VIETA POLYNOMIALS AND THEIR CLOSE RELATIVES

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Abstract. We will investigate Vieta and related polynomials, and bridges linking them. We will then employ the links for extracting interesting properties of Vieta and related families.

1. Introduction

Gibonacci polynomials $g_n(x)$ are defined by the recurrence $g_n(x) = xg_{n-1}(x) + g_{n-2}(x)$, where x is an arbitrary complex variable, $g_0(x)$ and $g_1(x)$ are arbitrary complex polynomials, and $n \geq 2$. When $g_0(x) = 0$ and $g_1(x) = 1$, $g_n(x) = f_n(x)$, the nth *Fibonacci polynomial*; and when $g_0(x) = 2$ and $g_1(x) = x$, $g_n(x) = l_n(x)$, the nth *Lucas polynomial*. In particular, $f_n(1) = F_n$, the nth Fibonacci number; and $l_n(1) = L_n$, the nth Lucas number [1, 6, 13].

Pell polynomials $p_n(x)$ and *Pell-Lucas polynomials* $q_n(x)$ are defined by $p_n(x) = f_n(2x)$ and $q_n(x) = l_n(2x)$, respectively. The *Pell numbers* P_n and *Pell-Lucas numbers* Q_n are given by $P_n = p_n(1) = f_n(2)$ and $2Q_n = q_n(1) = f_n(2)$, respectively [5, 8].

Jacobsthal polynomials $J_n(x)$ and *Jacobsthal-Lucas polynomials* $j_n(x)$ satisfy the recurrence $z_n(x) = z_{n-1}(x) + xz_{n-2}(x)$, where $n \geq 2$. When $z_0(x) = 0$ and $z_1(x) = 1$, $z_n(x) = J_n(x)$; and when $z_0 = 2$ and $z_1 = 1$, $z_n(x) = j_n(x)$ [2, 3]. Clearly, $J_n(1) = F_n$; and $j_n(1) = L_n$ and hence the name Jacobsthal-Lucas polynomials for $j_n(x)$. The numbers $J_n = J_n(2)$ and $j_n = j_n(2)$ are the nth Jacobsthal and Jacobsthal-Lucas numbers, respectively.

The Chebyshev family satisfies the recurrence $z_n(x) = 2xz_{n-1} - z_{n-2}(x)$, where $n \geq 2$. When $z_0(x) = 1$ and $z_1(x) = x$, $z_n(x) = T_n(x)$, the nth *Chebyshev polynomial of the first kind*; and when $z_0(x) = 1$ and $z_1(x) = 2x$, $z_n(x) = U_n(x)$, the nth *Chebyshev polynomial of the second kind* [4, 8, 10].

Interestingly, the numbers $a_{nk} = \binom{n-k-1}{k}$ appear in one set of explicit formulas for $f_n(x)$, $p_n(x)$, $J_n(x)$, and $U_n(x)$. Likewise, the numbers $b_{nk} = \frac{n}{n-k}\binom{n-k}{k}$ appear in their counterparts for $l_n(x)$, $q_n(x)$, $j_n(x)$, and $T_n(x)$ [1, 6, 3, 5, 8, 11]. Robbins attributes the discovery of array (b_{nk}) to the French mathematician François Vièta (1540–1603) [4, 11].

The numbers $(-1)^k a_{nk}$ and $(-1)^k b_{nk}$ can be used to study two additional and related classes of polynomials: *Vieta polynomials* $V_n(x)$ and *Vieta-Lucas polynomials* $v_n(x)$. They satisfy the recurrence $h_n(x) = xh_{n-1}(x) - h_{n-2}(x)$, where $n \geq 2$. When $h_0(x) = 0$ and $h_1(x) = 1$, $h_n(x) = V_n(x)$; and when $h_0(x) = 2$ and $h_1(x) = x$, $h_n(x) = v_n(x)$. E. Jacobsthal, N. Robbins, and A. G. Shannon and A. F. Horadam studied them extensively [4, 11].

In the interest of brevity and convenience, we will omit the argument in the functional notation, when there is no ambiguity; so g_n will mean $g_n(x)$.

