NEW IDENTITIES SATISFIED BY POWERS OF FIBONACCI AND LUCAS NUMBERS

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Abstract. The impetus for this research came from previous work of the author and others. This work centered around finding generalizations of the identities

$$F_{n+1}^2 + F_n^2 = F_{2n+1},$$
$$F_{n+1}^3 + F_n^3 - F_{n-1}^3 = F_{3n},$$

and of their higher power analogues. The main result in this paper represents an addition to the literature of such identities. Specifically, the main result is an identity satisfied by mth powers of Fibonacci numbers in which the subscripts of the Fibonacci numbers involved are arbitrarily spaced. From this main result, additional (similar) identities that involve the Fibonacci/Lucas numbers arise as so-called *dual* identities.

1. Introduction

The Fibonacci and Lucas numbers are defined, respectively, for all integers n, by

$$F_n = F_{n-1} + F_{n-2}, F_0 = 0, F_1 = 1,$$
$$L_n = L_{n-1} + L_{n-2}, L_0 = 2, L_1 = 1.$$

We begin by stating an identity that involves sums of powers of Fibonacci numbers. To do this, we require some notation. The Fibonomial coefficient $\binom{m}{i}_F$ is defined for integers $m \geq 0$ by

$$\binom{m}{i}_F = \begin{cases} 0, & \text{if } i < 0 \text{ or } i > m; \\ 1, & \text{if } i = 0 \text{ or } i = m; \\ \frac{F_m \cdots F_{m-i+1}}{F_1 \cdots F_i}, & \text{if } 0 < i < m. \end{cases}$$

For a discussion on generalized binomial coefficients, we recommend [5] and the more recent paper [8] .

Taking $m \geq 1$ to be an integer, the identity to which we refer in the preceding paragraph is

$$\sum_{i=0}^{m} (-1)^{\frac{i(i+3)}{2}} \binom{m}{i}_F F_{n+m-i}^{m+1} = F_1 \cdots F_m F_{(m+1)(n+\frac{m}{2})}. \tag{1.1}$$

Identity (1.1) is a special case of identity (5) in Torretto and Fuchs [12].

The first two cases of (1.1), corresponding to $m = 1$ and $m = 2$, are respectively

$$F_{n+1}^2 + F_n^2 = F_{2n+1}, \tag{1.2}$$
$$F_{n+1}^3 + F_n^3 - F_{n-1}^3 = F_{3n}. \tag{1.3}$$

Identity (1.2) occurs as I_{11} in [6, page 56]. Dickson [2, page 395] attributes identity (1.3) to Edouard Lucas.

In [4], Ginsburg gave the identity $F_{n+2}^3 - 3F_n^3 + F_{n-2}^3 = 3F_{3n}$. It was this identity of Ginsburg that prompted the present author to seek generalizations of (1.2) and (1.3). This led to (see

[9])

$$F_{n+k+1}^2 + F_{n-k}^2 = F_{2k+1}F_{2n+1}, \tag{1.4}$$

$$F_{3k+1}F_{n+k+1}^3 + F_{3k+2}F_{n+k}^3 - F_{n-2k-1}^3 = F_{3k+1}F_{3k+2}F_{3n}, \tag{1.5}$$

in which k and n are integers. In [9], the author also presented a conjecture giving analogues of (1.4) and (1.5) for higher powers. This conjecture was proved in [8]. In [7], Howard proved that (1.4) is equivalent to $F_n^2 + (-1)^{n+k+1}F_k^2 = F_{n-k}F_{n+k}$, which occurs as I_{19} in [6, page 19].

In [10], the present author proved that

$$F_mF_{n+k}^3 + (-1)^{k+m+1}F_kF_{n+m}^3 + (-1)^{k+m}F_{k-m}F_n^3 = F_{k-m}F_kF_mF_{3n+k+m}, \tag{1.6}$$

which generalizes (1.5). In [11], the present author proposed a fourth power identity analogous to (1.6). This identity was proved in [1]. Indeed, in [1] the authors went much further, working with sequences more general than the Fibonacci sequence, and giving mth power identities satisfied by these sequences.

