ON A SCALED BALANCED-POWER PRODUCT RECURRENCE

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Abstract. A power product recurrence (due to M. W. Bunder) is extended here by the introduction of a scaling factor, and delivers a sequence whose general term closed form is derived for both degenerate and non-degenerate characteristic root cases. It is shown how recurrence parameter conditions dictate the nature of each solution type, and a fundamental link between them is highlighted together with some other observations and results.

1. Introduction

Let $c \in \mathbb{Z}^+$ be an arbitrary scaling variable. Consider, given $z_0 = a$, $z_1 = b$, the scaled power product recurrence

$$z_{n+1} = c(z_n)^p (z_{n-1})^q, \qquad n \geq 1, \tag{1.1}$$

which defines a sequence $\{z_n\}_{n=0}^{\infty} = \{z_n\}_0^{\infty} = \{z_n(a,b,p,q;c)\}_0^{\infty}$ with first few terms

$$\{z_n(a,b,p,q;c)\}_0^{\infty} = \{a, b, a^q b^p c, a^{pq} b^{p^2+q} c^{p+1}, a^{p^2q+q^2} b^{p^3+2pq} c^{p^2+p+q+1}, a^{p^3q+2pq^2} b^{p^4+3p^2q+q^2} c^{p^3+p^2+p(2q+1)+q+1}, \ldots\}. \tag{1.2}$$

In 1975 Bunder [1] considered the $c = 1$ instance of (1.1) and noted that the general $(n+1)$th term has, for $n \geq 0$, a closed form

$$z_n(a,b,p,q;1) = a^{w_n(1,0;p,-q)} b^{w_n(0,1;p,-q)} \tag{1.3}$$

featuring terms of two particular (initial values specific) Horadam sequences $\{w_n(1,0;p,-q)\}_0^{\infty}$ and $\{w_n(0,1;p,-q)\}_0^{\infty}$. The Horadam sequence $\{w_n(w_0,w_1;p,q)\}_0^{\infty}$ is defined, for given w_0, w_1, by the order two linear recurrence

$$w_{n+1} = p w_n - q w_{n-1}, \qquad n \geq 1, \tag{1.4}$$

the notation for which, and the standard form of the recursion (1.4), having been fixed by the originator Alwyn F. Horadam in the 1960's [4].

Recently, Bunder's observation (1.3) has been proved inductively and generalized [2], and subsequently reproved again from first principles [5]. The article [3] looks at the case $p = q = 1/2$ of (1.1), developing results when $c = 1$ in the main but also (in an Appendix) discussing the version where c is held intact symbolically; both the resulting sequences themselves, and their growth rates, are of interest in [3] wherein (1.1) delivers a scaled ($c > 1$) or non-scaled ($c = 1$) so called geometric mean sequence. In this paper the retention, along with p, q, of the multiplicative scalar c as a generic variable in (1.1) constitutes a collective extension of previous publications which also include those works [6, 8, 9] that are referenced more fully in [3] and afford this paper additional background context for the reader. Using standard difference equations theory we formulate—according to characteristic root types—closed forms for $z_n(a,b,p,q;c)$ that are conditional on the recurrence parameters p and q, and give a fundamental connection between them together with some other observations and results. We base our paper on the premise that the powers of (1.1) satisfy the constraint

$$p + q = 1, \tag{1.5}$$

and we refer to the recurrence as a *balanced* one such as that instance studied in [3].

2. Results and Analysis

Let $t_n = \ln(z_n)$ (assuming $a, b, c > 0$). Then (1.1) reads

$$t_{n+1} - pt_n - qt_{n-1} = \ln(c), \tag{2.1}$$

the characteristic equation of which is

$$0 = \lambda^2 - p\lambda - q \tag{2.2}$$

with roots

$$\hat{\alpha}(p,q) = (p + \sqrt{p^2+4q})/2, \qquad \hat{\beta}(p,q) = (p - \sqrt{p^2+4q})/2. \tag{2.3}$$

Clearly, there are two separate characteristic root cases to consider in formulating closed forms for $z_n(a,b,p,q;c)$ and, as we shall see, they arrive with their own additional attendant conditions on the power parameters p, q of (1.1) as part of the process.

2.1. Non-Degenerate Roots Case ($p^2 + 4q \neq 0$).

For $p^2 \neq -4q$ the roots $\hat{\alpha}(p,q), \hat{\beta}(p,q)$ are distinct, and the homogeneous solution of (2.1) takes the form $E\hat{\alpha}^n(p,q) + F\hat{\beta}^n(p,q)$ with E, F arbitrary constants. A particular solution to (2.1) is chosen to be $Cn\ln(c)$ which, when substituted into it, yields

$$C[(1-p-q)n + 1 + q] = 1 \tag{2.4}$$

and, with $p + q = 1$ imposed (1.5), gives $C(q) = 1/(1+q)$ where $q \neq -1$ (or, equivalently, $C(p) = 1/(2-p)$ where $p \neq 2$).

