A TWO PARAMETER PELL DIOPHANTINE EQUATION THAT GENERALIZES A FIBONACCI CLASSIC

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Abstract. In this paper, we present the results of our investigation into a two parameter Pell Diophantine equation. With certain constraints on the two parameters, we present the positive integer solutions of the Pell equation in question. Indeed, assuming these constraints, we express the positive integer solutions in terms of a second order recurring sequence. For certain values of the parameters, the Pell equation in question reduces to a classic Pell equation, whose solutions are expressed in terms of Fibonacci and Lucas numbers.

1. Introduction

In this paper, we consider the two parameter Pell Diophantine equation

$$x^2 - \left(b^2 + a^2\right) y^2 = -a^2, \tag{1.1}$$

in which a and b are positive integers. We consider only those a and b for which $\sqrt{b^2 + a^2}$ is irrational, and for which $a|(2b)$. The irrationality of $\sqrt{b^2 + a^2}$ is assumed throughout, and henceforth we do not restate this condition. When the divisibility condition is assumed, we express the positive integer solutions (which we define in the paragraph that follows) of (1.1) in terms of a second order recurring sequence. Without the divisibility condition, we are unable to give the positive integer solutions of (1.1).

We take an *integer* solution (x, y) of (1.1) to be a solution in which both x and y are integers. We take a *rational* solution (x, y) of (1.1) to be a solution in which both x and y are rational. An integer solution is a rational solution, but a rational solution is not necessarily an integer solution. A positive solution is one where $x > 0$ and $y > 0$. Throughout this paper, we always indicate the type of solution that we are considering. As is customary in this topic, we refer to the solution (x, y), or to the solution $x + y\sqrt{b^2 + a^2}$, interchangeably.

In this paper, we take the Fibonacci and Lucas numbers to have starting values $(F_0, F_1) = (0, 1)$, and $(L_0, L_1) = (2, 1)$, respectively. The identity

$$L_n^2 - 5F_n^2 = 4(-1)^n, \tag{1.2}$$

which occurs in [3, p. 56], inspires the two Pell equations

$$x^2 - 5y^2 = 4, \text{ and } x^2 - 5y^2 = -4. \tag{1.3}$$

All the positive integer solutions of the first equation in (1.3) are (L_{2n}, F_{2n}), $n \geq 1$. All the positive integer solutions of the second equation in (1.3) are (L_{2n-1}, F_{2n-1}), $n \geq 1$.

Long and Jordan [6] solve the equations in (1.3) with the use of continued fractions. Later, Lind [5] solves these equations by working in the quadratic field $Q\left(\sqrt{5}\right)$. Then, in a letter to the editor, Ferguson [2] simultaneously solves these equations with a clever method of descent.

In [1], Euler and Sadek initiate study into the positive *rational* solutions of the generalized Pell equation

$$x^2 - \left(b^2 - a^2\right) y^2 = a^2, \tag{1.4}$$

with appropriate constraints on the positive integers a and b. When $b = 3$ and $a = 2$, (1.4) reduces to the first of the Pell equations in (1.3), which Euler and Sadek solve. Euler and Sadek [1, pp. 245–246], also show how the procedures outlined in their paper are used to solve the second of the Pell equations in (1.3). In [7], we extend the results of Euler and Sadek by giving all the positive integer solutions of (1.4), subject to the constraint $a|(2b)$.

In this paper, our motivation for studying (1.1) is that, for $b = 1$ and $a = 2$, this equation reduces to the second of the equations in (1.3). In Section 2, we define a second order recurring sequence that is central to our analysis of (1.1). Indeed, in Section 3, under the assumption that $a|(2b)$, we express certain positive integer solutions of (1.1) in terms of this recurring sequence. In Section 4, we exhibit all positive integer solutions of (1.1) when $a|(2b)$, with a even. In Section 5, we do likewise, under the assumption that $a|(2b)$, with a odd. Finally, in Section 6, we state our main theorem, which summarizes our findings concerning the positive integer solutions of (1.1) when $a|(2b)$.