Of interest to us in the present paper is Theorem 3 in [1], which we now state in the context of the Fibonacci numbers.

Theorem 1.1. *Let $m \geq 2$ be an integer, and let $a_1, a_2, \ldots, a_m$ be distinct integers. Define*

$$P(i,m) = \prod_{j=1}^{m} F_{a_i - a_j}, \text{ in which } j \neq i.$$

Then for any integer $m \geq 2$, we have

$$\sum_{i=1}^{m} F_{n+a_i}^m / P(i,m) = F_{mn+a_1+a_2+\cdots+a_m}. \tag{1.7}$$

The first two cases of (1.7), corresponding to $m = 2$ and $m = 3$, are respectively

$$F_{n+a_1}^2/F_{a_1-a_2} + F_{n+a_2}^2/F_{a_2-a_1} = F_{2n+a_1+a_2}, \tag{1.8}$$

and

$$\begin{aligned} &F_{n+a_1}^3/\left(F_{a_1-a_2}F_{a_1-a_3}\right) + F_{n+a_2}^3/\left(F_{a_2-a_1}F_{a_2-a_3}\right) \\ &\quad + F_{n+a_3}^3/\left(F_{a_3-a_1}F_{a_3-a_2}\right) = F_{3n+a_1+a_2+a_3}. \end{aligned} \tag{1.9}$$

It is easy to see that (1.8) generalizes (1.2), and (1.9) generalizes (1.3). In fact, each of (1.1)–(1.6) arises from (1.7).

On its left side, identity (1.7) consists of a linear combination of mth powers of Fibonacci numbers that are spaced according to the values of the a_i. Furthermore, the subscript of F on the right side of (1.7) contains each a_i.

Let $m \geq 2$ be an integer and let k be an integer. Also let $a_1, a_2, \ldots, a_m$ be integers. Define

$$S(i,m,k) = F_{a_1+a_2+\cdots+a_m-a_i-k}.$$

We now state our main result, which is the theorem that follows.

Theorem 1.2. *Let $m \geq 2$ be an integer. Let k be an integer, and let $a_1, a_2, \ldots, a_{m+1}$ be distinct integers. Then*

$$\sum_{i=1}^{m+1} (-1)^{a_i} S(i,m+1,k) F_{n+a_i}^m / P(i,m+1) = (-1)^{a_1+a_2+\cdots+a_{m+1}+k+1} F_{mn+k}, \tag{1.10}$$

where $P(i,m)$ is defined in the statement of Theorem 1.1.

Notice that (1.10) is analogous to (1.7), but the subscript of F on the right side of (1.10) is *independent* of the a_i. However, compared to (1.7), (1.10) has an extra mth power on the left. So each mth power identity that arises from (1.10) involves $m+1$ distinct integers $a_1, a_2, \ldots, a_{m+1}$, and the subscript of F on the right can be chosen to be $mn+k$, where k is an arbitrary integer.

To illustrate the discussion in the previous paragraph, let $m = 3$ and let $k = 1$. Take (a_1, a_2, a_3, a_4)=$(2, 4, 7, 10)$. Then (1.10) becomes

$$902F_{n+2}^3 - 2261F_{n+4}^3 - 427F_{n+7}^3 + 6F_{n+10}^3 = -14F_{3n+1}. \tag{1.11}$$

Next take $k = 5$, and let the remaining parameters be the same. Then (1.10) becomes

$$752F_{n+2}^3 - 1885F_{n+4}^3 - 356F_{n+7}^3 + 5F_{n+10}^3 = -80F_{3n+5}. \tag{1.12}$$

Indeed, for $m = 3$ and $(a_1, a_2, a_3, a_4) = (2, 4, 7, 10)$, (1.11) and (1.12) belong to an infinite family, parametrized by k, of third power identities produced by (1.10).

