, and for the inductive step,

$$\begin{aligned} 2\sum_{j=1}^{k+1} F_j^2 F_{j+1} &= 2\sum_{j=1}^{k} F_j^2 F_{j+1} + 2F_{k+1}^2 F_{k+2} \\ &= F_k F_{k+1} F_{k+2} + 2F_{k+1}^2 F_{k+2} \\ &= F_{k+1} F_{k+2}(F_k + 2F_{k+1}) \\ &= F_{k+1} F_{k+2} F_{k+3}. \end{aligned}$$

□

We note that Proposition 5.1 is as geometrically impressive as it algebraically impressive. For if we take two copies of each of the boxes with dimensions

$$(F_1^2 \times F_2), (F_2^2 \times F_3), \ldots, (F_n^2 \times F_{n+1})$$

then these $2n$ boxes can be packed into a box of dimensions $F_n \times F_{n+1} \times F_{n+2}$. This fact is *not* implied by formula (5.1), but by the preceding calculation that demonstrates its truth. Indeed, assume that the first $2k$ boxes can be packed into a box of dimensions $F_k \times F_{k+1} \times F_{k+2}$. We take this box along with two boxes of dimensions $F_{k+1}^2 \times F_{k+2}$ and pack them into a box with dimensions $F_{k+1} \times F_{k+2} \times F_{k+3}$. This is shown in Figure 5 below.

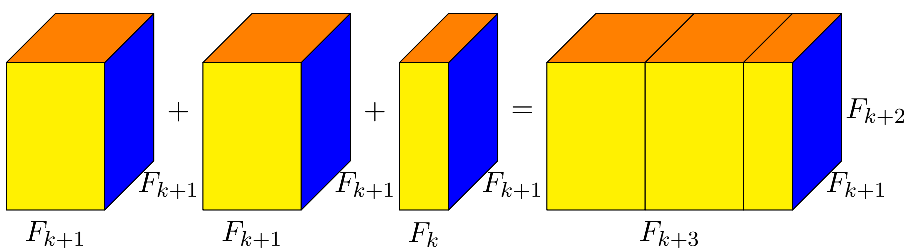

FIGURE 5. Packing the first $2k$ boxes inductively.

6. Sums of Products of Three Consecutive Fibonacci Numbers

Our second example considers Figure 6 below, depicted by Brousseau in [3]. It shows four rectangles with sides F_{n+1} and F_n arranged into the corners of a square of side length F_{n+2}. At the square's center is a smaller square of side length $F_{n+1} - F_n = F_{n-1}$. Brousseau notes that the diagram provides a one-glance demonstration of the identity

$$F_{n+2}^2 = 4F_n F_{n+1} + F_{n-1}^2. \tag{6.1}$$

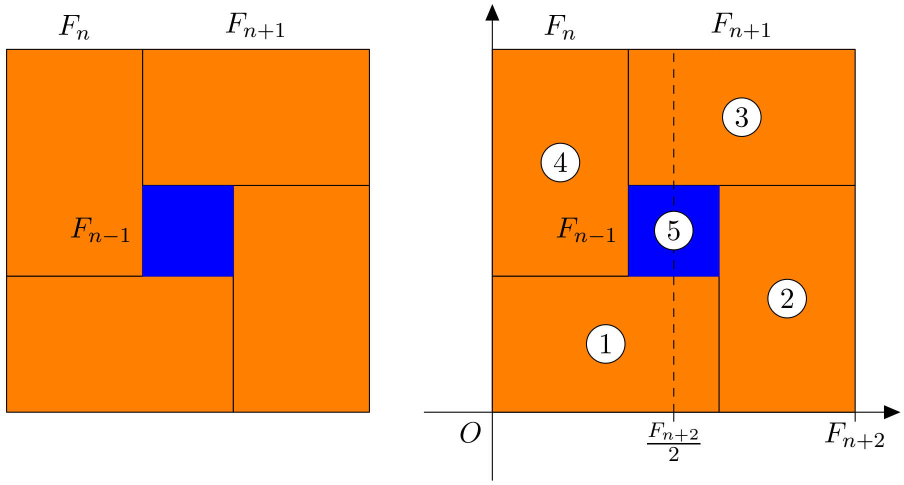

FIGURE 6. A spiral of tiles yields two formulas.

Before considering the x-mean, we first note that equation (6.1) and a telescoping sum can be used to find a formula for the sum of the products of two consecutive Fibonacci numbers,

$$\sum_{j=1}^{n} F_j F_{j+1} = \frac{1}{4} \sum_{j=1}^{n} \left(F_{j+2}^2 - F_{j-1}^2 \right) = \frac{F_{n+2}^2 + F_{n+1}^2 + F_n^2 - 2}{4}.$$

Surprisingly, the same diagram can be used to find a formula for the sum of products of three consecutive Fibonacci numbers. To this end, we first note that the x-mean is located at

$$\overline{x} = \frac{F_{n+2}}{2}. \tag{6.2}$$

As before, this can also be found by considering a weighted average of its parts. Foregoing the algebraic details, we can show that

$$\overline{x} = \frac{\overline{x}_1 A_1 + \overline{x}_2 A_2 + \overline{x}_3 A_3 + \overline{x}_4 A_4 + \overline{x}_5 A_5}{A_1 + A_2 + A_3 + A_4 + A_5} = \frac{F_{n-1}^3 + 2F_{n-1}^2 F_n + 4F_n F_{n+1} F_{n+2}}{2F_{n+2}^2}. \tag{6.3}$$

