SOME PROPERTIES OF THE EQUATION $x^2 = 5y^2 - 4$

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Abstract. The Diophantine equation $x^2 = 5y^2 - 4$ and its three classes of solutions for automorphs will be discussed. For n an odd positive integer, any ordered pair $(x, y) = (L_{2n-1}, F_{2n-1})$ is a solution to the equation and all of the solutions are $(\pm L_{2n-1}, \pm F_{2n-1})$. We will demonstrate how to create a parameter k linking $k^3 + 3k$ to the terms x and y of such a solution (x, y). This will produce some new identities involving the Fibonacci numbers and Lucas numbers.

1. Introduction

This article deals with the solutions of the Diophantine equation

$$x^2 = 5y^2 - 4. \tag{1.1}$$

These solutions are well-known [6, Vol. 1, Theorem 8.7, p. 148] and will be classified using a group of automorphs of the form $x^2 - 5y^2$ [13, p. 165]. With the help of the number theory program [8], we will find three classes of solutions, one improper solution (gcd(x,y)=2) and two proper solutions (gcd(x,y)=1), to (1.1) with

$$(x, y) = (4, 2), \quad (x, y) = (1, 1), \quad (x, y) = (-1, 1). \tag{1.2}$$

For each class, the solutions can be described by the automorph

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \pm \begin{bmatrix} 9 & 20 \\ 4 & 9 \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix}. \tag{1.3}$$

More information on this classical subject can be found at [6, Vol. 1, Chapter 8], [13, Section 9.3, p. 161–168], [4, Theorem 2.2.9, p. 44], [5, 11, 12]. In order to explain the concept of attached number to each class, as defined by [13, p. 165], we will use a parameter $k_n \in \mathbb{Z}$ linking $k_n^3 + 3k_n$ to x_n and y_n. To accomplish this task, we will make use of Sloane's On-line Encyclopedia of Integer Sequences (OEIS) [9]. In particular, we will use extensively the Fibonacci and Lucas sequences. It is well-known [7, Theorem 7, p. 91], [15, Fundamental identity], [3, p. 29], [14, p. 30] that for all $n \in \mathbb{N}$

$$L_n^2 = 5F_n^2 + 4(-1)^n. \tag{1.4}$$

Thus, for any odd positive integer, the solutions of equation (1.1) are obtained. We will explore the three cases given in (1.2) using $(L_n, F_n) = (x_n, y_n)$.

2. Observations Within The Class Of $(4, 2)$

Let $(x_1, y_1) = (4, 2)$. If we set $x_1 = k_1^3 + 3k_1$, $k_1 = 1$. Using (1.3), $(x_2, y_2) = (76, 34)$. If we set $x_2 = k_2^3 + 3k_2$, $k_2 = 4$. Using (1.3), $(x_3, y_3) = (1364, 610)$. If we set $x_3 = k_3^3 + 3k_3$, $k_3 = 11$. Continuing this process using (1.3), we have the following table.

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n	1	2	3	4	5	...
x_n	4	76	1364	24476	439204	...
y_n	2	34	610	10946	196418	...
k_n	1	4	11	29	76	...

From [9], $\{k_n\}$ is the sequence A002878 defined by $k_1 = 1$, $k_2 = 4$, and for $n \geq 1$,

$$k_{n+2} = 3k_{n+1} - k_n.$$

The $\{k_n\}$ sequence is just the odd indexed terms of the Lucas sequence and the $\{x_n\}$ sequence is every sixth Lucas number starting at $L_3 = 4$. These relationships are stated and proved in the following proposition.

Proposition 2.1. *Let n be a positive integer. Then*

$$x_n = L_{6n-3} = L_{2n-1}^3 + 3L_{2n-1} = k_n^3 + 3k_n. \tag{2.1}$$

Proof. If j is a positive integer, the combination of the two conditions from [1, p. 41, p. 37]

$$L_{3j} = L_j(L_{2j} + (-1)^{j-1}) \text{ and } L_{2j} = L_j^2 + 2(-1)^{j+1}.$$

produces

$$L_{3j} = L_j^3 + 3(-1)^{j+1}L_j. \tag{2.2}$$

If $j = 2n - 1$, the relation (2.1) is established. □

It should be noted that the $\{y_n\}$ sequence is just every sixth Fibonacci number starting with $F_3 = 2$.