From hereon we will absorb the relationship between p and q as $p(q) = 1 - q$ for convenience. Writing $f_n(q) = C(q)n = n/(1+q)$, and the characteristic roots as $\hat{\alpha}(q) = \hat{\alpha}(p(q),q)$, $\hat{\beta}(q) = \hat{\beta}(p(q),q)$, we have a general solution

$$t_n(\hat{\alpha}(q), \hat{\beta}(q), q, c) = E\hat{\alpha}^n(q) + F\hat{\beta}^n(q) + f_n(q)\ln(c), \tag{2.5}$$

from which, applying the initial values $t_0 = \ln(a)$, $t_1 = \ln(b)$, we obtain simultaneous equations

$$\begin{aligned} \ln(a) &= E + F, \\ \ln(b) - f_1(q)\ln(c) &= E\hat{\alpha}(q) + F\hat{\beta}(q), \end{aligned} \tag{2.6}$$

for E, F, with solutions

$$\begin{aligned} E &= \frac{\ln(b) \; - \hat{\beta}(q)\ln(a) - f_1(q)\ln(c)}{\hat{\alpha}(q) - \hat{\beta}(q)}, \\ F &= \frac{\hat{\alpha}(q)\ln(a) - \ln(b) \; + f_1(q)\ln(c)}{\hat{\alpha}(q) - \hat{\beta}(q)}, \end{aligned} \tag{2.7}$$

giving $t_n(\hat{\alpha}(q), \hat{\beta}(q), q, a, b, c) = N_a(\hat{\alpha}(q), \hat{\beta}(q), n)\ln(a) + N_b(\hat{\alpha}(q), \hat{\beta}(q), n)\ln(b) + N_c(\hat{\alpha}(q), \hat{\beta}(q), q, n)\ln(c)$ as a full solution, where

$$\begin{aligned} N_a(\hat{\alpha}(q), \hat{\beta}(q), n) &= [\hat{\alpha}(q)\hat{\beta}^n(q) - \hat{\beta}(q)\hat{\alpha}^n(q)]/[\hat{\alpha}(q) - \hat{\beta}(q)], \\ N_b(\hat{\alpha}(q), \hat{\beta}(q), n) &= [\hat{\alpha}^n(q) - \hat{\beta}^n(q)]/[\hat{\alpha}(q) - \hat{\beta}(q)], \\ N_c(\hat{\alpha}(q), \hat{\beta}(q), q, n) &= f_n(q) - f_1(q)N_b(\hat{\alpha}(q), \hat{\beta}(q), n). \end{aligned} \tag{2.8}$$

In other words,

$$z_n(a, b, p(q), q; c) = a^{N_a(\hat{\alpha}(q), \hat{\beta}(q), n)} b^{N_b(\hat{\alpha}(q), \hat{\beta}(q), n)} c^{N_c(\hat{\alpha}(q), \hat{\beta}(q), q, n)} \tag{2.9}$$

for $n \geq 0$, where the functional powers of a, b, c are seen as, respectively, N_a, N_b, N_c (these exponents are denoted D_a, D_b, D_c in the degenerate roots case which follows). Note that it is known, from [5, p. 176], that

$$N_a(\hat{\alpha}(q), \hat{\beta}(q), n) = w_n(1, 0; p(q), -q) \tag{2.10}$$

and

$$N_b(\hat{\alpha}(q), \hat{\beta}(q), n) = w_n(0, 1; p(q), -q), \tag{2.11}$$

so that, from (2.8),

$$N_c(\hat{\alpha}(q), \hat{\beta}(q), q, n) = [n - w_n(0, 1; p(q), -q)]/(1 + q). \tag{2.12}$$

Finally, since $p(q) = 1 - q$ the characteristic roots in this instance simplify conveniently as $\hat{\alpha}(q) = 1$, $\hat{\beta}(q) = -q$, and (2.8) reduces to

$$\begin{aligned} N_a(q, n) &= \frac{q + (-q)^n}{1 + q}, \\ N_b(q, n) &= \frac{1 - (-q)^n}{1 + q}, \\ N_c(q, n) &= \frac{1}{1 + q}\left(n - \frac{[1 - (-q)^n]}{1 + q}\right); \end{aligned} \tag{2.13}$$

hence, (2.9) reads

$$z_n(a, b, p(q), q; c) = a^{N_a(q,n)} b^{N_b(q,n)} c^{N_c(q,n)}, \qquad n \geq 0, \tag{2.14}$$

as our final form for the general term of the sequence delivered by (1.1) with $p(q) = 1 - q$, $q \neq -1$ (the initial elements of which are, of course, given explicitly by (1.2) with $p = p(q)$).

