SOMMERVILLE'S SYMMETRICAL CYCLIC COMPOSITIONS OF A POSITIVE INTEGER WITH PARTS AVOIDING MULTIPLES OF AN INTEGER

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Abstract. A linear composition of a positive integer N is an ordered list of positive integers (called parts) whose sum equals N. A linear composition of N is called palindromic of type I if it stays the same when it is read in reverse order, while it is called palindromic of type II if it becomes a palindromic composition of type I (of an integer smaller than N) when we remove the first part. By considering all cyclic shifts of a linear composition of N as equivalent linear compositions, we may define a cyclic composition of N. Cyclic compositions were originally studied by D. M. Y. Sommerville more than a century ago, who also considered symmetrical cyclic compositions of N. In this paper, we prove that the equivalence class of every symmetrical cyclic composition of N with length K (excluding the one with all parts equal when K divides N) contains exactly two linear palindromic compositions of type I or II. Using this result, we derive generating functions for the cardinalities of classes of symmetrical cyclic compositions of N that avoid integers in a set A. We then derive general recurrences for the cardinalities of these classes of symmetrical cyclic compositions. When A consists of all multiples of a positive integer r, we use these recurrences to derive Fibonacci-type recurrences. We also indicate that the number of dihedral compositions of N with K parts in A is the average of the corresponding numbers of cyclic compositions and Sommerville's symmetrical cyclic compositions.

1. Introduction

Linear compositions of positive integers were studied by many mathematicians in the 19th century, but the first systematic study was made by MacMahon [11, 12]. A *linear composition* of a positive integer N of *length* K is a K-tuple $(\lambda_1, \lambda_2, \ldots, \lambda_K) \in \mathbb{Z}_{>0}^K$ such that

$$N = \lambda_1 + \lambda_2 + \cdots + \lambda_K. \tag{1.1}$$

Here the numbers $\lambda_1, \lambda_2, \ldots, \lambda_K$ are called the *parts* of the composition. We may define *cyclic compositions* of N of length K as equivalence classes on the set of all linear compositions of length K such that two compositions are equivalent, that is, they belong to the same class, if and only if one can be obtained from the other by cyclic shifts. If $(\lambda_1, \ldots, \lambda_K)$ is a representative of an equivalence class, we denote the class by $[(\lambda_1, \ldots, \lambda_K)]_R$. For example, when $N = 4$, there are five equivalence classes (cyclic compositions):

- with length 1: $[(4)]_R$;
- with length 2: $[(1,3)]_R = [(3,1)]_R$ and $[(2,2)]_R$;
- with length 3: $[(1,1,2)]_R = [(1,2,1)]_R = [(2,1,1)]_R$;
- with length 4: $[(1,1,1,1)]_R$.

A *type I linear palindromic (or self-inverse)* composition of N with length K is a linear composition $(\lambda_1, \ldots, \lambda_K)$ of N such that

$$(\lambda_1, \lambda_2, \ldots, \lambda_K) = (\lambda_K, \lambda_{K-1}, \ldots, \lambda_1),$$

i.e., $\lambda_i = \lambda_{K+1-i}$ for $i = 1, \ldots, K$. Denote by $P_A^{L_1}(N; K)$ the number of type I linear palindromic compositions of N with length K and parts in the set $A \subseteq \mathbb{Z}_{>0}$.

A *type II linear palindromic (or self-inverse)* composition of N with length K is a linear composition $(\lambda_1, \ldots, \lambda_K)$ of N such that

$$(\lambda_1, \lambda_2, \ldots, \lambda_K) = (\lambda_1, \lambda_K, \ldots, \lambda_2),$$

i.e, $\lambda_i = \lambda_{K+2-i}$ for $i = 2, \ldots, K$. For $K = 1$, we assume that $(\lambda_1) = (N)$ is a linear palindromic composition of both types. We denote by $P_A^{L_2}(N; K)$ the number of type II linear palindromic compositions of N with length K and parts in the set $A \subseteq \mathbb{Z}_{>0}$.

Sommerville [17, pp. 301-304] examined the number of *symmetrical cyclic compositions* of N with length K. In the terminology of this paper, a cyclic composition of N with length K is called symmetrical if and only if its equivalence class contains at least one type I or II linear palindromic composition. It just so happens that in the previous example with $N = 4$, all cyclic compositions are symmetrical.

Denote by $P_A^R(N; K)$ the number of symmetric cyclic compositions of N with length K and parts in the set $A \subseteq \mathbb{Z}_{>0}$. When $A = \mathbb{Z}_{>0}$ and $0 \leq k \leq n$, Sommerville [17] proved that

$$P_{\mathbb{Z}_{>0}}^R(2n+1; 2k+1) = P_{\mathbb{Z}_{>0}}^R(2n+2; 2k+1) = P_{\mathbb{Z}_{>0}}^R(2n+1; 2k) = P_{\mathbb{Z}_{>0}}^R(2n; 2k) = \binom{n}{k}. \quad (1.2)$$

We have to exclude the cases where either $K = 2k = 0$ or $N = 2n = 0$.

In this paper, we generalize Sommerville's results. One of our main results is the following theorem.

