NIVEN REPUNITS IN GENERAL BASES

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ABSTRACT. We show that the generalized repunit $(b^n-1)/(b-1)$ is divisible by n if and only if n is divisible by the multiplicative order of b modulo every prime factor of n. This fact is a generalization of an older result which holds for $b=10$. A few consequences of this theorem concerning base-b Niven numbers are also discussed.

1. Introduction

A repunit, or a decimal repunit, is a number composed of a string of ones, e.g., 11, 111, 1111, etc. A repunit of length n is denoted by R_n, which can be algebraically expressed as

$$R_n = \frac{10^n - 1}{9}.$$

There are well-known interesting properties of repunits, many of which deal with divisibility. For instance, it is known that if n is a multiple of m, then R_n is a multiple of R_m. Therefore, for R_n to be a prime repunit, it is necessary that n be a prime number.

The name repunit was introduced over half a century ago by Albert H. Beiler [3, p. 83]. Then almost twenty years later, in 1982, W. M. Snyder [12] wrote an article in which base-b repunits were first considered.

Definition 1.1. *For integers $n \geq 1$ and $b \geq 2$, we define the number $R_{n,b}$ by the formula*

$$R_{n,b} = \frac{b^n - 1}{b - 1}.$$

The numbers $R_{n,b}$ are now called generalized repunits, because for each b, we may represent $R_{n,b}$ by a string of n ones when the base-b number system is considered.

Note that the sequence $R_{n,b}$ generalizes not only the decimal repunits $R_{n,10}$ but also the Mersenne numbers $R_{n,2} = 2^n - 1$. This relation seems to have made divisibility of repunits and their generalization of greater interest, in particular where factorization and primality testing are concerned, e.g., the works by Harvey Dubner [5] and John H. Jaroma [7].

In 1989 Kennedy and Cooper [8] gave a complete classification of all repunits R_n which are divisible by their digital length n. This article is now a straight-forward generalization of this result, i.e., we are seeking for a necessary and sufficient condition for the number n to divide the generalized repunit $R_{n,b}$.

As a preliminary observation, if $\gcd(n,b) > 1$, then we have a prime p dividing both n and b. In this case, $R_{n,b} = 1 + b + b^2 + \cdots + b^{n-1} \equiv 1 \pmod{p}$, and neither p nor n can divide $R_{n,b}$. Therefore, a necessary condition for $R_{n,b}$ to be divisible by n is that $\gcd(n,b) = 1$. We shall assume this condition in stating the main result, i.e., Theorem 1.3, given after the following definition.

Definition 1.2. *When the number $a \geq 1$ is relatively prime to $m \geq 2$, we denote by $|a|_m$ the multiplicative order of a modulo m.*

Throughout this article, we agree that whenever we write $|a|_m$, we also implicitly assume that $\gcd(a, m) = 1$, for otherwise such notation makes no sense.

Theorem 1.3. *Let $n, b \geq 2$. The number $R_{n,b}$ is a multiple of n if and only if n is a multiple of $|b|_p$ for every prime p dividing n.*

Incidentally, a number that is divisible by its sum of digits is called a Niven number. This name, after the mathematician Ivan M. Niven (1915–1999), was first revealed in 1980 by Kennedy *et al.* [9] with regards to decimal representation. But naturally, we may also say that N is a base-b Niven number when N is divisible by the sum of its digits when expressed in base b. This generalization of Niven numbers seemed to appear for the first time in an article by Grundman [6] in 1994.

Thus Theorem 1.3 gives a criterion for identifying all Niven numbers among the generalized repunits to the same base. We will first present several corollaries of the theorem which are of some interest, and then we shall provide the proof and discuss some applications of Theorem 1.3 to other problems related to base-b Niven numbers.

In passing, we should mention that repunits belong to the family of repdigits, which, in base b, are given by $dR_{n,b}$ with $1 \leq d \leq b-1$. Now $dR_{n,b}$ is divisible by its digital sum dn if and only if $R_{n,b}$ is divisible by n. Hence, Theorem 1.3 may well be extended to read as follows: *A repdigit of length n is a Niven number with respect to a fixed base b if and only if n is a multiple of $|b|_p$ for every prime p dividing n.*

2. Immediate Consequences of the Theorem

Corollary 2.1. *Suppose that $R_{n,b}$ is a multiple of n. Then $\gcd(n, b-1) > 1$. In particular, the least prime factor of n is a divisor of $b-1$.*

Proof. Let p be the smallest prime dividing n. Also let $d = |b|_p$, hence, $d < p$. But by Theorem 1.3, d is also a factor of n. So in order not to contradict the minimality of p, we must conclude that $d = 1$. This implies $b \equiv 1 \pmod{p}$, which gives the claim. □

The next Corollary 2.2 is an old result concerning the Mersenne number $2^n - 1$ which can be found, for instance, in the text by W. Sierpiński [11, p. 219], whose proof was attributed to A. Schnizel.