Remark 2.1. We note from (2.13) that, for $n \geq 0$, $N_a(q, n) + N_b(q, n) = 1$ by inspection, in addition to which the relation $N_b(q, n) + (1 + q)N_c(q, n) = n$ holds. By way of example, for $n = 4$ we see, using the term $z_4(a, b, p(q), q; c)$ of (1.2), $N_a(q, 4) = p^2(q)q + q^2 = (1-q)^2 q + q^2 = q - q^2 + q^3$ and $N_b(q, 4) = p^3(q) + 2p(q)q = (1 - q)^3 + 2(1 - q)q = 1 - q + q^2 - q^3$, for which $N_a(q, 4) + N_b(q, 4) = 1$ trivially. Furthermore, with $N_c(q, 4) = p^2(q) + p(q) + q + 1 = (1-q)^2 + (1-q) + q + 1 = 3 - 2q + q^2$ then $N_b(q, 4) + (1+q)N_c(q, 4) = 1 - q + q^2 - q^3 + (1+q)(3 - 2q + q^2) = \cdots = 4$ as expected.

Remark 2.2. Writing J_n as the general $(n + 1)$th term of the familiar Jacobsthal sequence $\{J_n\}_0^\infty = \{J_0, J_1, J_2, J_3, J_4, \ldots\} = \{0, 1, 1, 3, 5, \ldots\}$ (with, for $n \geq 0$, closed form $J_n = [2^n - (-1)^n]/3$), and S_n as that for the sequence $\{S_n\}_0^\infty = \{S_0, S_1, S_2, S_3, S_4, \ldots\} = \{0, 1, 3, 9, 23, \ldots\}$ (Sequence No. A045883 on the OEIS [7], with $S_n = [(3n+1)2^n - (-1)^n]/9$), we remark, as a further check on (2.13), that $N_a(1/2, n) = J_{n-1}/2^{n-1}$ $(n \geq 1)$ and $N_b(1/2, n) = J_n/2^{n-1}$ $(n \geq 0)$, together with $N_c(1/2, n) = S_{n-1}/2^{n-2}$ $(n \geq 1)$; these recover results for the geometric mean recursion examined previously (that is, for the case $p = q = 1/2$ of (1.1))—see the Appendix of [3], where $N_a(1/2, n), N_b(1/2, n), N_c(1/2, n)$ are denoted as, respectively, $\Omega_a(n), \Omega_b(n), \Omega_c(n)$.

Remark 2.3. Closed forms for $N_a(1/2, n)$, $N_b(1/2, n)$ and $N_c(1/2, n)$ combine, from the above remarks, to give immediate identities $J_{n-1} + J_n = 2^{n-1}$ and $J_n + 3S_{n-1} = n2^{n-1}$ for $n \geq 1$, which are easily verified; these two, in turn, yield $3S_n - J_n = n2^n$ $(n \geq 0)$ as a further one, together with $S_n + S_{n-1} = n2^{n-1}$ $(n \geq 1)$.

2.2. **Degenerate Roots Case ($p^2 + 4q = 0$).** For $p^2 = -4q$ the characteristic roots $\hat{\alpha}(p) = \hat{\beta}(p) = \frac{1}{2}p$ are non-distinct and the homogeneous solution of (2.1) is, with G, H arbitrary, of the form $(Gn + H)\hat{\alpha}^n(p)$. This time we must choose a particular solution $Cn^2\ln(c)$ to (2.1), which on substitution yields

$$C[(1 - p - q)n^2 + 2(1 + q)n + 1 - q] = 1. \tag{2.15}$$

Thus $q = -1$, which combines with $p+q = 1$ (1.5) to give $p = 2$ (with the governing constraint $p^2 = -4q$ satisfied) and $\hat{\alpha}(2) = \hat{\beta}(2) = 1$, and so a general solution

$$t_n(c) = Gn + H + n^2\ln(c)/2 \tag{2.16}$$

to (2.1). Initial values for t_0, t_1 give G, H readily as $G = \ln[b/(ac^{1/2})]$, $H = \ln(a)$, and (2.16) reads $t_n = \ln(a^{1-n}b^n c^{n(n-1)/2})$ after a little rearrangement, that is

$$z_n(a, b, 2, -1; c) = a^{D_a(n)} b^{D_b(n)} c^{D_c(n)}, \qquad n \geq 0, \tag{2.17}$$

where

$$\begin{aligned} D_a(n) &= 1 - n, \\ D_b(n) &= n, \\ D_c(n) &= n(n - 1)/2; \end{aligned} \tag{2.18}$$

these exponent functions agree with computations to independently generate

$$\{z_n(a, b, 2, -1; c)\}_0^\infty = \{a, b, b^2c/a, b^3c^3/a^2, b^4c^6/a^3, b^5c^{10}/a^4, b^6c^{15}/a^5, \ldots\} \tag{2.19}$$

using (1.1) or (1.2).