As before, we equate (6.2) and (6.3) to discover a far-from-obvious identity,

$$2F_{n-1}^2 F_n + 4F_n F_{n+1} F_{n+2} = F_{n+2}^3 - F_{n-1}^3. \tag{6.4}$$

Now with the aid of a telescoping sum we obtain,

$$2\sum_{j=1}^{n} F_{j-1}^2 F_j + 4\sum_{j=1}^{n} F_j F_{j+1} F_{j+2} = F_n^3 + F_{n+1}^3 + F_{n+2}^3 - 2. \tag{6.5}$$

And finally, equations (6.5) and (5.1) together yield a delightful formula for the sum of products of three consecutive Fibonacci numbers.

Proposition 6.1.

$$\sum_{j=1}^{n} F_j F_{j+1} F_{j+2} = \frac{F_n^3 + F_{n+1}^3 + F_{n+2}^3 - F_{n-1} F_n F_{n+1} - 2}{4}. \tag{6.6}$$

Armed with this result, we find a formula for the sum of consecutive cubes.

7. Sums of Cubes of Consecutive Fibonacci Numbers

For this we now consider the tiling shown in Figure 7 below.

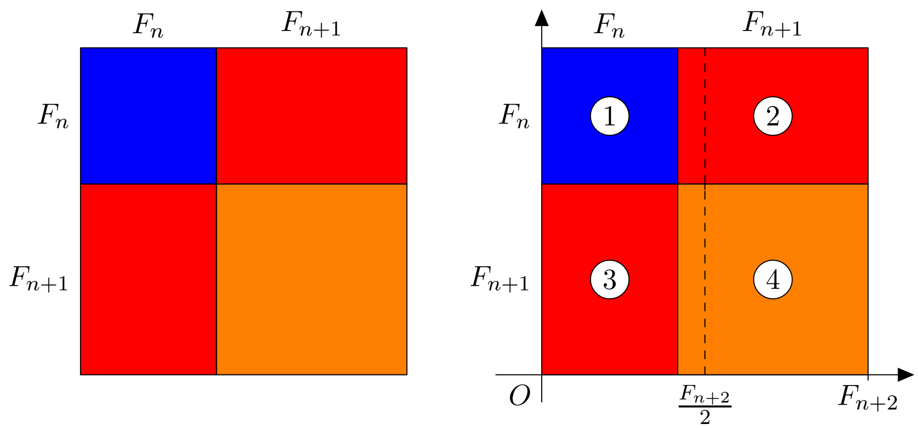

Figure 7. The above picture also demonstrates two identities.

The x-mean is located at

$$\overline{x} = \frac{F_{n+2}}{2}. \tag{7.1}$$

Considering a weighted average of its parts, a minor calculation shows that

$$\overline{x} = \frac{\overline{x}_1 A_1 + \overline{x}_2 A_2 + \overline{x}_3 A_3 + \overline{x}_4 A_4}{A_1 + A_2 + A_3 + A_4} = \frac{F_n^3 + F_{n+1}^3 + 3F_n F_{n+1} F_{n+2}}{2F_{n+2}^2}. \tag{7.2}$$

By equating (7.1) and (7.2) we've established that

$$F_n^3 + 3F_n F_{n+1} F_{n+2} = F_{n+2}^3 - F_{n+1}^3. \tag{7.3}$$

Using equation (7.3), it follows in the now customary fashion that

$$\sum_{j=1}^{n} F_j^3 + 3\sum_{j=1}^{n} F_j F_{j+1} F_{j+2} = \sum_{j=1}^{n} \left(F_{j+2}^3 - F_{j+1}^3\right) = F_{n+2}^3 - 1. \tag{7.4}$$

Finally, equations (7.4) and (6.6) together yield a closed formula for the sum of consecutive cubes of Fibonacci numbers,

$$\sum_{j=1}^{n} F_j^3 = \frac{F_{n+2}^3 - 3F_{n+1}^3 - 3F_n^3 + 3F_{n-1}F_n F_{n+1} + 2}{4}. \tag{7.5}$$

A cleaner expression is obtained by substituting $F_{n+2} = F_{n+1} + F_n$ and $F_{n-1} = F_{n+1} - F_n$ into equation (7.5). We obtain the following result.

Proposition 7.1.

$$\sum_{j=1}^{n} F_j^3 = \frac{3F_{n+1}^2 F_n - F_{n+1}^3 - F_n^3 + 1}{2}. \tag{7.6}$$

Certainly, equation (7.6) isn't as impressive as the closed and factored expression found in [4], However, the grace of the geometrical argument perhaps offsets its ungainly answer. Interestingly, the formula is comparable to that obtained in [1] using a combinatorial argument, which is equivalent to the identity

$$\sum_{j=1}^{n} F_j^3 = \frac{F_{n+1}F_{n+2}^2 + (-1)^n F_n - 2F_n^3}{2}.$$

The reader is invited to discover further identities using this approach, perhaps beginning with Figure 1 in the introduction.

Acknowledgement

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