In this article, we will find a close relationship between V_n and f_n; V_n and p_n; V_n and J_n; V_n and U_n; and between v_n and l_n; v_n and q_n; v_n and j_n; and v_n and T_n. We will then employ them to extract interesting properties of Vieta and Vieta-Lucas polynomials.

2. Vieta Links with Other Families

Let $i = \sqrt{-1}$. Clearly, $i^{n-1}f_n$ satisfies the Vieta recurrence. This, coupled with the initial conditions $V_0(ix) = 0 = f_0$ and $v_1(ix) = 1 = f_1$, implies that $V_n(ix) = i^{n-1}f_n$. Consequently,

$$V_n(x) = i^{n-1}f_n(-ix). \tag{2.1}$$

Likewise, we have

$$v_n(x) = i^n l_n(-ix) \tag{2.2}$$
$$V_n(x) = i^{n-1}p_n(-ix/2) \tag{2.3}$$
$$v_n(x) = i^{n-1}q_n(-ix/2) \tag{2.4}$$
$$V_n(x) = x^{n-1}J_n(-1/x^2) \tag{2.5}$$
$$v_n(x) = x^n j_n(-1/x^2) \tag{2.6}$$
$$V_n(x) = U_{n-1}(x/2) \tag{2.7}$$
$$v_n(x) = 2T_n(x/2). \tag{2.8}$$

For example, since $f_{n+1} = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} x^{n-2k}$, it follows by (2.1) that

$$V_{n+1} = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n-k}{k} x^{n-2k}.$$

2.1. **Jacobsthal-Fibonacci-Lucas Links.**

It follows from identities (2.1) and (2.5) that

$$x^n J_{n+1}(-1/x^2) = i^n f_{n+1}(-ix).$$

Replacing x with $i/\sqrt{x}$ yields

$$J_{n+1}(x) = x^{n/2} f_{n+1}(1/\sqrt{x}). \tag{2.9}$$

Likewise,

$$j_n(x) = x^{n/2} l_n(1/\sqrt{x}). \tag{2.10}$$

It follows from identity (2.9) that $J_{n+1}(1) = F_{n+1}$ and $J_{n+1} = 2^{n/2} f_{n+1}(1/\sqrt{2})$. Similarly, $j_n(1) = L_n$ and $j_n = 2^{n/2} l_n(1/\sqrt{2})$.

2.2. **Jacobsthal-Chebyshev Links.**

Identities (2.5) and (2.8) imply that

$$x^n J_{n+1}(-1/x^2) = U_n(x/2).$$

Consequently,

$$J_{n+1}(x) = (-i\sqrt{x})^n U_n(i/2\sqrt{x}). \tag{2.11}$$

Similarly,

$$j_n(x) = 2(-i\sqrt{x})^n T_n(i/2\sqrt{x}). \tag{2.12}$$

It follows by identity (2.11) that $F_{n+1} = (-i)^n U_n(i/2)$ and $J_{n+1} = (-\sqrt{2}i)^n U_n(i/2\sqrt{2})$. Similarly, $L_n = 2(-i)^n T_n(i/2)$ and $j_n = 2(-\sqrt{2}i)^n T_n(i/2\sqrt{2})$.

Next we will extract a few interesting Vieta identities and their byproducts.

3. Interesting Vieta Properties

To begin with, suppose we replace x with $-ix$ in the well-known identity $l_n^2 - (x^2+4)f_n^2 = 4(-1)^n$. By identities (2.1) and (2.2), it then yields

$$v_n^2 - (x^2-4)V_n^2 = 4. \tag{3.1}$$

It follows by identity (3.1) that (v_n, V_n) is a solution of the (Pell's) equation $u^2-(x^2-4)v^2 = 4$, where x is an integer ≥ 3 and is nonsquare.

The next theorem establishes links between V_n and f_{2n}, and v_n and l_{2n}.

Theorem 3.1. *Let $n \geq 0$. Then*

$$xV_n(x^2+2) = f_{2n} \tag{3.2}$$

$$v_n(x^2+2) = l_{2n}. \tag{3.3}$$

Proof. We will establish identity (3.2) using induction. Clearly, it is true when $n = 0$ and $n = 1$.

