COMBINATORIAL INTERPRETATIONS OF SOME CONVOLUTION IDENTITIES

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Abstract. We obtain, by way of combinatorial arguments, a number of convolution identities involving the Jacobsthal numbers, the Jacobsthal-Lucas numbers and various generalizations of the Fibonacci numbers.

1. Introduction

Suppose that both $f(n)$ and $g(n)$ are functions defined on the nonnegative integers. We term $(f * g)(n)$ the nth *convolution* of f and g, where

$$(f * g)(n) = \sum_{k=0}^{n} f(k)g(n-k).$$

In [3] we obtained, via the manipulation of certain generating functions, the convolution identity given below in Theorem 1.1.

Theorem 1.1.

$$\sum_{k=0}^{n} L_k J_{n-k} = j_{n+1} - L_{n+1},$$

where L_n denotes the nth Lucas number, and J_n and j_n denote the nth Jacobsthal number and the nth Jacobsthal-Lucas number, respectively [5].

Subsequently, we asked for a combinatorial proof of this result. The purpose of this paper is to provide just such a proof and then to obtain combinatorial interpretations of other convolution identities involving sequences arising from recurrence relations. The proof of the above result is given in Section 2, while proofs of further identities are obtained in Sections 3 and 4.

2. A Combinatorial Proof

In this section we prove Theorem 1.1 by combinatorial means. Let $\mathcal{B}_n$ denote a $1 \times n$ board. It is well-known that the nth Fibonacci number F_n enumerates the ways of tiling $\mathcal{B}_{n-1}$ using uncolored 1×1 squares and 1×2 dominos. It is not quite so well-known, however, that L_n counts the number of ways of tiling a circular n-board $\mathcal{C}_n$ with squares and dominoes, on which the cells are numbered 1 through n and a tiling is termed an *n-bracelet* [2]. Such an n-bracelet is termed *out-of-phase* if the same domino covers cells n and 1, and *in-phase* if this is not the case. We refer to the kth cell of $\mathcal{C}_n$ as $\mathcal{T}_k$.

Note, incidentally, that these are not bracelets in the conventional mathematical sense. Indeed, in the field of combinatorics, bracelets of length n are mathematical objects that may be regarded as equivalence classes of n-character strings over some alphabet of size k, in which rotations and reflections are taken as equivalent. Such bracelets represent, for example, structures with n circularly-connected beads of up to k different colors. However, the fact that the cells of $\mathcal{C}_n$ are numbered means that there are no such symmetry considerations in our scenario. It is also worth pointing out here that there are F_{n+1} in-phase and F_{n-1} out-of-phase uncolored tilings of $\mathcal{C}_n$, leading to the well-known identity $L_n = F_{n+1} + F_{n-1}$.

The nth Jacobsthal number J_n enumerates the ways of tiling $\mathcal{B}_{n-1}$ using white 1×1 squares, white 1×2 dominos and black 1×2 dominos, while the nth Jacobsthal-Lucas number j_n counts the number of ways of tiling $\mathcal{C}_n$ with these white squares, white dominos and black dominoes [1]. Again, there arises the possibility of both in-phase and out-of-phase n-bracelets. Now, we give the proof of Theorem 1.1.

Proof. Let us count the number of ways of tiling $\mathcal{C}_n$ with white squares, white dominos and black dominoes under the restriction that there must be at least one black domino present in each of the resultant n-bracelets. This enumeration will be carried out in two different ways.

First, consider the in-phase n-bracelets containing at least one black domino. For each such n-bracelet there exists some k with $1 \le k \le n-1$ for which a black domino covers both $\mathcal{T}_k$ and $\mathcal{T}_{k+1}$, all the leftmost $k-1$ cells in the n-bracelet are covered by white tiles and the rightmost $n-(k+1)$ ones can be tiled using white squares, white dominos and black dominos. Bearing in mind the combinatorial interpretations of F_n and J_n mentioned previously, it follows that the number of such n-bracelets is given by $F_k J_{n-k}$. Summing over all possible values of k gives us

$$\sum_{k=1}^{n} F_k J_{n-k}$$

as the number of in-phase n-bracelets.

We now enumerate the out-of-phase n-bracelets containing at least one black domino. Suppose first that a white domino covers cells n and 1. This leaves us with what is essentially a $1\times(n-2)$ board to tile using white squares, white dominos and black dominoes under the restriction that there must be at least one black domino present in each of these $(n-2)$-bracelets. A similar argument to that given in the previous paragraph leads to

$$\sum_{k=1}^{n-2} F_k J_{n-k-2}$$

as the number of out-of-phase n-bracelets in which at least one black domino is present and a white domino covers cells n and 1. The remaining out-of-phase n-bracelets arise from the situation in which a black domino occupies cells n and 1. Enumerating such n-bracelets is equivalent to counting the number of tilings of $\mathcal{B}_{n-2}$ using white squares, white dominos and black dominoes. This is equal to $F_1 J_{n-1}$, where the factor $F_1 = 1$ has been included for ease and consistency of notation.