Theorem 1.1. *Assume $N, K \in \mathbb{Z}_{>0}$.*

(a) *If $N, K > 1$, then the equivalence class of every symmetrical cyclic composition of N with length K (but excluding the one with all the parts being equal when K divides N) contains exactly two linear palindromic compositions of type I or II.*

(b) *If $A \subseteq \mathbb{Z}_{>0}$, then*

$$P_A^R(N; K) = \frac{P_A^{L_1}(N; K) + P_A^{L_2}(N; K)}{2}.$$

Remark 1: The two results in Theorem 1.1 are simple and interesting, and they are useful for studying *dihedral compositions* of integers as well — see further discussion in Section 4.

In Section 2.2, for a general $A \subseteq \mathbb{Z}_{>0}$, we provide generating functions for $P_A^R(2n+1; 2k+1)$, $P_A^R(2n+2; 2k+1)$, $P_A^R(2n+1; 2k)$, $P_A^R(2n; 2k)$, and other similar quantities. In Section 2.3, we present recursive relations for these quantities. Finally, when A is the set of positive integers that avoid multiples of an integer, we obtain Fibonacci-type recursive formulas in Section 2.4. For example, when A avoids multiples of integer $r \geq 2$, and

$$f_n := P_A^R(2n+1; \text{odd}) := \sum_{k=0}^{n} P_A^R(2n+1; 2k+1) \text{ or } f_n := P_A^R(2n+2; \text{odd}) := \sum_{k=0}^{n} P_A^R(2n+2; 2k+1)$$

we prove in Theorem 2.11 of Section 2.4 that

$$f_n = f_{n-1} + f_{n-2} + \cdots + f_{n-r} \quad \text{for } n \geq r.$$

On the other hand, if $f_n := P_A^R(2n+1; \text{even}) := \sum_{k=1}^{n} P_A^R(2n+1; 2k)$ and r is even, we prove that

$$f_n = f_{n-1} + f_{n-2} + \cdots + f_{n-r} + r - 2 \quad \text{for } n \geq r.$$

Proofs of all the results appear in Section 3. Section 4 gives concluding remarks.

2. Main results

2.1. **Some lemmas.** In the following lemma (and in other results later in the paper), we define

$$I(x \in A) = 1 \quad \text{if } x \in A, \quad \text{and 0 otherwise.}$$

MacMahon [11] proved that, for $n, k \in \mathbb{Z}_{>0}$ with $1 \leq k \leq n$ and $A = \mathbb{Z}_{>0}$,

$$P^{L_1}_{\mathbb{Z}_{>0}}(2n; 2k) = P^{L_1}_{\mathbb{Z}_{>0}}(2n; 2k-1) = P^{L_1}_{\mathbb{Z}_{>0}}(2n-1; 2k-1) = \binom{n-1}{k-1}, \tag{2.1}$$

while $P^{L_1}_{\mathbb{Z}_{>0}}(2n-1; 2k) = 0$. Regarding type II linear palindromic compositions, we obtain the following result.

Lemma 2.1. *Let $A \subseteq \mathbb{Z}_{>0}$.*

(a) *For any $N, K \in \mathbb{Z}_{>1}$ with $2 \leq K \leq N$,*

$$P^{L_2}_A(N; K) = \sum_{i=1}^{N-K+1} P^{L_1}_A(N-i; K-1)\, I(i \in A).$$

(b) *If $A = \mathbb{Z}_{>0}$ and $0 \leq k \leq n$, then*

$$P^{L_2}_{\mathbb{Z}_{>0}}(2n+1; 2k+1) = P^{L_2}_{\mathbb{Z}_{>0}}(2n+2; 2k+1) = \binom{n}{k}, \quad P^{L_2}_{\mathbb{Z}_{>0}}(2n+1; 2k) = 2\binom{n}{k},$$

and

$$P^{L_2}_{\mathbb{Z}_{>0}}(2n; 2k) = \binom{n-1}{k} + \binom{n}{k}.$$

(*We exclude the cases where either $N = 2n = 0$ or $K = 2k = 0$.*)

In Lemmas 2.2–2.4 below, we present results for **general** type I and type II palindromic strings. By **general**, we mean the strings do not necessarily have to be compositions of a positive integer. Before we give these results, we introduce our notation as follows.

Let K be a positive integer and consider the column vectors (of K components)

$$\begin{aligned} \boldsymbol{e}_1 &= (1, 0, \ldots, 0, 0)', \\ \boldsymbol{e}_2 &= (0, 1, \ldots, 0, 0)', \\ &\vdots \\ \boldsymbol{e}_K &= (0, 0, \ldots, 0, 1)'. \end{aligned}$$

Consider also the $K \times K$ matrices

$$\boldsymbol{S} = (\boldsymbol{e}_K, \boldsymbol{e}_{K-1}, \ldots, \boldsymbol{e}_2, \boldsymbol{e}_1), \quad \boldsymbol{T} = (\boldsymbol{e}_1, \boldsymbol{e}_K, \boldsymbol{e}_{K-1}, \ldots, \boldsymbol{e}_2),$$

and

$$\boldsymbol{P} = \begin{pmatrix} \boldsymbol{e}_2' \\ \boldsymbol{e}_3' \\ \vdots \\ \boldsymbol{e}_K' \\ \boldsymbol{e}_1' \end{pmatrix}.$$

