A DIRECT PROOF THAT F_n DIVIDES F_{mn} EXTENDED TO DIVIDISIBILITY PROPERTIES OF RELATED NUMBERS

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Abstract. A direct proof that F_n divides F_{mn} uses the quotient of the division to derive divisibility properties for Fibonacci, Lucas, Pell, and Pell-Lucas numbers.

1. Introduction

For positive integers m and n, the standard method of proving $F_n \mid F_{mn}$ is by first establishing the relationship $F_{m+n} = F_{m-1}F_n + F_mF_{n+1}$, then using induction. Several interesting proofs were given in [1] and [2] including one that gives the quotient of the division in terms of a Lucas sum. An "unusual" proof that involves the use of hyperbolic functions was published in [5]. In [3], a unified approach to divisibility properties for Fibonacci (F_n), Lucas (L_n), Pell (P_n), and Pell-Lucas (Q_n) numbers was given. The proofs of these properties utilized the fact that $Z[\sqrt{2}]$ and $Z[\sqrt{5}]$ are closed under addition and multiplication. This paper extends the divisibility properties in [3] to a larger family of integers and the proofs do not use the properties of $Z[\sqrt{2}]$ or $Z[\sqrt{5}]$. In addition, the quotient of the division is given explicitly in each case. Also, new divisibility properties are given. In addition, comparing a result from [1] gives rise to a new identity (Corollary 3.3).

2. Preliminary Results

For nonnegative integers n, consider the recurrence relation defined by

$$x_{n+2} = cx_{n+1} + x_n \qquad (2.1)$$
$$x_0 = a,\ x_1 = b$$

where a, b, and c are integers. Following the standard procedures for solving second order homogeneous recurrence relations with constant coefficients [6], the Binet formula for the integer family $\{x_n\}$ defined by (2.1) is

$$x_n = \frac{1}{u-v}\left([b-av]u^n - [b-au]v^n\right), \qquad (2.2)$$

where $u = \frac{c+\sqrt{c^2+4}}{2}$ and $v = \frac{c-\sqrt{c^2+4}}{2}$ are the roots of $\lambda^2 - c\lambda - 1 = 0$. These roots satisfy $u + v = c$, $u - v = \sqrt{c^2+4}$, and $uv = -1$. In particular, if $a = 0$ and $b = c = 1$, then $x_n = F_n$. If $a = 2$ and $b = c = 1$, then $x_n = L_n$. If $a = 0$, $b = 1$, and $c = 2$, then $x_n = P_n$. If $a = b = c = 2$, then $x_n = Q_n$. Also, for F_n and L_n, $u = \alpha = \frac{1}{2}(1+\sqrt{5})$ and $v = \beta = \frac{1}{2}(1-\sqrt{5})$. For P_n and Q_n, $u = \gamma = 1 + \sqrt{2}$ and $v = \delta = 1 - \sqrt{2}$. For the four special cases we consider, the Binet formulas are

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta},\ L_n = \alpha^n + \beta^n, P_n = \frac{\gamma^n - \delta^n}{\gamma - \delta},\ \text{and } Q_n = \gamma^n + \delta^n,$$

for $n \geq 0$.

Lemma 2.1. *$x_n u + x_{n-1} = (b - av)u^n$ and $x_n v + x_{n-1} = (b - au)v^n$ for $n \geq 1$.*

Proof. Using the facts that $uv = -1$, $\frac{1+u^2}{u-v} = u$, and the Binet formula for x_n we have

$$\begin{aligned} x_n u + x_{n-1} &= \frac{1}{u-v}\left[(b-av)u^n - (b-au)v^n\right]u + \frac{1}{u-v}\left[(b-av)u^{n-1} - (b-au)v^{n-1}\right] \\ &= \frac{1}{u-v}\left[(b-av)u^{n+1} + (b-au)v^{n-1} + (b-av)u^{n-1} - (b-au)v^{n-1}\right] \\ &= \left[(b-av)u^{n-1}\frac{(1+u^2)}{u-v}\right] \\ &= (b-av)u^{n-1}u \\ &= (b-av)u^n. \end{aligned}$$

Similarly, $x_n v + x_{n-1} = (b - au)v^n$. □

3. Divisibility Properties

Now we provide our proofs of some known results and some new divisibility properties. In the sequel, a, b, and c will always be as in (2.1).