We are now able to enumerate the ways of tiling $\mathcal{C}_n$ with white squares, white dominos and black dominoes under the restriction that there must be at least one black domino present in each of these n-bracelets. This is is given by

$$\begin{aligned}\sum_{k=1}^{n} F_k J_{n-k} + F_1 J_{n-1} + \sum_{k=1}^{n-2} F_k J_{n-k-2} &= \sum_{k=1}^{n} F_k J_{n-k} + F_1 J_{n-1} + \sum_{k=3}^{n} F_{k-2} J_{n-k} \\ &= \sum_{k=1}^{n} \left(F_k + F_{k-2}\right) J_{n-k} + F_1 J_{n-1} - F_0 J_{n-2} - F_{-1} J_{n-1}\end{aligned}$$

$$= \sum_{k=1}^{n} L_{k-1} J_{n-k}$$
$$= \sum_{k=0}^{n-1} L_k J_{n-1-k}.$$

From the combinatorial interpretations of j_n and L_n we see that $j_n - L_n$ gives the total number of ways of tiling $\mathcal{C}_n$ with white squares, white dominos and black dominoes under the restriction that each of the resultant n-bracelets must contain at least one black domino. We thus have

$$\sum_{k=0}^{n-1} L_k J_{n-1-k} = j_n - L_n,$$

from which the statement of the theorem follows, noting that the n-bracelets generated via the initial enumeration are indeed unique. □

3. Further Identities

We now go on to obtain two further results by way of combinatorial proofs. First, let $\mathcal{S}_r = \{s_1, \ldots, s_r, d\}$ be the set composed of r 1×1 squares of different colors and a single 1×2 domino. Next, for some fixed $r \in \mathbb{N}$, let $(F_r(n))_{n \geq 0}$ be the generalized Fibonacci sequence given by

$$F_r(n) = rF_r(n-1) + F_r(n-2),$$

where $F_r(0) = 0$ and $F_r(1) = 1$. It is well-known that $F_r(n)$ enumerates the ways of tiling $\mathcal{B}_{n-1}$ using elements from $\mathcal{S}_r$. In our next theorem we obtain a result concerning a convolution involving this generalized Fibonacci sequence.

Theorem 3.1. *Let $a, b, n \in \mathbb{N}$, where $b > a$. Then*

$$(F_a * F_b)(n) = \frac{F_b(n) - F_a(n)}{b-a}.$$

Proof. Let $\mathcal{S} = \mathcal{S}_b \backslash \mathcal{S}_a$. In order to prove this result we count, in two different ways, the number of tilings of $\mathcal{B}_{n-1}$ using elements from $\mathcal{S}_b$, where we impose the restriction that each tiling must contain at least one element from $\mathcal{S}$.

First, let us denote the kth cell, going from left to right, of $\mathcal{B}_{n-1}$ by $\mathcal{U}_k$, where $1 \leq k \leq n-1$. We enumerate the restricted tilings of $\mathcal{B}_{n-1}$ in a systematic way, as follows. Any one of the $b-a$ elements of $\mathcal{S}$, each of which is a square tile, is placed on $\mathcal{U}_k$. The section of $\mathcal{B}_{n-1}$ to the left of $\mathcal{U}_k$ is then tiled with elements from $\mathcal{S}_a$ while the section to the right of $\mathcal{U}_k$ is tiled with elements from $\mathcal{S}_b$. Note then that this gives us a tiling of $\mathcal{B}_{n-1}$ for which the first appearance of a tile from $\mathcal{S}$ occurs at $\mathcal{U}_k$. From this we see that the total number of tilings for which the first appearance of a tile from $\mathcal{S}$ occurs, when moving from left to right, at $\mathcal{U}_k$ is given by

$$(b-a)F_a((k-1)+1)F_b(((n-1)-k)+1) = (b-a)F_a(k)F_b(n-k).$$

Summing over all k such that $1 \leq k \leq n-1$, we see that the total number of tilings of $\mathcal{B}_{n-1}$ using elements from $\mathcal{S}_b$, with the restriction that each tiling must contain at least one element from $\mathcal{S}$, is given by

$$(b-a)\sum_{k=1}^{n-1} F_a(k)F_b(n-k) = (b-a)\sum_{k=0}^{n} F_a(k)F_b(n-k)$$
$$= (b-a)(F_a * F_b)(n),$$

where, in the first step, we have used the fact that $F_a(0) = F_b(0) = 0$. Notice that the tilings generated by the above process are unique.