Now assume it is true for all nonnegative integers $< n$. Since $f_{2n} = (x^2+2)f_{2n-2} - f_{2n-4}$, by the Vieta recurrence we have

$$\begin{aligned} V_n(x^2+2) &= (x^2+2)V_{n-1}(x^2+2) - V_{n-2}(x^2+2) \\ &= (x^2+2)\cdot\frac{1}{x}f_{2n-2} - \frac{1}{x}f_{2n-4} \\ xV_n(x^2+2) &= f_{2n}. \end{aligned}$$

So the given result is true for n also. Thus, by induction, it is true for all $n \geq 0$.

Identity (3.3) follows similarly. □

The next corollary follows from Theorem 3.1.

Corollary 1 (Shannon and Horadam, 1999)**.** *Let $n \geq 0$. Then $V_n(3) = F_{2n}$ and $v_n(3) = L_{2n}$.*

Theorem 3.1 also yields the following results.

Corollary 2.

$$p_{2n}(x) = 2xV_n(4x^2+2) \tag{3.4}$$

$$q_{2n}(x) = v_n(4x^2+2). \tag{3.5}$$

The next two results follow from Corollary 2.

Corollary 3. *Let $n \geq 0$. Then $P_{2n} = 2V_n(6)$ and $2Q_{2n} = v_n(6)$.*

It also follows by Theorem 3.1, and identities (2.9) and (2.10) that

$$J_{2n}(x) = x^{n-1}V_n\left(\frac{2x+1}{x}\right) \tag{3.6}$$

$$j_{2n}(x) = x^n v_n\left(\frac{2x+1}{x}\right). \tag{3.7}$$

Identities (3.6) and (3.7) imply that $J_{2n} = 2^{n-1}V_n(5/2)$ and $j_{2n} = 2^n v_n(5/2)$.

The next theorem presents two charming identities involving Vieta polynomials.

Theorem 3.2. *Let $n \geq 0$. Then*

$$v_n(x^2-2) - (x^2-4)V_n^2(x) = 2 \tag{3.8}$$

$$v_n(x^2-2) - v_n^2(x) = -2. \tag{3.9}$$

Proof. We have $v_n(-x) = (-1)^n v_n(x)$, and $V_n(x) = i^{n-1} f_n(-ix)$. Since $v_n(x^2+2) = l_{2n}(x)$, $v_n(-x^2+2) = l_{2n}(-ix)$; that is, $(-1)^n v_n(x^2-2) = l_{2n}(-ix)$. Since $l_{2n}(u) = (u^2+4)f_n^2(u) + 2(-1)^n$, we then have

$$(-1)^n v_n(x^2-2) = -(x^2-4)\frac{V_n^2(x)}{i^{2n-2}} + 2(-1)^n.$$

This yields identity (3.8).

Identity (3.9) follows similarly. □

Horadam attributes identity (3.9) to Jacobsthal [4].

It follows from identities (3.8) and (3.9) that $v_n(2) = 2$. It also follows from them that $Q_{2n} = 4P_n^2 + (-1)^n = 2Q_n^2 - (-1)^n$.

Theorem 3.2 has interesting Pell consequences, as the next two corollaries show.

Corollary 4. *Let $n \geq 0$. Then*

$$q_{2n}(x) - 4(x^2+1)p_n^2(x) = 2(-1)^n \tag{3.10}$$

$$q_{2n}(x) - q_n^2(x) = 2(-1)^{n+1}. \tag{3.11}$$

Proof. We have $v_n(-x) = (-1)^n v_n(x), V_n(ix) = i^{n-1} p_n(x/2)$, and $q_{2n}(x) = v_n(4x^2+2)$. It then follows by identity (3.8) that

$$\begin{aligned} v_n(-x^2-2) + (x^2+4)V_n^2(ix) &= 2 \\ (-1)^n v_n(x^2+2) + (x^2+4)\cdot i^{2n-2} p_n^2(x/2) &= 2 \\ q_{2n}(x/2) - (x^2+4)p_n^2(x/2) &= 2(-1)^n \\ q_{2n}(x) - 4(x^2+1)p_n^2(x) &= 2(-1)^n. \end{aligned}$$

Identity (3.11) can be established similarly. □

The next corollary follows from identities (3.10) and (3.11).