2. A Second Order Recurring Sequence

Let $p > 0$ be a rational number. We define the sequence $\{U_n\} = \{U_n(p)\}$, for all integers n, by

$$U_n = \left(p^2 + 2\right) U_{n-1} - U_{n-2}, U_0 = 0, U_1 = 1. \tag{2.1}$$

In the sequel, we express the solutions of (1.1) with the use of terms from the sequence (2.1). We now show that $\{U_n\}$, $n \geq 0$, is a strictly increasing sequence of non-negative rational numbers. To begin, $U_1 - U_0 > 0$ and $U_1 > 0$. Now suppose that, for some integer $k \geq 1$, $U_k - U_{k-1} > 0$, and $U_k > 0$. Then

$$U_{k+1} - U_k = \left(p^2 + 2\right) U_k - U_{k-1} - U_k = U_k - U_{k-1} + p^2 U_k. \tag{2.2}$$

By induction on k, we can therefore conclude that $\{U_n\}$, $n \geq 0$, is indeed a strictly increasing sequence of non-negative rational numbers.

We also require the identity

$$U_n^2 - U_{n-1}U_{n+1} = 1, n \geq 0, \tag{2.3}$$

which we now prove. It is true that $U_0^2 - U_{-1}U_1 = 1$, and $U_1^2 - U_0U_2 = 1$. Suppose, for some integer $k \geq 1$, that $U_k^2 - U_{k-1}U_{k+1} = 1$. Then

$$\begin{aligned} U_{k+1}^2 - U_kU_{k+2} &= U_{k+1}^2 - U_k\left(\left(p^2 + 2\right) U_{k+1} - U_k\right) \\ &= U_k^2 + U_{k+1}\left(U_{k+1} - \left(p^2 + 2\right) U_k\right) \\ &= U_k^2 - U_{k-1}U_{k+1} \\ &= 1, \end{aligned}$$

and so (2.3) follows by induction.

3. The Condition $a|(2b)$ and Positive Integer Solutions of (1.1)

In (1.1), assume that $a|(2b)$. Also, in the sequence (2.1), let $p = \frac{2b}{a}$. Then, with our analysis in Section 2 in mind, we state and prove a lemma that gives certain positive integer solutions of (1.1) in terms of the sequence (2.1).

Lemma 3.1. *Suppose $a|(2b)$, and in the sequence (2.1), let $p = \frac{2b}{a}$. Then*

$$(x_n, y_n) = \left(b\left(U_n + U_{n-1}\right), U_n - U_{n-1}\right), n \geq 1, \tag{3.1}$$

are positive integer solutions of (1.1).

Proof. In (3.1), x_n and y_n are positive integers. In (1.1), transposing a^2 to the left, and substituting $x = b\left(U_n + U_{n-1}\right)$ and $y = U_n - U_{n-1}$, we obtain upon expansion

$$2\left(a^2 + 2b^2\right) U_n U_{n-1} + a^2\left(1 - U_n^2 - U_{n-1}^2\right). \tag{3.2}$$

In (2.1), replace p by $\frac{2b}{a}$, then multiply through by a^2 and rearrange terms to obtain

$$2\left(a^2 + 2b^2\right) U_{n-1} = a^2\left(U_n + U_{n-2}\right). \tag{3.3}$$

In (3.3), multiply both sides by U_n, and use the result to substitute for the leftmost product in (3.2). The expression in (3.2) then becomes

$$-a^2\left(U_{n-1}^2 - U_{n-2}U_n - 1\right). \tag{3.4}$$

With the use of (2.3), we see that the expression in (3.4) reduces to zero. This completes the proof of Lemma 3.1. □

4. All Positive Integer Solutions of (1.1) When $a|(2b)$ with a Even

Let c be a positive integer, and consider the Pell equation

$$x^2 - \left(c^2 + 4\right) y^2 = -4. \tag{4.1}$$

We begin with a known result that occurs as part of Theorem 3 in [4]. In [4], the authors define (in notation adapted for the present paper) the non-negative integer sequence

$$r_n = cr_{n-1} + r_{n-2}, r_0 = 0, r_1 = 1.$$

They then prove that all the positive integer solutions of (4.1) are given by

$$(r_{n+1} + r_{n-1}, r_n), n = 1, 3, 5, \ldots.$$

The solutions given in the previous line are precisely

$$(r_{2n} + r_{2n-2}, r_{2n-1}), n \geq 1.$$

It is easy to prove by induction that $r_{2n} = cU_n(c)$, $n \geq 1$. Keeping in mind that $cr_{2n-1} = r_{2n} - r_{2n-2}$, we see that the aforementioned result of Hoggatt and Bicknell [4] translates as the following theorem.