Remark 2.4. As a mark of self-consistency in this degenerate roots case we note that, given the governing condition $p^2 + 4q = 0$, the constraint $p + q = 1$ is alone sufficient to force $q = -1$, $p = 2$ as explicit recurrence parameter values, for (i) writing $p(q) = 1 - q$ then $0 = p^2(q) + 4q = (1 - q)^2 + 4q = (1 + q)^2 \Rightarrow q = -1$ and $p(-1) = 2$, while (ii) writing $q(p) = 1 - p$ then $0 = p^2 + 4q(p) = p^2 + 4(1 - p) = (p - 2)^2 \Rightarrow p = 2$ and $q(2) = -1$.

2.3. **A Solutions Connection.** We finish by noting that (2.18) is available directly from (2.13) using simple limiting arguments, thus establishing a basic link between the two solution types described by the closed forms of (2.14) and (2.17). Letting $q(\varepsilon) = -1 + \varepsilon$, then $D_a(n) = \lim_{\varepsilon\to 0}\{N_a(q(\varepsilon), n)\} = \lim_{\varepsilon\to 0}\{N_a^u(\varepsilon, n)/N_a^l(\varepsilon)\}$, where $N_a^u(\varepsilon, n) = \varepsilon - 1 + (1-\varepsilon)^n$ and $N_a^l(\varepsilon) = \varepsilon$. Since $N_a^u(0, n)/N_a^l(0)$ has the indeterminate form $0/0$ then, applying L'Hôpital's Rule, $D_a(n) = \lim_{\varepsilon\to 0}\{\frac{d}{d\varepsilon}[N_a^u(\varepsilon, n)]/\frac{d}{d\varepsilon}[N_a^l(\varepsilon)]\} = \lim_{\varepsilon\to 0}\{1 - n(1 - \varepsilon)^{n-1}\} = 1 - n$; it is left as a straightforward reader exercise to confirm that the functions $D_b(n)$ and $D_c(n)$ are given as, respectively, $\lim_{\varepsilon\to 0}\{N_b(q(\varepsilon), n)\}$ and $\lim_{\varepsilon\to 0}\{N_c(q(\varepsilon), n)\}$ in a similar fashion.

Had N_a, N_b, N_c been expressed in terms of $q(p) = 1 - p$ and n, these results would have been forthcoming by considering the limit as $\varepsilon \to 0$ in each, having set $p(\varepsilon) = 2 + \varepsilon$.

Remark 2.5. The identities of Remark 2.1 are valid in the limit $q \to -1$ and so with reference to the counterpart exponent functions of (2.18) in this degenerate roots case, for consider $\lim_{q\to -1}\{N_a(q, n) + N_b(q, n)\} = \lim_{q\to -1}\{N_a(q, n)\} + \lim_{q\to -1}\{N_b(q, n)\} = D_a(n) + D_b(n) = (1 - n) + n = 1$. Also, we see trivially that $\lim_{q\to -1}\{N_b(q, n) + (1 + q)N_c(q, n)\} = D_b(n) + 0 \cdot D_c(n) = D_b(n) = n$.

3. Summary

In this paper a scaled power product recursion has been analyzed, with balanced powers in the sense described. Recurrence parameter-conditional closed forms for the general term of resulting sequences have been derived in degenerate and non-degenerate characteristic root cases and, along with other observations given, a fundamental link between them highlighted. Note that the restriction of positivity for a, b, c at the outset of the formulation procedures has no bearing on either (2.14) or (2.17), which are valid for any non-zero values of these variables. We remark, too, that a natural particular solution to (2.1), of form $C\ln(c)$, has $C = [1 - (p+q)]^{-1}$ which is inadmissible since the recurrence (1.1) is balanced; should this be relaxed (that is, $p + q \neq 1$), then a new solution scenario emerges which is beyond the chosen remit of this article and within which the relatively tractable nature of the exponent functions of (2.13) is lost.

It remains to be seen if this type of scaled power product recurrence can be examined by any method(s) other than the routine application of difference equations theory presented here.

Dedication

This paper is dedicated to David Evans by the author PJL, in memory of his gentle personality and demeanor that made the lives of those with whom he interacted professionally so much the better. He will be sadly missed as a colleague for his warmth, charm, wit and kindness, and his untimely death is a great loss to all who worked with him.

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