3. Observations Within The Class Of $(1, 1)$

Let $(x_1, y_1) = (1, 1) = (L_1, F_1)$. Extending the Fibonacci numbers to negative indices, we set $k_1 = 1 = F_{-1}$. Using (1.3), $(x_2, y_2) = (29, 13) = (L_7, F_7)$. Set $k_2 = 5 = F_5$. Using (1.3), $(x_3, y_3) = (1364, 610) = (L_{13}, F_{13})$. Set $k_3 = 89 = F_{11}$. Continuing this process using (1.3), we have the following table.

n	1	2	3	4	5	...
x_n	1	29	521	9349	167761	...
y_n	1	13	233	4181	75025	...
k_n	1	5	89	1597	28657	...

This data suggests the following congruence to be true. Let n be a positive integer. Then

$$x_n \equiv y_n^2 - k_n^3 - 3k_n \pmod{2y_n^2}.$$

We will now prove this congruence with the following proposition.

Proposition 3.1. *Let n be a positive integer. Then*

$$L_{6n-5} \equiv F_{6n-5}^2 - F_{6n-7}^3 - 3F_{6n-7} \pmod{2F_{6n-5}^2}. \tag{3.1}$$

Before proving Proposition 3.1 we consider the values of expression

$$\frac{y_n^2 - k_n^3 - 3k_n - x_n}{2y_n^2}.$$

We obtain the following table.

Table 1

n	1	2	3	4	5
$\frac{y_n^2-k_n^3-3k_n-x_n}{2y_n^2}$	-2	0	-6	-116	-2090

All the numbers in the above expression are even. Using [9] to find the sequence A049661, we have the following lemma.

Lemma 3.2.

$$\frac{F_{6n-5}^2 - F_{6n-7}^3 - 3F_{6n-7} - L_{6n-5}}{2F_{6n-5}^2} = -2\frac{F_{6n-11}-1}{4}. \tag{3.2}$$

Proof. The two conditions to demonstrate, implying the former relation, are

$$4 \mid (F_{6n-11}-1) \text{ and } L_{6n-5} = F_{6n-11}F_{6n-5}^2 - F_{6n-7}^3 - 3F_{6n-7}.$$

For the divisibility by 4, use [3, p. 56 and Theorem III, p. 39] to state the following for all positive integers j.

$$F_{6j+1} = F_{3j+1}^2 + F_{3j}^2, \quad F_3 = 2 \mid F_{3j} \text{ even}, \quad F_{3j+1} \text{ odd}, \quad F_{6j+1} \equiv 1 \pmod 4.$$

This proves the divisibility by 4. Using $L_{6n-5} = F_{6n-3} - F_{6n-7}$ from [3, Problem 11, p. 29] or [15], the above equality is equivalent to

$$0 = F_{6n-11}F_{6n-5}^2 - F_{6n-7}^3 - 2F_{6n-7} - F_{6n-3}.$$

Now we use (1.3) to establish the equality. We begin with

$$\begin{bmatrix} L_{6n-5} \\ F_{6n-5} \end{bmatrix} = \begin{bmatrix} 9 & 20 \\ 4 & 9 \end{bmatrix} \begin{bmatrix} L_{6n-11} \\ F_{6n-11} \end{bmatrix} = \begin{bmatrix} 20F_{6n-11} + 9L_{6n-11} \\ 9F_{6n-11} + 4L_{6n-11} \end{bmatrix}.$$

Again, using $L_{6n-5} = F_{6n-3} - F_{6n-7}$, gives

$$\begin{bmatrix} F_{6n-3} - F_{6n-7} \\ F_{6n-5} \end{bmatrix} = \begin{bmatrix} 9F_{6n-9} + 20F_{6n-11} - 9F_{6n-13} \\ 4F_{6n-9} + 9F_{6n-11} - 4F_{6n-13} \end{bmatrix}.$$