In Section 2, we briefly summarize those parts of a paper of L. A. G. Dresel [3] that we require in Sections 3 and 4. In Section 3, we prove our main result. In Section 4, from our main result, we generate results that are similar in nature to our main result.

2. Elements of a Seminal Paper of L. A. G. Dresel

From our point of view, the importance of a 1992 paper of L. A. G. Dresel [3] cannot be overstated. For one thing, Dresel's paper outlines a method of proof for certain types of Fibonacci/Lucas identities. It is this method that we use to prove our main result. In order to make the present paper self contained, we outline, in the next few paragraphs, those elements of Dresel's paper that we require in the present paper.

Let $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2$, so that $\alpha\beta = -1$. Then the Binet (closed) forms for F_n and L_n are

$$\begin{aligned} F_n &= (\alpha^n - \beta^n)/\sqrt{5}, \\ L_n &= \alpha^n + \beta^n, \end{aligned} \tag{2.1}$$

and these Binet forms are valid for all integers n.

In (2.1), we make the substitutions

$$X = \alpha^n,\ Y = \beta^n,\ XY = \alpha^n\beta^n = (-1)^n. \tag{2.2}$$

The substitutions (2.2) transform any Fibonacci/Lucas identity (FL-identity) into an identity of algebraic forms in the variables X and Y. Dresel calls this identity of algebraic forms the *XY-transform* of the FL-identity in question. Conversely, if the XY-transform is an identity, it proves the corresponding FL-identity. Dresel defines an FL-identity, or expression, as being *homogeneous* if its XY-transform is an homogeneous algebraic form in X and Y.

To illustrate, we use the example in [3, page 170]. The algebraic identity

$$X^3 + Y^3 = (X+Y)^3 - 3XY(X+Y) \tag{2.3}$$

proves the FL-identity

$$L_{3n} = L_n^3 - 3(-1)^n L_n. \tag{2.4}$$

Furthermore, since (2.3) is homogeneous of degree 3, then (2.4) is an homogeneous FL-identity of degree 3.

Next, let $j \geq 0$ and k be integers. Then, with the substitutions (2.2), we have

$$\begin{aligned} F_{jn+k} &= \left(\alpha^k X^j - \beta^k Y^j\right)/\sqrt{5}, \\ L_{jn+k} &= \alpha^k X^j + \beta^k Y^j, \\ (-1)^{jn} &= X^j Y^j. \end{aligned} \tag{2.5}$$

The known identities $F_{-n} = (-1)^{n+1}F_n$ and $L_{-n} = (-1)^n L_n$ justify the restriction $j \geq 0$ above. From (2.5), we see that F_{jn+k} and L_{jn+k} transform to homogeneous forms of degree j in X and Y, and $(-1)^{jn}$ transforms to an homogeneous form of degree $2j$ in X and Y. Accordingly, we say that F_{jn+k} and L_{jn+k} are of degree j in the variable n, while $(-1)^{jn}$ is of degree $2j$ in the variable n. Of course, implicit in this discussion is an underlying field of coefficients, usually $Q(\sqrt{5})$.

In any FL-expression, we take a *term* to be a product of its *factors*. We take the *degree* of a term to be the sum of the degrees of its factors. If each term of an FL-expression has the same degree, then the FL-expression is said to be homogeneous. Via (2.5), we can see when an FL-expression is homogeneous, and determine its degree without writing out its XY-transform.