Corollary 5. *Let $n \geq 0$. Then $q_n^2 - 4(x^2+1)p_n^2 = 4(-1)^n$.*

This corollary has a magnificent byproduct. It follows by the corollary that $Q_n^2 - 2P_n^2 = (-1)^n$. Consequently, (Q_n, P_n) is a solution of the Pell's equation $u^2 - 2v^2 = (-1)^n$; its converse is also true [8].

Theorem 3.1, coupled with Theorem 3.2, yields the next theorem; it provides a link between $f_{2n}(x)$ and $J_n(x)$, and $l_{2n}(x)$ and $j_n(x)$.

Theorem 3.3. *Let $n \geq 0$. Then*

$$f_{2n}(x) = x(x^2+2)^{n-1} J_n\left(-\frac{1}{(x^2+2)^2}\right) \tag{3.12}$$

$$l_{2n}(x) = (x^2+2)^n j_n\left(-\frac{1}{(x^2+2)^2}\right). \tag{3.13}$$

Theorem 3.3 has consequences to the Pell family, as the following corollary shows.

Corollary 6. *Let $n \geq 0$. Then*

$$p_{2n}(x) = 2x(4x^2+2)^{n-1} J_n\left(-\frac{1}{(4x^2+2)^2}\right)$$

$$q_{2n}(x) = (4x^2+2)^n j_n\left(-\frac{1}{(4x^2+2)^2}\right).$$

The next corollary follows from Theorem 3.3 and Corollary 6.

Corollary 7. *Let $n \geq 0$. Then*

$$F_{2n} = 3^{n-1} J_n(-1/9) \qquad L_{2n} = 3^n j_n(-1/9)$$
$$P_{2n} = 2 \cdot 6^{n-1} J_n(-1/36) \qquad 2Q_{2n} = 6^n j_n(-1/36).$$

Theorems 3.2 and 3.3 together yield the next two results. Their proofs are straightforward.

Theorem 3.4. *Let $n \geq 0$. Then*

$$(x^2-2)^n j_n\left(-\frac{1}{(x^2-2)^2}\right) - x^{2n-2}(x^2-4)J_n^2\left(-\frac{1}{x^2}\right) = 2$$
$$(x^2-2)^n j_n\left(-\frac{1}{(x^2-2)^2}\right) - x^{2n} j_n^2\left(-\frac{1}{x^2}\right) = -2.$$

The next result follows from Theorem 3.4.

Corollary 8. *Let $n \geq 0$. Then*

$$x^{2n} j_n^2(-1/x^2) - x^{2n-2}(x^2-4)J_n^2(-1/x^2) = 4. \tag{3.14}$$

Identity (3.14) implies

$$j_n^2(x) - (4x+1)J_n^2(x) = 4(-x)^n.$$

Since $j_n(2) = j_n$ and $J_n(2) = J_n$, identity (3.14) yields the following result, linking Jacobsthal and Jacobsthal-Lucas numbers.

Corollary 9. *Let $n \geq 0$. Then*

$$j_n^2 - 9J_n^2 = 4(-2)^n. \tag{3.15}$$

Identity (3.15) has a delightful byproduct. Since $3J_{2n} = 4^n - 1$ and $j_{2n} = 4^n + 1$, it implies that $3J_{2n} - 2^{n+1} - j_{2n} = (4^n - 1) - 2^{n+1} - (4^n + 1)$ is a Pythagorean triple; clearly, it is primitive. The area of the Pythagorean triangle is $3 \cdot 2^n J_{2n} = 2^n(4^n - 1)$.

Next we will investigate a close relationship between Vieta and Chebyshev polynomials.

3.1. **Vieta-Chebyshev Bridges.**

Using the links between Vieta and Chebyshev families, we can translate Vieta identities into Chebyshev ones, and vice versa.