Theorem 3.1. *For a given c, let $x_0 = a = 0$ and $x_1 = b = 1$ in (2.1). Then $x_n \mid x_{mn}$ for all nonnegative integers m.*

Proof. Since $a = 0$ and $b = 1$, (2.2) gives

$$x_n = \frac{1}{u-v}\left(u^n - v^n\right).$$

Thus, by Lemma 2.1 and the Binomial Theorem,

$$\begin{aligned} x_{mn} &= \frac{1}{u-v}\left(u^{mn} - v^{mn}\right) \\ &= \frac{1}{u-v}\left[(u^n)^m - (v^n)^m\right] \\ &= \frac{1}{u-v}\left[(x_n u + x_{n-1})^m - (x_n v + x_{n-1})^m\right] \\ &= \frac{1}{u-v}\left[\sum_{i=0}^{m}\binom{m}{i}(x_n u)^i x_{n-1}^{m-i} - \sum_{i=0}^{m}\binom{m}{i}(x_n v)^i x_{n-1}^{m-i}\right] \\ &= \frac{1}{u-v}\left[\sum_{i=1}^{m}\binom{m}{i}x_n^i x_{n-1}^{m-i}(u^i - v^i)\right] \\ &= \sum_{i=1}^{m}\binom{m}{i}x_n^i x_{n-1}^{m-i} x_i \\ &= x_n \sum_{i=1}^{m}\binom{m}{i}x_n^{i-1} x_{n-1}^{m-i} x_i. \end{aligned} \tag{3.1}$$

The result follows since $\sum_{i=1}^{m}\binom{m}{i}x_n^{i-1}x_{n-1}^{m-i}x_i$ is an integer. □

Notice that (3.1) shows what the quotient would be if x_{mn} is divided by x_n.

Corollary 3.2. *Let $c = 1$ in (2.1). Then $F_n \mid F_{mn}$.*

In [1], the authors gave several proofs that $F_n \mid F_{mn}$. Their first proof provided the quotient $M = L_{(m-1)n} + (-1)^n L_{(m-3)n} + (-1)^{2n} L_{(m-5)n} + \cdots$, if m is even, and $M = (-1)^{\frac{(m-1)n}{2}} + L_{(m-1)n} + (-1)^n L_{(m-3)n} + (-1)^{2n} L_{(m-5)n} + \cdots$, if m is odd. By comparing the quotient from our proof to M, the following corollary follows.

Corollary 3.3.

$$\sum_{i=1}^{m} \binom{m}{i} F_n^{i-1} F_{n-1}^{m-i} F_i = L_{(m-1)n} + (-1)^n L_{(m-3)n} + (-1)^{2n} L_{(m-5)n} + \cdots$$

$$= \sum_{k=1}^{\frac{m}{2}} (-1)^{(k-1)n} L_{(m-[2k-1])n}, \tag{3.2}$$

if m is even.

$$\sum_{i=1}^{m} \binom{m}{i} F_n^{i-1} F_{n-1}^{m-i} F_i = (-1)^{\frac{(m-1)n}{2}} + L_{(m-1)n} + (-1)^n L_{(m-3)n} + (-1)^{2n} L_{(m-5)n} + \cdots$$

$$= (-1)^{\frac{(m-1)n}{2}} + \sum_{k=1}^{\frac{m-1}{2}} (-1)^{(k-1)n} L_{(m-[2k-1])n}, \tag{3.3}$$

if m is odd.

Example 3.4. *If F_{10n} is divided by F_n, the quotient is*

$$\sum_{i=1}^{10} \binom{10}{i} F_n^{i-1} F_{n-1}^{10-i} F_i = L_{9n} + (-1)^n L_{7n} + L_{5n} + (-1)^n L_{3n} + L_n.$$

Example 3.5. *If F_{3n} is divided by F_n, the quotient is*

$$\sum_{i=1}^{3} \binom{3}{i} F_n^{i-1} F_{n-1}^{3-i} F_i = (-1)^n + L_{2n}.$$

Corollary 3.6. *If $c = 2$ in (2.1), then $P_n \mid P_{mn}$.*

Theorem 3.7. $F_n \mid \left(L_{mn} - 2F_{n-1}^m\right)$.