Now suppose we have an homogeneous FL-equation of degree s. By definition, the associated XY-transform of this FL-equation is homogeneous, and has the form

$$\sum_{i=0}^{s} a_i X^{s-i} Y^i = \sum_{i=0}^{s} b_i X^{s-i} Y^i. \tag{2.6}$$

In (2.6), the coefficients a_i and b_i are independent of n, and some of these coefficients may be zero. Dividing (2.6) by X^s, we obtain the polynomial equation

$$\sum_{i=0}^{s} a_i Z^i = \sum_{i=0}^{s} b_i Z^i, \ Z = X^{-1}Y, \tag{2.7}$$

which has degree s, and which Dresel calls the *polynomial* form of the FL-equation. If the FL-equation in question is an identity, then (2.7) is satisfied by infinitely many values of Z. This leads to Dresel's *verification theorem* [3, page 171], which follows. To bring together the concepts that we have outlined above, we give Dresel's short proof of this theorem.

Theorem 2.1. *If an FL-equation is homogeneous of degree s in the variable n, and if this equation is satisfied for $s+1$ different values of n, then it is an identity that is true for all values of n.*

Proof. The homogeneous FL-equation in question can be reduced to a polynomial equation of the form (2.7). Since the homogeneous FL-equation is satisfied for $s+1$ different values of n, its corresponding polynomial equation is satisfied for $s+1$ different values of Z. Since the polynomials on each side of this polynomial equation are of degree s, it follows from the fundamental of algebra that they are identical. Therefore, the FL-equation is an identity that is true for all n. □

When required, we can insert appropriate powers of -1 to write identities in homogeneous form, thus rendering them provable by Dresel's verification theorem. As an example, in the identity [6, page 59]

$$F_{n-2}F_{n-1}F_{n+1}F_{n+2} - F_n^4 = -1, \tag{2.8}$$

we express the right side as $-(-1)^{2n}$, so that (2.1) becomes an homogeneous identity of degree 4 in the variable n. Thus, we can prove (2.1) by verifying it for the five values $n = 0, 1, 2, 3, 4$.

Now suppose that we have a homogeneous FL-identity, in the variable n, with its polynomial form as given in (2.7). Then, being an identity in Z, (2.7) remains true if Z is replaced by $-Z$. That is

$$\sum_{i=0}^{s} a_i(-Z)^i \equiv \sum_{i=0}^{s} b_i(-Z)^i. \tag{2.9}$$

But then (2.9) corresponds to a new FL-identity, in the variable n, with an XY-transform in which Y has been replaced by $-Y$. Dresel calls this new FL-identity the *dual* identity of the original FL-identity. By replacing Y by $-Y$ in (2.5), it follows that the dual identity can be obtained by making the following changes in the original FL-identity.

- when j is odd, replace F_{jn+k} by $L_{jn+k}/\sqrt{5}$,
- when j is odd, replace L_{jn+k} by $\sqrt{5}F_{jn+k}$,
- when j is odd, replace $(-1)^{jn}$ by $-(-1)^{jn}$.

Clearly, when j is even, no changes need to be made when finding the dual identity because then $(-Y)^j = Y^j$.

For instance, the dual of (1.3) is

$$L_{n+1}^3 + L_n^3 - L_{n-1}^3 = 5L_{3n}.$$

As a second example, Simson's identity is

$$F_{n-1}F_{n+1} - F_n^2 = (-1)^n,$$

and its dual identity is

$$L_{n-1}L_{n+1} - L_n^2 = 5(-1)^{n+1}.$$

In Section 4, we use the concept of a dual identity to obtain further results from our main result.

3. Proof of Our Main Result

We begin this section by proving a lemma that we employ in the proof of our main result. In fact, this lemma corresponds to the case $m = 2$ of our main result, and as such is the first case of our main result.

Lemma 3.1. *Let k be an arbitrary integer, and let a_1, a_2, a_3 be distinct integers. Then*

$$\begin{aligned}(-1)^{a_1} F_{a_2+a_3-k}F_{n+a_1}^2/\left(F_{a_1-a_2}F_{a_1-a_3}\right) + (-1)^{a_2} F_{a_1+a_3-k}F_{n+a_2}^2/\left(F_{a_2-a_1}F_{a_2-a_3}\right)\\ + (-1)^{a_3} F_{a_1+a_2-k}F_{n+a_3}^2/\left(F_{a_3-a_1}F_{a_3-a_2}\right) = (-1)^{a_1+a_2+a_3+k+1}F_{2n+k}.\end{aligned} \tag{3.1}$$

Proof. In the terminology of Section 2, (3.1) is homogeneous of degree 2 in the variable n. Accordingly, we prove (3.1) by verifying its validity for $n = -a_1, -a_2, -a_3$.