Theorem 4.1. *The only positive integer solutions of (4.1) are given by*

$$\left(c\left(U_n(c) + U_{n-1}(c)\right), U_n(c) - U_{n-1}(c)\right), n \geq 1. \tag{4.2}$$

We now consider (1.1), where $a|(2b)$ with a even. Accordingly, set $a = 2m$ for m a positive integer. Then $a|(2b) \Rightarrow m|b \Rightarrow b = mc$, for c a positive integer. Then the Pell equation (1.1) becomes

$$x^2 - m^2\left(c^2 + 4\right) y^2 = -4m^2. \tag{4.3}$$

Our next lemma gives a connection between the solutions of (4.1) and the solutions of (4.3).

Lemma 4.2. *The mapping $(x, y) \to (mx, y)$ takes positive integer solutions of (4.1) to positive integer solutions of (4.3). The mapping $(x, y) \to \left(\frac{x}{m}, y\right)$ takes positive integer solutions of (4.3) to positive integer solutions of (4.1).*

Proof. Suppose (x_0, y_0) is a positive integer solution of (4.1). Then $x_0^2 - \left(c^2 + 4\right) y_0^2 = -4$, so that $m^2 x_0^2 - m^2\left(c^2 + 4\right) y_0^2 = -4m^2$. That is, (mx_0, y_0) is a positive integer solution of (4.3).

Now suppose (x_0, y_0) is a positive integer solution of (4.3). Then substitution gives $x_0^2 - m^2\left(c^2 + 4\right) y_0^2 = -4m^2$. This implies that $m^2|x_0^2$, and so $m|x_0$. Therefore, $\left(\frac{x_0}{m}\right)^2 - \left(c^2 + 4\right) y_0^2 = -4$, so that $\left(\frac{x_0}{m}, y_0\right)$ is a positive integer solution of (4.1). This completes the proof of Lemma 4.2. □

The mappings in Lemma 4.2 can be represented by 2×2 matrices with non-vanishing determinants. Furthermore, these matrices are inverses of one another. Therefore, based on Lemma 4.2 and Theorem 4.1, *all* the positive integer solutions of (4.3) are

$$(mc\,(U_n(c) + U_{n-1}(c))\,, U_n(c) - U_{n-1}(c))\,, n \geq 1. \tag{4.4}$$

Expressed in terms of a and b, the solutions in (4.4) are

$$\left(b\left(U_n\left(\frac{2b}{a}\right) + U_{n-1}\left(\frac{2b}{a}\right)\right), U_n\left(\frac{2b}{a}\right) - U_{n-1}\left(\frac{2b}{a}\right)\right), n \geq 1.$$

These solutions are of the same form as those presented in Lemma 3.1. Therefore, we can conclude that, when $a|(2b)$ with a even, all the positive integer solutions of (1.1) are those given in Lemma 3.1.

5. All Positive Integer Solutions of (1.1) When $a|(2b)$ with a Odd

Let c be a positive integer, and consider the Pell equation

$$x^2 - \left(c^2 + 1\right) y^2 = -1. \tag{5.1}$$

The *fundamental solution* (the smallest solution) of (5.1) is $(c, 1)$, or $c + \sqrt{c^2+1}$. By a theorem in Nagell [8, Theorem 106, page 201], we then have the following lemma.

Lemma 5.1. *All the positive integer solutions, $x_n + \sqrt{c^2+1}y_n$, of (5.1) are given by*

$$x_n + \sqrt{c^2+1}y_n = \left(c + \sqrt{c^2+1}\right)^{2n-1}, n \geq 1. \tag{5.2}$$

Now consider (1.1), where $a|(2b)$ with a odd. This divisibility condition implies that $a|b$. Accordingly, set $b = ac$ for c a positive integer. The equation (1.1) then becomes

$$x^2 - a^2\left(c^2 + 1\right) y^2 = -a^2. \tag{5.3}$$

Our next lemma gives a connection between the solutions of (5.1) and the solutions of (5.3). Since the proof of this lemma follows the same lines as the proof of Lemma 4.2, we state it without proof.

Lemma 5.2. *The mapping $(x, y) \to (ax, y)$ takes positive integer solutions of (5.1) to positive integer solutions of (5.3). The mapping $(x, y) \to \left(\frac{x}{a}, y\right)$ takes positive integer solutions of (5.3) to positive integer solutions of (5.1).*

The mappings in Lemma 5.2 are of course similar in nature to the mappings in Lemma 4.2, a fact that we soon exploit.