Eliminating F_{6n-13}, we obtain

$$F_{6n-11} = -4F_{6n-3} + 9F_{6n-5} + 4F_{6n-7}. \tag{3.3}$$

But adding

$$F_{6n-5} + F_{6n-4} = F_{6n-3}, \quad F_{6n-6} + F_{6n-5} = F_{6n-4}, \quad -F_{6n-7} - F_{6n-6} = -F_{6n-5},$$

we have

$$3F_{6n-5} - F_{6n-7} = F_{6n-3}. \tag{3.4}$$

Hence, combining 3.3) and (3.4), we obtain

$$\begin{aligned}
&F_{6n-11}F_{6n-5}^2 - F_{6n-7}^3 - 2F_{6n-7} - F_{6n-3} \\
&= (-4F_{6n-3} + 9F_{6n-5} + 4F_{6n-7})F_{6n-5}^2 - F_{6n-7}^3 - 2F_{6n-7} - F_{6n-3} \\
&= (-4(3F_{6n-5} - F_{6n-7}) + 9F_{6n-5} + 4F_{6n-7})F_{6n-5}^2 - F_{6n-7}^3 - 2F_{6n-7} - (3F_{6n-5} - F_{6n-7}) \\
&= -3F_{6n-5}^3 + 8F_{6n-5}^2F_{6n-7} - F_{6n-7}^3 - 3F_{6n-5} - F_{6n-7} \\
&= -(3F_{6n-5} + F_{6n-7})\left(F_{6n-5}^2 + F_{6n-7}^2 + 1 - 3F_{6n-5}F_{6n-7}\right).
\end{aligned}$$

This expression is 0, justified by the Markoff relation [2, Ch. 11],

$$F_{6n-5}^2 + F_{6n-7}^2 + 1 = 3F_{6n-5}F_{6n-7}.$$

This last relation is true from the statements

$$F_3^2 + F_1^2 + 1 - 3F_3F_1 = 2^2 + 1^2 + 1 - 3 \times 2 \times 1 = 0,$$

and using (3.4), for all positive integers n

$$\begin{aligned} &F_{6n-3}^2 + F_{6n-5}^2 + 1 - 3F_{6n-3}F_{6n-5} \\ &= F_{6n-5}^2 + (3F_{6n-5} - F_{6n-3})^2 + 1 - 3F_{6n-5}(3F_{6n-5} - F_{6n-3}) \\ &= F_{6n-5}^2 + F_{6n-7}^2 + 1 - 3F_{6n-5}F_{6n-7}. \end{aligned}$$

This proves relation (3.2) and as a consequence our last proposition. □

4. Observations Within The Class Of $(-1, 1)$

For this class, we will extend both the Fibonacci and Lucas numbers to negative indices. Let $(x_1, y_1) = (-1, 1) = (L_{-1}, F_{-1})$. Set $k_1 = 2 = F_{-3}$. Using (1.3), $(x_2, y_2) = (11, 5) = (L_5, F_5)$. Set $k_2 = 2 = F_3$. Using (1.3), $(x_3, y_3) = (199, 89) = (L_{11}, F_{11})$. Set $k_3 = 34 = F_9$. Continuing this process using (1.3), we have the following table.

n	1	2	3	4	5	...
x_n	−1	11	199	3571	64079	...
y_n	1	5	89	1597	28657	...
k_n	2	2	34	610	10946	...

This data suggests the following congruence to be true. Let n be a positive integer. Then

$$x_n \equiv y_n^2 - k_n^3 - 3k_n \pmod{2y_n^2}.$$

We will now prove this congruence with the following proposition.

Proposition 4.1. *Let n be a positive integer. Then*

$$L_{6n-7} \equiv F_{6n-7}^2 - F_{6n-9}^3 - 3F_{6n-9} \mod 2F_{6n-7}^2. \tag{4.1}$$

Before proving Proposition 4.1 we consider the values of expression

$$\frac{y_n^2 - k_n^3 - 3k_n - x_n}{2y_n^2}.$$

We obtain the following table.