In (3.1), substitute $n = -a_3$. Then, upon clearing fractions, and with repeated use of the well-known identity $F_{-n} = (-1)^{n+1}F_n$, we see that we are required to prove that

$$(-1)^{a_2+1}F_{a_1-a_3}F_{k-a_2-a_3} + (-1)^{a_1}F_{a_2-a_3}F_{k-a_1-a_3} + (-1)^{a_2}F_{a_1-a_2}F_{k-2a_3} = 0. \tag{3.2}$$

Identity (3.2) is homogeneous of degree 1 in the variable k. Therefore, to prove (3.2), we need only verify its validity for $k = a_1 + a_3$ and for $k = a_2 + a_3$. These verifications follow immediately, and so (3.1) is true for $n = -a_3$.

The validity of (3.1) for $n = -a_1$ and for $n = -a_2$ follows in the same manner demonstrated in the previous paragraph. This completes the proof of Lemma 3.1. □

We now prove Theorem 1.2, the main result in this paper. We proceed by induction on m. Recall that Lemma 3.1 is the first case, corresponding to $m = 2$, of our main result. To begin, suppose that for some integer $m_0 \geq 2$ we always have

$$\sum_{i=1}^{m_0+1} (-1)^{a_i} S(i, m_0+1, k) F_{n+a_i}^{m_0} / P(i, m_0+1) = (-1)^{a_1+a_2+\cdots+a_{m_0+1}+k+1} F_{m_0 n+k} \tag{3.3}$$

whenever k is an integer, and $a_1, a_2, \ldots, a_{m_0+1}$ are distinct integers. By Lemma 3.1, (3.3) is true for $m_0 = 2$.

We assert that

$$\sum_{i=1}^{m_0+2} (-1)^{a_i} S(i, m_0+2, k) F_{n+a_i}^{m_0+1} / P(i, m_0+2) = (-1)^{a_1+a_2+\cdots+a_{m_0+2}+k+1} F_{(m_0+1)n+k} \tag{3.4}$$

whenever a_{m_0+2} is an integer that is distinct from $a_1, a_2, \ldots, a_{m_0+1}$.

Identity (3.4) is an homogeneous identity of degree $m_0 + 1$ in the variable n. Therefore, we can prove (3.4) by verifying its validity for $m_0 + 2$ distinct values of n. In (3.4), substitute $n = -a_{m_0+2}$. Then the rightmost term on the left side of (3.4) vanishes, and cancellation reduces (3.4) to

$$\begin{aligned} &\sum_{i=1}^{m_0+1} (-1)^{a_i} S(i, m_0+1, k - a_{m_0+2}) F_{-a_{m_0+2}+a_i}^{m_0} / P(i, m_0+1) \\ &= (-1)^{a_1+a_2+\cdots+a_{m_0+2}+k+1} F_{-(m_0+1)a_{m_0+2}+k}. \end{aligned} \tag{3.5}$$

In (3.3), replace k by $k - a_{m_0+2}$, and n by $-a_{m_0+2}$. This yields (3.5), which must therefore be true. So we have verified (3.4) for $n = -a_{m_0+2}$. By symmetry, (3.4) is true if n is any of the $m_0 + 1$ values $-a_1, -a_2, \ldots, -a_{m_0+1}$. Therefore (3.4) is true by the verification theorem of Dresel.

Finally, as stated earlier, (3.3) is true for $m_0 = 2$, and so we have proved Theorem 1.2 by induction on m.