Finally, on noting that $F_b(n) - F_a(n)$ gives the number of total number of tilings of $\mathcal{B}_{n-1}$ using elements from $\mathcal{S}_b$, with the restriction that each tiling must contain at least one element from $\mathcal{S}$, we have completed the proof of the theorem. □

Our next result is related to Theorem 3.1 in the sense that the roles of the squares and dominos are interchanged. Indeed, we adopt a similar approach here to that used in Theorem 3.1, but include the proof for the sake of completeness.

Let $\mathcal{D}_r = \{d_1, \ldots, d_r, s\}$ be the set composed of r 1×2 dominos of different colors and a single 1×1 square. For some fixed $r \in \mathbb{N}$, let $(G_r(n))_{n \geq 0}$ be the generalized Fibonacci sequence given by

$$G_r(n) = G_r(n-1) + rG_r(n-2),$$

where $G_r(0) = 0$ and $G_r(1) = 1$. It is known that $G_r(n)$ enumerates the ways of tiling $\mathcal{B}_{n-1}$ using elements from $\mathcal{D}_r$.

Theorem 3.2. *Let $a, b, n \in \mathbb{N}$, where $b > a$. Then*

$$\left(G_a * G_b\right)(n) = \frac{G_b(n+1) - G_a(n+1)}{b-a}.$$

Proof. We start by letting $\mathcal{D} = \mathcal{D}_b \backslash \mathcal{D}_a$. This time we consider the tilings of $\mathcal{B}_n$, where $\mathcal{U}_k$ denotes the kth cell of $\mathcal{B}_n$. When enumerating the restricted tilings, any one of the $b - a$ elements of $\mathcal{D}$, each of which is a domino, covers both $\mathcal{U}_k$ and $\mathcal{U}_{k+1}$, where $1 \leq k \leq n-1$. The section of $\mathcal{B}_n$ to the left of $\mathcal{U}_k$ is then tiled with elements from $\mathcal{D}_a$ while the section to the right of $\mathcal{U}_{k+1}$ is tiled with elements from $\mathcal{D}_b$. Note then that this gives us a tiling of $\mathcal{B}_n$ for which the first appearance, when moving from left to right, of a tile from $\mathcal{D}$ occurs at $\mathcal{U}_k$. From this we see that the total number of tilings for which the first appearance of a tile from $\mathcal{D}$ occurs at $\mathcal{U}_k$ is given by

$$(b-a)G_a((k-1)+1)G_b((n-(k+1))+1) = (b-a)G_a(k)G_b(n-k).$$

Now, summing over all k such that $1 \leq k \leq n-1$, we see that the total number of tilings of $\mathcal{B}_n$ using elements from $\mathcal{D}_b$, with the restriction that each tiling must contain at least one element from $\mathcal{D}$, is given by

$$\begin{aligned}(b-a)\sum_{k=1}^{n-1} G_a(k)G_b(n-k) &= (b-a)\sum_{k=0}^{n} G_a(k)G_b(n-k) \\ &= (b-a)\left(G_a * G_b\right)(n),\end{aligned}$$

where, in the first step, we have used the fact that $G_a(0) = G_b(0) = 0$. Notice once again that the tilings generated by the above process are unique.

Then, since $G_b(n+1) - G_a(n+1)$ gives the total number of tilings of $\mathcal{B}_n$ using elements from $\mathcal{D}_b$, with the restriction that each tiling must contain at least one element from $\mathcal{D}$, the theorem has been proved. □

4. A Generalization

The results given in Theorems 3.1 and 3.2 can in fact be generalized somewhat. To this end, we let $\mathcal{M}_{r,t} = \{s_1, \ldots, s_r, d_1, \ldots, d_t\}$ be the set composed of r 1×1 squares of different colors and t 1×2 dominos of different colors. Note that, for $b > a$, the differences $\mathcal{M}_{b,t} \backslash \mathcal{M}_{a,t}$ and $\mathcal{M}_{r,b} \backslash \mathcal{M}_{r,a}$ are just the sets $\mathcal{S}$ and $\mathcal{D}$, respectively, as defined previously in the proofs of Theorems 3.1 and 3.2. With $H_{r,t}(n)$ defined via the recurrence $H_{r,t}(n) = rH_{r,t}(n-1) + tH_{r,t}(n-2)$, where $H_{r,t}(0) = 0$ and $H_{r,t}(1) = 1$, we obtain, by similar reasoning to that used in the proofs of Theorems 3.1 and 3.2, the following results:

$$(H_{a,t} * H_{b,t})(n) = \frac{H_{b,t}(n) - H_{a,t}(n)}{b-a}$$

and

$$(H_{r,a} * H_{r,b})(n) = \frac{H_{r,b}(n+1) - H_{r,a}(n+1)}{b-a}.$$

Interested readers might also like to look at [4], which is in some sense related to the work carried out in the current paper.

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