Next, we show that the positive integer solutions of (5.1), as given in (5.2), are precisely those given in (3.1). That is, for the sequence $\{U_n\} = \{U_n(2c)\}$, defined by

$$U_n = \left(4c^2 + 2\right) U_{n-1} - U_{n-2}, U_0 = 0, U_1 = 1, \tag{5.4}$$

we show that, for $n \geq 1$,

$$\left(c + \sqrt{c^2+1}\right)^{2n-1} = c\,(U_n + U_{n-1}) + (U_n - U_{n-1})\sqrt{c^2+1}. \tag{5.5}$$

That (5.5) is true for $n = 1$ is immediate. Now suppose that (5.5) is true for $n = k \geq 1$. Then

$$\left(c + \sqrt{c^2+1}\right)^{2k+1} = \left(c + \sqrt{c^2+1}\right)^2 \left(c + \sqrt{c^2+1}\right)^{2k-1}. \tag{5.6}$$

By the induction assumption, the right side of (5.6) is

$$\begin{aligned}&\left(2c^2+1+2c\sqrt{c^2+1}\right)\left(c\left(U_k+U_{k-1}\right)+\left(U_k-U_{k-1}\right)\sqrt{c^2+1}\right)\\&=\left(4c^3+3c\right)U_k-cU_{k-1}+\left(\left(4c^2+1\right)U_k-U_{k-1}\right)\sqrt{c^2+1}\\&=c\left(U_{k+1}+U_k\right)+\left(U_{k+1}-U_k\right)\sqrt{c^2+1}.\end{aligned}\tag{5.7}$$

We arrive at last line in the array (5.7) by using the recurrence (5.4) to replace each occurrence of $-U_{k-1}$ by $U_{k+1}-\left(4c^2+2\right)U_k$. This establishes (5.5).

Now, by what we have just proved, all the positive integer solutions of (5.1) are given by

$$\left(c\left(U_n(2c)+U_{n-1}(2c)\right),U_n(2c)-U_{n-1}(2c)\right),n\geq 1.\tag{5.8}$$

By Lemma 5.2, all the positive integer solutions of (5.1), given in (5.8), are mapped onto the positive integer solutions of (5.3). The positive integer solutions of (5.3) are therefore

$$\left(ac\left(U_n(2c)+U_{n-1}(2c)\right),U_n(2c)-U_{n-1}(2c)\right),n\geq 1.\tag{5.9}$$

Finally, in (5.9) we replace c by $\frac{b}{a}$ to obtain the positive integer solutions of (1.1), which are

$$\left(b\left(U_n\left(\frac{2b}{a}\right)+U_{n-1}\left(\frac{2b}{a}\right)\right),U_n\left(\frac{2b}{a}\right)-U_{n-1}\left(\frac{2b}{a}\right)\right),n\geq 1.\tag{5.10}$$

These solutions are of the same form as those presented in Lemma 3.1. Therefore, we can conclude that, when $a|(2b)$ with a odd, all the positive integer solutions of (1.1) are those given in Lemma 3.1.

6. A Summary and Concluding Comments

In the theorem that follows, we summarize our conclusions concerning the positive integer solutions of (1.1).

Theorem 6.1. *Suppose $a|(2b)$. Then, with $p=\frac{2b}{a}$ in the sequence (2.1), all the positive integer solutions of (1.1) are given by*

$$\left(x_n,y_n\right)=\left(b\left(U_n+U_{n-1}\right),U_n-U_{n-1}\right),n\geq 1.\tag{6.1}$$

When $a=2$ and $b=1$, the sequence $\{U_n\}$ in Theorem 6.1 is

$$U_n=3U_{n-1}-U_{n-2},U_0=0,U_1=1.\tag{6.2}$$

That is, $U_n=F_{2n}$, so that the solutions of (1.1), given in Theorem 6.1, become

$$\left(F_{2n}+F_{2n-2},F_{2n}-F_{2n-2}\right),n\geq 1,$$

or more simply, (L_{2n-1},F_{2n-1}), $n\geq 1$. This coincides with what we state in the introduction.

By induction, it is easy to prove that $U_{-n}=-U_n$, for all integers n. This means that if we allow negative integer values of n, then (6.1) produces all the integer solutions of (1.1) that lie in the first and third quadrants.

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