4. Further Identities Obtained as Duals

Beginning with identity (1.10), we now take up Dresel's idea of writing down dual identities. Here, we simply take a known identity, and make the changes listed in the three dot points given at the end of Section 2. We begin by considering identity (1.10) as an homogeneous identity of degree m in the variable n. Therefore, with the same assumptions on the various parameters, we can write down the dual identity of (1.10) with respect to the variable n. There are two possibilities that depend upon the parity of m. For m even, the dual identity in question is

$$\sum_{i=1}^{m+1} (-1)^{a_i} S(i, m+1, k) L_{n+a_i}^{m} / P(i, m+1) = (-1)^{a_1+a_2+\cdots+a_{m+1}+k+1} 5^{\frac{m}{2}} F_{mn+k}. \tag{4.1}$$

For m odd, the dual identity in question is

$$\sum_{i=1}^{m+1} (-1)^{a_i} S(i, m+1, k) L_{n+a_i}^{m} / P(i, m+1) = (-1)^{a_1+a_2+\cdots+a_{m+1}+k+1} 5^{\frac{m-1}{2}} L_{mn+k}. \tag{4.2}$$

We next find the dual identity of (1.10) with respect to the variable k. From (2.5), we see that (1.10) is homogeneous of degree 3 in the variable k. To proceed, we require the analogue of $S(i,m,k)$ for the Lucas sequence. Accordingly, given an integer $m \geq 2$, an integer k, and integers $a_1, a_2, \ldots, a_m$, define

$$T(i,m,k) = L_{a_1+a_2+\cdots+a_m-a_i-k}.$$

The dual identity of (1.10) with respect to the variable k is then

$$\sum_{i=1}^{m+1} (-1)^{a_i} T(i,m+1,k) F_{n+a_i}^m / P(i,m+1) = (-1)^{a_1+a_2+\cdots+a_{m+1}+k} L_{mn+k}. \tag{4.3}$$

Finally, we find the dual identity of (4.3) with respect to the variable n. Once again, this dual identity depends upon the parity of m. For m even, the dual identity in question is

$$\sum_{i=1}^{m+1} (-1)^{a_i} T(i,m+1,k) L_{n+a_i}^m / P(i,m+1) = (-1)^{a_1+a_2+\cdots+a_{m+1}+k} 5^{\frac{m}{2}} L_{mn+k}. \tag{4.4}$$

For m odd, the dual identity in question is

$$\sum_{i=1}^{m+1} (-1)^{a_i} T(i,m+1,k) L_{n+a_i}^m / P(i,m+1) = (-1)^{a_1+a_2+\cdots+a_{m+1}+k} 5^{\frac{m+1}{2}} F_{mn+k}. \tag{4.5}$$

5. Concluding Comments

Regarding identities (1.7) and (1.10), neither seems to be obtainable from the other. We therefore believe that (1.10), together with (4.1)–(4.5), add to the plethora of existing identities satisfied by powers of Fibonacci/Lucas numbers.

In Section 4, we used Dresel's method of writing down new (dual) identities from known identities to write down an additional five identities that are similar in nature to our main result. However, this is not the end of the story, as we demonstrate via a simple example. Consider Lemma 3.1, which is the first instance of our main result. The dual identity of (3.1) with respect to a_1 is

$$\begin{aligned} &(-1)^{a_1+1} F_{a_2+a_3-k} L_{n+a_1}^2 / (L_{a_1-a_2} L_{a_1-a_3}) + (-1)^{a_2} L_{a_1+a_3-k} F_{n+a_2}^2 / (L_{a_2-a_1} F_{a_2-a_3}) \\ &+ (-1)^{a_3} L_{a_1+a_2-k} F_{n+a_3}^2 / (L_{a_3-a_1} F_{a_3-a_2}) = (-1)^{a_1+a_2+a_3+k} F_{2n+k}. \end{aligned} \tag{5.1}$$

The possibilities seem limitless.

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