Corollary 2.2. *The Mersenne number $R_{n,2}$ is not divisible by n for any value of $n \geq 2$.*

Proof. This follows from Corollary 2.1 with $b = 2$, since no prime factor of n would divide $b-1$. □

Corollary 2.3. *Let $b \geq 3$ and let p be any prime factor of $b-1$. If n is a power of p, then the number $R_{n,b}$ is divisible by n.*

Proof. This too follows immediately from Theorem 1.3, since $|b|_p = 1$ and p is the unique prime factor of n. □

We remark that Corollary 2.3 is equivalent to an earlier theorem by Trojovský and Tobiáš [13, Theorem 1] which is given as a rather isolated result on generalized repunits, not within the context of Niven numbers.

Corollary 2.4. *Let the factorization of n into prime numbers be written as $n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$ with $e_1, e_2, \ldots, e_k \geq 1$ and such that $p_1 < p_2 < \cdots < p_k$. Then $n \mid R_{n,b}$ if and only if*

(1) *p_1 is a factor of $b-1$ and*

(2) *for* $2 \le i \le k$, *either* $p_i \mid b-1$ *or else* $|b|_{p_i} = p_1^{f_1} p_2^{f_2} \cdots p_{i-1}^{f_{i-1}}$ *with non-negative integers* $f_1, f_2, \ldots, f_{i-1}$ *such that* $f_j \le e_j$ *for all* j *in the range* $1 \le j \le i-1$.

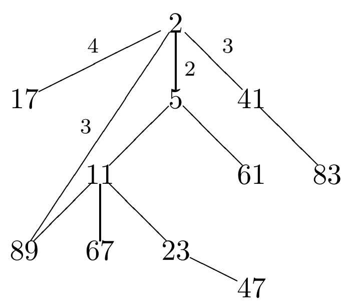
Proof. It is clear that the stated set of conditions is equivalent to having $|b|_{p_i}$ divides n for $i = 1, 2, \ldots, k$. □

Corollary 2.4 is the practical version of Theorem 1.3 which leads to the construction of a number n that divides $R_{n,b}$. For example, let us consider $b = 3$. Keeping the notation in Corollary 2.4, we must have $p_1 = 2$. We then evaluate $|3|_p$, say for $p = 5, 7, \ldots, 97$, and record the primes p for which $|3|_p$ factors into the primes that are already in the list:

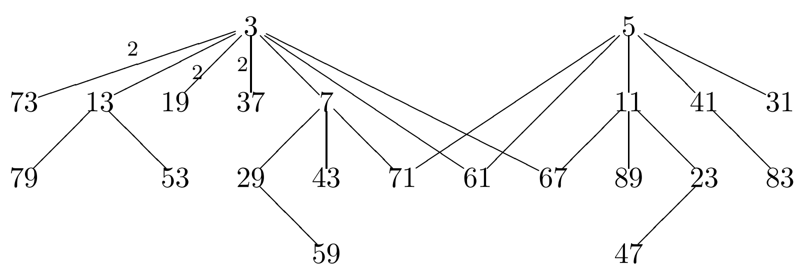
p	$\lvert 3\rvert_p$	p	$\lvert 3\rvert_p$
5	2^2	47	23
11	5	61	2×5
17	2^4	67	2×11
23	11	83	41
41	2^3	89	$2^3 \times 11$

These are the primes $p < 100$ which qualify to form the product n as described in Corollary 2.4. Hence, for instance, we have $n \mid R_{n,3}$ for $n = 2^3 \cdot 5 \cdot 11 \cdot 89$ or $n = 2^4 \cdot 17 \cdot 41 \cdot 83$.

To see the transitive dependencies among these primes in a more readable manner, we can transform the above list to a lattice diagram as follows.



Similarly, we may consider $b = 16$ and, after computing $|16|_p$ using all primes $p < 100$, we come up with a more crowded lattice diagram below, with only the primes 2, 17, and 97 being ineligible.



Note that for each value of b, we can construct an infinite number of n satisfying the criteria of Corollary 2.4, e.g., by considering arbitrarily large exponents $e_1, e_2, \ldots, e_k$. Not only so, but the list of qualifying prime factors of such n is also endless. We state this claim as follows.

Corollary 2.5. *Let* $b \ge 3$ *and* $k \ge 1$ *be fixed. Then there exists a number* n *with at least* k *distinct prime factors such that* $R_{n,b}$ *is a multiple of* n.