Proof. Proceeding as in the proof of Theorem 3.1,

$$L_{mn} = \sum_{i=0}^{m} \binom{m}{i} F_n^i F_{n-1}^{m-i} L_i = 2F_{n-1}^m + \sum_{i=1}^{m} \binom{m}{i} F_n^i F_{n-1}^{m-i} L_i. \tag{3.4}$$

So, $L_{mn} - 2F_{n-1}^m = F_n \sum_{i=1}^{m} \binom{m}{i} F_n^{i-1} F_{n-1}^{m-i} L_i$. The result follows since $\sum_{i=1}^{m} \binom{m}{i} F_n^{i-1} F_{n-1}^{m-i} L_i$ is an integer. □

Theorem 3.8. $P_n \mid \left(Q_{mn} - 2P_{n-1}^m\right)$.

Proof. Since

$$Q_{mn} = \sum_{i=0}^{m} \binom{m}{i} P_n^i P_{n-1}^{m-i} Q_i = 2P_{n-1}^m + P_n \sum_{i=1}^{m} \binom{m}{i} P_n^{i-1} P_{n-1}^{m-i} Q_i, \tag{3.5}$$

the result follows. □

Lemma 3.9. *Let $a = 2$ and $b = c$ in (2.1). Then*

(i) $x_{mn} = u^{mn} + v^{mn}$ *and*
(ii) $x_{mn} = \frac{1}{(u-v)^m}\left(\sum_{i=0}^{m}\binom{m}{i}x_n^i x_{n-1}^{m-i}[u^i + (-1)^m v^i]\right)$.

Proof. If $a = 2$ and $b = c$, then simple calculations yield $b - av = u - v$ and $b - au = -(u - v)$. Thus,

$$\begin{aligned} x_{mn} &= \frac{1}{(u-v)}\left([b-av]u^{mn} - [b-au]v^{mn}\right) \\ &= u^{mn} + v^{mn} \\ &= \left(\frac{x_n u + x_{n-1}}{u-v}\right)^m + \left(\frac{x_n v + x_{n-1}}{-(u-v)}\right)^m \\ &= \frac{1}{(u-v)^m}\left(\sum_{i=0}^{m}\binom{m}{i}(x_n u)^i x_{n-1}^{m-i} + (-1)^m\sum_{i=0}^{m}\binom{m}{i}(x_n v)^i x_{n-1}^{m-i}\right) \\ &= \frac{1}{(u-v)^m}\left(\sum_{i=0}^{m}\binom{m}{i}x_n^i x_{n-1}^{m-i}[u^i + (-1)^m v^i]\right). \end{aligned}$$

The proof of the lemma is complete. □

Theorem 3.10. *Let $a = 2$ and $b = c = 1$ in (2.1), and assume m is odd. Then $L_n \mid L_{mn}$.*

Proof. The assumptions imply that $x_n = L_n$. Since m is odd, $m - 1 = 2j$, where j is a nonnegative integer. Now Lemma 2.1 yields

$$\begin{aligned} L_{mn} &= \frac{1}{(u-v)^m}\left(\sum_{i=0}^{m}\binom{m}{i}L_n^i L_{n-1}^{m-i}[u^i + (-1)^m v^i]\right) \\ &= \frac{1}{(u-v)^{m-1}}\left(\sum_{i=1}^{m}\binom{m}{i}L_n^i L_{n-1}^{m-i}\left[\frac{u^i - v^i}{u-v}\right]\right) \\ &= \frac{1}{(u-v)^{2j}}\left(\sum_{i=1}^{m}\binom{m}{i}L_n^i L_{n-1}^{m-i}F_i\right) \\ &= \frac{1}{5^j}\left(\sum_{i=1}^{m}\binom{m}{i}L_n^i L_{n-1}^{m-i}F_i\right). \end{aligned}$$

Thus, $5^j L_{mn} = \sum_{i=1}^{m}\binom{m}{i}L_n^i L_{n-1}^{m-i}F_i$. Since L_n divides the right-hand side and since L_n and 5 are relatively prime [4], $L_n \mid L_{mn}$. □

A similar argument yields the following theorem.