Table 2

$n =$	1	2	3	4	5
$\frac{y_n^2 - k_n^3 - 3k_n - x_n}{2y_n^2}$	−6	0	−2	−44	−798

In fact, we will demonstrate more precisely the following lemma.

Lemma 4.2.

$$\frac{F_{6n-7}^2 - F_{6n-9}^3 - 3F_{6n-9} - L_{6n-7}}{2F_{6n-7}^2} = -2\frac{F_{6n-13} - 1}{4}. \tag{4.2}$$

Proof. The derivation is identical to the one used for Lemma 3.2. For all positive integers j

$$F_{6j-1} = F_{3j-1}^2 + F_{3j}^2, \quad F_3 = 2 \mid F_{3j} \text{ even}, \quad F_{3j-1} \text{ odd}, \quad F_{6j-1} \equiv 1 \pmod 4.$$

From $L_{6n-7} = F_{6n-5} - F_{6n-9}$ with [3, Problem 11, p. 29] or [15], the equality is found to be equivalent to (4.2).

$$F_{6n-13}F_{6n-7}^2 - F_{6n-9}^3 - 2F_{6n-9} - F_{6n-5} = 0. \tag{4.3}$$

Relation (1.3) gives

$$\begin{bmatrix} L_{6n-7} \\ F_{6n-7} \end{bmatrix} = \begin{bmatrix} 9 & 20 \\ 4 & 9 \end{bmatrix} \begin{bmatrix} L_{6n-13} \\ F_{6n-13} \end{bmatrix} = \begin{bmatrix} 20F_{6n-13} + 9L_{6n-13} \\ 9F_{6n-13} + 4L_{6n-13} \end{bmatrix}.$$

Eliminating L_{6n-13}, we obtain

$$F_{6n-13} = -4F_{6n-5} + 9F_{6n-7} + 4F_{6n-9}, \tag{4.4}$$

and

$$3F_{6n-7} - F_{6n-9} = F_{6n-5}. \tag{4.5}$$

Combining (4.3), (4.4), and (4.5), then factoring the resulting expression leads to a Markoff relation. A recurrence easily shows that this relation is true, using (4.3). This proves relation (4.2) and as a consequence our last proposition. □

5. Conclusion

For equation $x^2 = 5y^2 - 4$ we have considered some ordered pairs of solutions (L_{6n-3}, F_{6n-3}), (L_{6n-5}, F_{6n-5}), and (L_{6n-7}, F_{6n-7}), from the set of all ordered pairs (L_{2n-1}, F_{2n-1}). These solutions are distributed into three classes associated to a specific identity between Fibonacci numbers and Lucas numbers 2.1, 3.2, and 4.2.

Table 3

Improper solution $(4, 2)$	$L_{6n-3} = L_{2n-1}^3 + 3L_{2n-1}$	$k_n = L_{2n-1}$
Proper solution $(1, 1)$	$L_{6n-5} = -F_{6n-7}^3 - 3F_{6n-7} + F_{6n-11}F_{6n-5}^2$	$k_n = F_{6n-7}$
Proper solution $(-1, 1)$	$L_{6n-7} = -F_{6n-9}^3 - 3F_{6n-9} + F_{6n-13}F_{6n-7}^2$	$k_n = F_{6n-9}$

The method we used for proving (3.2) and (4.2) gives the first line of Table 3,

$$L_{6n-3} = -F_{6n-5}^3 - 3F_{6n-5} + F_{6n-9}F_{6n-3}^2. \tag{5.1}$$

This provides a new identity that we could also verify with [10], that is,

$$(L_{2n-1}^3 + F_{6n-5}^3) + 3(L_{2n-1} + F_{6n-5}) = F_{6n-9}F_{6n-3}^2,$$

where

$$\frac{L_{2n-1} + F_{6n-5}}{F_{2n-3}} = \frac{F_{6n-3}}{F_{2n-1}} = L_{2n-1}^2 + 1.$$

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