Proof. An 1886 result by Bang [2] implies that for a fixed $b \ge 2$, the sequence $b^m - 1$ has a primitive prime divisor p for every m, i.e., where p divides $b^m - 1$ but not $b^r - 1$ with $1 \le r < m$. For such prime p, we have $|b|_p = m$.

Let q denote any prime factor of $b-1$, and simply let $n = q^k p_2 p_3 \cdots p_k$, where for each prime p_j, $2 \le j \le k$, we have $|b|_{p_j} = q^j$. Note that $p_j > q$ and, after reordering as necessary, it is clear that $n \mid R_{n,b}$ by Corollary 2.4.

□

3. Proof of the Theorem

Following the proof for the case $b = 10$ [8, p. 140], here we shall likewise prove Theorem 1.3 by introducing an intermediate criterion for the divisibility of $R_{n,b}$ by n.

Theorem 3.1. *The number $R_{n,b}$ is divisible by n if and only if n is divisible by $|b|_n$.*

Proof. One way is easy, for if n divides $R_{n,b}$, then n divides the numerator $b^n - 1$, so $b^n \equiv 1 \pmod{n}$ and $|b|_n$ divides n.

Conversely, assume that $|b|_n \mid n$, or equivalently, that $n \mid b^n - 1$. Let p be an arbitrary prime factor of n and write $n = mp^k$ with $\gcd(m,p) = 1$. By the Chinese Remainder Theorem, we will have $n \mid R_{n,b}$, provided that $p^k \mid R_{n,b}$. We proceed in two cases.

Case 1. Suppose $b \not\equiv 1 \pmod{p}$. Then p^k divides $R_{n,b} = (b^n - 1)/(b-1)$ if and only if p^k divides $b^n - 1$, which holds since n also divides $b^n - 1$.

Case 2. Suppose $b \equiv 1 \pmod{p}$. Let $\Phi_m(X)$ stand for the mth cyclotomic polynomial. Recall that $\Phi_1(X) = X - 1$ and $\Phi_p(X) = 1 + X + X^2 + \cdots + X^{p-1}$. We also have the identity

$$X^n - 1 = \prod_{d \mid n} \Phi_d(X), \tag{3.1}$$

which implies that $(b^n - 1)/(b-1)$ is divisible by $\Phi_p(b)\Phi_{p^2}(b)\cdots\Phi_{p^k}(b)$. Then by the well-known reduction formula $\Phi_{p^m}(X) = \Phi_{p^{m-1}}(X^p)$, we can show that for each $j = 1, 2, \ldots, k$,

$$\Phi_{p^j}(b) = \Phi_p(b^{p^{j-1}}) \equiv \Phi_p(1) = p \equiv 0 \pmod{p}.$$

Hence, $p^k \mid R_{n,b}$ as desired. □

And now evidently, Theorem 1.3 will be established upon proving the next equivalent statement.

Theorem 3.2. *Let $b, n \ge 2$. Then $|b|_n$ divides n if and only if $|b|_p$ divides n for every prime p such that $p \mid n$.*

Proof. Again, one way is trivial since $b^n \equiv 1 \pmod{n}$ implies that $b^n \equiv 1 \pmod{p}$ when p is a factor of n. For the converse, now assume that $n = mp^k$ for some prime p, such that $\gcd(m,p) = 1$ and $|b|_p \mid n$. It will suffice to show that $p^k \mid b^n - 1$.

Since $|b|_p \le p - 1$, we have $\gcd(|b|_p, p) = 1$ and so $|b|_p \mid m$. That is, $b^m \equiv 1 \pmod{p}$. Using again Identity (3.1), we see that $b^n - 1$ is divisible by $M_1 M_2 \cdots M_k$, where

$$M_j = \prod_{d \mid m} \Phi_{d \cdot p^j}(b),$$

for $j = 1, 2, \ldots, k$. This time we call for the factorization formula $\Phi_r(X^s) = \prod_{d \mid s} \Phi_{d \cdot r}(X)$, which applies when $\gcd(r,s) = 1$ (see the article by Cheng *et al.* [4, Corollary 2] for a proof of this identity), and we obtain

$$M_j = \Phi_{p^j}(b^m) = \Phi_p((b^m)^{p^{j-1}}) \equiv \Phi_p(1) \equiv 0 \pmod{p}.$$

Thus, $p \mid M_j$ and $p^k \mid b^n - 1$. □

Alternately, one may wish to prove Theorem 3.1 or Theorem 3.2 without resorting to cyclotomic polynomials. In particular, demonstrating divisibility by p^k in both cases can be done via a congruence property that is also used in the original proof by Kennedy and Cooper [8, Lemma 2.1].