Theorem 3.11. *Let $a = b = c = 2$ in (2.1) and assume m is odd. Then $Q_n \mid Q_{mn}$.*

Proof. The assumptions imply that $x_n = Q_n$. Let $m = 2j+1$, where j is a nonnegative integer. Lemma 3.9 yields

$$\begin{aligned} Q_{mn} &= \frac{1}{(u-v)^m}\left(\sum_{i=0}^{m}\binom{m}{i}Q_n^iQ_{n-1}^{m-i}[u^i+(-1)^m v^i]\right) \\ &= \frac{1}{(u-v)^{m-1}}\left(\sum_{i=1}^{m}\binom{m}{i}Q_n^iQ_{n-1}^{m-i}\left[\frac{u^i-v^i}{u-v}\right]\right) \\ &= \frac{1}{(u-v)^{2j}}\left(\sum_{i=1}^{m}\binom{m}{i}Q_n^iQ_{n-1}^{m-i}P_i\right) \\ &= \frac{1}{8^j}\left(\sum_{i=1}^{m}\binom{m}{i}Q_n^iQ_{n-1}^{m-i}P_i\right). \end{aligned}$$

Thus, $8^jQ_{mn} = \sum_{i=1}^{m}\binom{m}{i}Q_n^iQ_{n-1}^{m-i}P_i$. Induction arguments using the recurrence formula for Q_n show that Q_n is even and $\frac{1}{2}Q_n$ is odd. Since Q_n divides $\sum_{i=1}^{m}\binom{m}{i}Q_n^iQ_{n-1}^{m-i}P_i$ and since $\frac{1}{2}Q_n$ and 8 are relatively prime, $Q_n \mid Q_{mn}$. □

Lemma 3.12. *Let $a = 2$ and $b = c$ in (2.1). If $m = 2j$, where j is a nonnegative integer, then* $x_n \Big| \left([c^2+4]^j x_{2jn} - 2x_{n-1}^{2j}\right)$.

Proof. By Lemma 3.9,

$$x_{mn} = \frac{1}{(u-v)^m}\left(\sum_{i=0}^{m}\binom{m}{i}x_n^ix_{n-1}^{m-i}[u^i+v^i]\right).$$

Thus,

$$\begin{aligned} (c^2+4)^j x_{2jn} &= \sum_{i=0}^{m}\binom{m}{i}x_n^ix_{n-1}^{m-i}x_i \\ &= 2x_{n-1}^m + \sum_{i=1}^{m}\binom{m}{i}x_n^ix_{n-1}^{m-i}x_i, \end{aligned}$$

and so

$$(c^2+4)^j x_{2jn} - 2x_{n-1}^{2j} = x_n\sum_{i=1}^{2j}\binom{2j}{i}x_n^{i-1}x_{n-1}^{2j-i}x_i.$$

Since $\sum_{i=1}^{2j}\binom{2j}{i}x_n^{i-1}x_{n-1}^{2j-i}x_i$ is an integer, the lemma follows. □

Corollary 3.13. *If $b = c = 1$ in (2.1), then* $L_n\Big|\left(5^jL_{2jn} - 2L_{n-1}^{2j}\right)$ *for $n \geq 1$ and $j \geq 0$.*

Corollary 3.14. *If $b = c = 2$ in (2.1), then* $Q_n\Big|\left(8^jQ_{2jn} - 2Q_{n-1}^{2j}\right)$ *for $n \geq 1$ and $j \geq 0$.*

Remark. The definition of the Pell-Lucas numbers is not consistent in the literature. Some authors define the Pell-Lucas numbers by the recurrence relation

$$x_{n+2} = 2x_{n+1} + x_n$$

with initial conditions

$$x_0 = 1 \qquad \text{and} \qquad x_1 = 1$$

($a = b = 1$ and $c = 2$). Using these values, the first seven Pell-Lucas numbers are $1, 1, 3, 7, 17, 41, 99$. Theorem 3.11 and Corollary 3.14 also hold for this sequence.

4. Acknowledgment

The authors would like to thank the anonymous referee for pointing out references [1] and [2], and for detailed comments and suggestions that improved the clarity and presentation of this paper to a great extent.

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MSC2010: 11B39, 11B99

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