For example, the Vieta identity $v_n^2 - (x^2-4)V_n^2 = 4$ can be translated into a Chebyshev identity:

$$4T_n^2(x/2) - (x^2-4)U_{n-1}^2(x/2) = 4$$
$$T_n^2(x) - (x^2-1)U_{n-1}^2(x) = 1.$$

It follows by Theorem 3.1 that

$$f_{2n}(x) = xU_{n-1}\left(\frac{x^2+2}{2}\right) \tag{3.16}$$

$$l_{2n}(x) = 2T_n\left(\frac{x^2+2}{2}\right). \tag{3.17}$$

Next we will investigate some properties linking Jacobsthal and Chebyshev polynomials.

3.2. **Jacobsthal-Chebyshev Bridges.** It follows by identities (2.5)–(2.7) that

$$2T_n(x) = (2x)^n j_n(-1/4x^2) \tag{3.18}$$

$$U_n(x) = (2x)^n J_{n+1}(-1/4x^2). \tag{3.19}$$

Interestingly, we can compute Jacobsthal and Jacobsthal-Lucas numbers from Chebyshev polynomials. Letting $x = \dfrac{-i}{2\sqrt{2}}$ in identity (3.18), we get

$$\left(\frac{-2i}{2\sqrt{2}}\right)^n j_n(2) = 2T_n\left(\frac{-i}{2\sqrt{2}}\right)$$

$$j_n = 2(\sqrt{2}i)^n T_n\left(\frac{-i}{2\sqrt{2}}\right). \tag{3.20}$$

Similarly,

$$J_{n+1} = (\sqrt{2}i)^n U_n\left(\frac{-i}{2\sqrt{2}}\right). \tag{3.21}$$

Next we will focus on a charming gibonacci identity. We will find its Vieta counterpart, and employ it to extract the corresponding Jacobsthal, Chebyshev, and Pell identities.

4. Two Charming Vieta Identities

Consider the gibonacci identity [7]

$$g_{n+k}^3 - (-1)^k l_k g_n^3 + (-1)^k g_{n-k}^3 = \begin{cases} f_k f_{2k} g_{3n} & \text{if } g_i = f_i \\ (x^2+4) f_k f_{2k} g_{3n} & \text{if } g_i = l_i. \end{cases} \tag{4.1}$$

The next theorem gives its equally beautiful counterpart for Vieta polynomials. The proof is really short and neat, and hinges on identities (2.1) and (2.2).

Theorem 4.1.

$$h_{n+k}^3 - v_k h_n^3 + h_{n-k}^3 = \begin{cases} h_k h_{2k} h_{3n} & \text{if } h_i = V_i \\ (x^2-4) V_k V_{2k} h_{3n} & \text{if } h_i = v_i. \end{cases} \tag{4.2}$$

Proof. Suppose $h_i = V_i$ and $g_i = f_i$. By identity (4.1), we have

$$f_{n+k}^3 - (-1)^k l_k f_n^3 + (-1)^k f_{n-k}^3 = f_k f_{2k} f_{3n}.$$

Now replace x with $-ix$ and multiply the resulting equation with i^{3n+3k}, where $i = \sqrt{-1}$. Since $V_n(x) = i^{n-1} f_n(-ix)$, this yields

$$-iV_{n+k}^3 + iv_k V_n^3 - iV_{n-k}^3 = -iV_k V_{2k} V_{3n}$$

$$V_{n+k}^3 - v_k V_n^3 + V_{n-k}^3 = V_k V_{2k} V_{3n}. \tag{4.3}$$

On the other hand, let $h_i = v_i$. Again, by identity (4.1), we have

$$l_{n+k}^3 - (-1)^k l_k l_n^3 + (-1)^k l_{n-k}^3 = (x^2+4) f_k f_{2k} l_{3n}.$$

Since $v_n(x) = i^n l_n(-ix)$, as before, this yields

$$v_{n+k}^3 - v_k v_n^3 + v_{n-k}^3 = (x^2-4) V_k V_{2k} v_{3n}. \tag{4.4}$$

Combining identities (4.3) and (4.4), we get the desired result. □

It follows by identity (4.2) that

$$h_{n+1}^3 - xh_n^3 + h_{n-1}^3 = \begin{cases} xh_{3n} & \text{if } h_i = V_i \\ x(x^2-4)h_{3n} & \text{if } h_i = v_i. \end{cases}$$