4. Applications on Base-b Niven Numbers

In a recent article on Smith numbers, the author [14, Definition 4] considered a different generalization of repunits by allowing a string of zeros of fixed length between each adjacent ones in R_n, e.g., numbers like 1010101 or 100010001. We now extend this definition to a general base b, noting that there are multiple ways by which we may express these quantities algebraically.

Definition 4.1. *For each $n, k \geq 1$ and $b \geq 2$, we define the number $R_{n,b,k}$ according to*

$$R_{n,b,k} = \sum_{j=0}^{n-1} b^{jk} = \frac{b^{nk}-1}{b^k-1} = \frac{R_{nk,b}}{R_{k,b}} = R_{n,b^k}.$$

We see that in base-b, the number $R_{n,b,k}$ is composed of n ones, every two of which are separated by $k-1$ zeros. In particular, the relation $R_{n,b,k} = R_{n,b^k}$ asserts that $R_{n,b,k}$ has sum of digits equals n, when evaluated in either base b or base b^k. We have the following result.

Theorem 4.2. *Let $n, b \geq 2$ and $k \geq 1$. Then $R_{n,b,k}$ is a base-b Niven number if and only if $|b^k|_n \mid n$. In particular, for any $k \geq 1$, $R_{n,b,k}$ is a base-b Niven number whenever $R_{n,b}$ is.*

Proof. The first claim is clear by Theorem 3.1 applied to R_{n,b^k}. The second follows by the same theorem since $|b^k|_n$ is always a factor of $|b|_n$. □

The converse of the second part of Theorem 4.2 does not hold: $R_{n,b}$ may not be a base-b Niven number even if $R_{n,b,k}$ is. For a counter-example, we have seen that there is no base-2 Niven number among the Mersenne numbers $R_{n,2}$; nevertheless, the preceding diagram can be used to generate infinitely many base-2 Niven numbers of the form $R_{n,2,4}$.

Now one Olympiad-type problem asks for a proof that given n, there exists a number m divisible by its sum of decimal digits, which equals n. This challenge is due to Sierpiński, according to the problem book by Andreescu and Andrica [1, Problem 7.2.1], and whose proposed solution is essentially the number $R_{n,10,\phi(n)}$, provided that $\gcd(n, 10) = 1$. We are in a position to generalize this result to any base b:

Theorem 4.3. *Given $n, b \geq 2$, we can find a base-b Niven number m whose sum of base-b digits is n.*

Proof. If $\gcd(n, b) = 1$, simply let $m = R_{n,b,\phi(n)}$. Since $|b^{\phi(n)}|_n = 1$ by Euler's Theorem, then m is a base-b Niven number by Theorem 4.2.

If $\gcd(n, b) > 1$, we may write $n = st$, where t is the largest factor of n that is relatively prime to b. In this case, every prime factor of s is also a factor of b, hence, $s \mid b^r$ for some integer $r \geq 1$. Now let $m = b^r \cdot R_{n,b,\phi(t)}$, whose sum of digits in base b equals n. Again by Euler's Theorem,

$$R_{n,b,\phi(t)} = 1 + b^{\phi(t)} + b^{2\phi(t)} + \cdots + b^{(n-1)\phi(t)} \equiv n \equiv 0 \pmod{t}.$$

Hence, $t \mid R_{n,b,\phi(t)}$ and $n \mid m$. □

Along the same line, we propose one last problem of finding a Niven number with a prescribed digital condition.

Theorem 4.4. *Given $N, b \geq 2$, we can find two base-b Niven numbers m_1 and m_2, such that*

$$\left\lfloor \frac{m_1}{b^v} \right\rfloor = N \quad \textit{and} \quad m_2 \bmod b^w = N$$

for some integers $v, w \geq 1$, i.e., when expressed in base b, the digits in N are reproduced in m_1 as the leading digits and in m_2 as the trailing digits.

Proof. A known result established by McDaniel [10, Lemma 3] states that if $b^n - 1 \geq N$, then the number $m = (b-1)R_{n,b} \cdot N$ has sum of base-b digits equal $(b-1)n$. By Corollary 2.4, we let m be such a number but where $R_{n,b}$ is chosen to be divisible by n–so m is a Niven number–and such that $n \geq d$, where d is the number of base-b digits in N.

Writing $m = N \cdot b^n - N$, we observe that the digits in m, as read from left to right, necessarily begin with those of $N - 1$ and end with those of $b^d - N$ (and repeated digits of $(b-1)$'s in the middle). Hence, to satisfy the claim, we just let $m_1 = (b^n - 1)(N + 1)$ and $m_2 = (b^n - 1)(b^d - N)$. □

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