Since $xV_n(x^2+2) = f_{2n}$ and $v_n(x^2+2) = l_{2n}$, it also follows by identity (4.2) that

$$h_{2n+2}^3 - (x^2+2)h_{2n}^3 + h_{2n-2}^3 = \begin{cases} x^2(x^2+2)h_{6n} & \text{if } h_i = f_i \\ x^2(x^2+2)(x^2+4)h_{6n} & \text{if } h_i = l_i. \end{cases}$$

Next we will extract the Jacobsthal counterparts from identity (4.2).

4.1. **Jacobsthal Counterparts.** We have $J_n(x) = (-i\sqrt{x})^{n-1}V_n(i/\sqrt{x})$ and $j_n(x) = (-i\sqrt{x})^n v_n(i/\sqrt{x})$. Now replace x with $i/\sqrt{x}$ in equation (4.2), and multiply the resulting equation with $(-i\sqrt{x})^{3n+3k}$. We then get

$$z_{n+k}^3 - (-x)^k j_k(x) z_n^3 + (-1)^k x^{3k} z_{n-k}^3 = \begin{cases} z_k z_{2k} z_{3n} & \text{if } z_i(x) = J_i(x) \\ (4x+1)J_k(x)J_{2k}(x)z_{3n} & \text{if } z_i(x) = j_i(x). \end{cases} \tag{4.5}$$

In particular, identity (4.5) implies that

$$J_{n+k}^3 - (-2)^k j_k J_n^3 + (-8)^k J_{n-k}^3 = J_k J_{2k} J_{3n}$$
$$j_{n+k}^3 - (-2)^k j_k j_n^3 + (-8)^k j_{n-k}^3 = 9J_k J_{2k} j_{3n}.$$

Next we will find the Chebyshev and Pell counterparts of identity (4.2).

4.2. **Chebyshev and Pell Counterparts.** Since $U_n(x) = V_{n+1}(2x)$ and $2T_n(x) = v_n(2x)$, it follows from identity (4.2) that

$$z_{n+k}^3 - 2T_k z_n^3 + z_{n-k}^3 = \begin{cases} z_{k-1} z_{2k-1} z_{3n+2} & \text{if } z_i = U_i \\ (x^2-1)U_{k-1}U_{2k-1}z_{3n} & \text{if } z_i = T_i. \end{cases} \tag{4.6}$$

Likewise, we have

$$z_{n+k}^3 - (-1)^k q_k z_n^3 + (-1)^k z_{n-k}^3 = \begin{cases} z_k z_{2k} z_{3n} & \text{if } z_i = p_i \\ 4(x^2+1)p_k p_{2k} z_{3n} & \text{if } z_i = q_i. \end{cases} \tag{4.7}$$

Using these techniques, we can transform gibonacci polynomial identities to Vieta, Pell, Jacobsthal, and Chebyshev polynomial identities. For example, we invite Fibonacci enthusiasts to find the Vieta, Pell, Jacobsthal, and Chebyshev counterparts of the following gibonacci identities [9]:

$$g_{n+3}^2 = (x^2+1)g_{n+2}^2 + (x^2+1)g_{n+1}^2 - g_n^2 \tag{4.8}$$

$$g_{n+4}^3 = (x^3+2x)g_{n+3}^3 + (x^4+3x^2+2)g_{n+2}^3 - (x^3+2x)g_{n+1}^3 - g_n^3 \tag{4.9}$$

$$\begin{aligned} g_{n+4}^4 = {} & (x^4+3x^2+1)g_{n+4}^4 + (x^6+5x^4+7x^2+2)g_{n+3}^4 - \\ & (x^6+5x^4+7x^2+2)g_{n+2}^4 - (x^4+3x^2+1)g_{n+1}^4 + g_n^4. \end{aligned} \tag{4.10}$$

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