Note that for $1 \le j \le K-1$,

$$\boldsymbol{P}^j = \begin{pmatrix} \boldsymbol{e}'_{j+1} \\ \vdots \\ \boldsymbol{e}'_K \\ \boldsymbol{e}'_1 \\ \vdots \\ \boldsymbol{e}'_j \end{pmatrix},$$

that is, $\boldsymbol{P}^j$ is a *cyclic shifter*. Note that the string $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_K)'$ is palindromic of type I if and only if $\boldsymbol{S\lambda} = \boldsymbol{\lambda}$, and it is palindromic of type II if and only if $\boldsymbol{T\lambda} = \boldsymbol{\lambda}$. (We treat all strings as column vectors.)

Next, we define the *period* of linear string $\boldsymbol{\lambda}= (\lambda_1, \ldots, \lambda_K)'$ to be the smallest positive integer d with the property that $\boldsymbol{\lambda}$ can be obtained by repeating K/d times the linear string $(\lambda_1, \ldots, \lambda_d)'$. In such a case, $\lambda_{jd+i} = \lambda_i$ for $i = 1, \ldots, d$ and $j = 0, \ldots, (K/d) - 1$.

The following three lemmas are important in proving Theorem 1.1.

Lemma 2.2. *Let d be the period of $\boldsymbol{\lambda}= (\lambda_1, \ldots, \lambda_K)'$. If $\boldsymbol{\lambda}$ is a palindromic string, then $(\lambda_1, \ldots, \lambda_d)'$ is a palindromic string of the same type.*

Lemma 2.3. *Let d denote the period of a palindromic string $\boldsymbol{\lambda}= (\lambda_1, \ldots, \lambda_K)'$. We assume $1 < d \le K$, i.e., $\boldsymbol{\lambda}$ has at least two distinct parts. Let j be an integer with $1 \le j \le d-1$.*

- **Case 1:** *$d = 2m$ with $m \in \mathbb{Z}_{>0}$.*
  - (1) *If $\boldsymbol{S\lambda} = \boldsymbol{\lambda}$ (i.e., $\boldsymbol{\lambda}$ is of type I), then:*
    - (a) *when $j = m$, $\boldsymbol{\lambda}$ is in the null space of $\boldsymbol{P}^j\boldsymbol{S}-\boldsymbol{SP}^j$; when $j \ne m$, $\boldsymbol{\lambda}$ is not in the null space of $\boldsymbol{P}^j\boldsymbol{S}-\boldsymbol{SP}^j$;*
    - (b) *on the other hand, $\boldsymbol{\lambda}$ is not in the null space of $\boldsymbol{P}^j\boldsymbol{S}-\boldsymbol{TP}^j$.*
  - (2) *If $\boldsymbol{T\lambda} = \boldsymbol{\lambda}$ (i.e., $\boldsymbol{\lambda}$ is of type II), then:*
    - (a) *when $j = m$, $\boldsymbol{\lambda}$ is in the null space of $\boldsymbol{P}^j\boldsymbol{T}-\boldsymbol{TP}^j$; when $j \ne m$, $\boldsymbol{\lambda}$ is not in the null space of $\boldsymbol{P}^j\boldsymbol{T}-\boldsymbol{TP}^j$;*
    - (b) *on the other hand, $\boldsymbol{\lambda}$ is not in the null space of $\boldsymbol{P}^j\boldsymbol{T}-\boldsymbol{SP}^j$.*
- **Case 2***: $d = 2m+1$ with $m \in \mathbb{Z}_{>0}$.*
  - (1) *If $\boldsymbol{S\lambda} = \boldsymbol{\lambda}$ (i.e., $\boldsymbol{\lambda}$ is of type I), then:*
    - (a) *when $j = m$, $\boldsymbol{\lambda}$ is in the null space of $\boldsymbol{P}^j\boldsymbol{S}-\boldsymbol{TP}^j$; when $j \ne m$, $\boldsymbol{\lambda}$ is not in the null space of $\boldsymbol{P}^j\boldsymbol{S}-\boldsymbol{TP}^j$;*
    - (b) *on the other hand, $\boldsymbol{\lambda}$ is not in the null space of $\boldsymbol{P}^j\boldsymbol{S}-\boldsymbol{SP}^j$.*
  - (2) *If $\boldsymbol{T\lambda} = \boldsymbol{\lambda}$ (i.e., $\boldsymbol{\lambda}$ is of type II), then:*
    - (a) *when $j = m+1$, $\boldsymbol{\lambda}$ is in the null space of $\boldsymbol{P}^j\boldsymbol{T}-\boldsymbol{SP}^j$; when $j \ne m+1$, $\boldsymbol{\lambda}$ is not in the null space of $\boldsymbol{P}^j\boldsymbol{T}-\boldsymbol{SP}^j$;*
    - (b) *on the other hand, $\boldsymbol{\lambda}$ is not in the null space of $\boldsymbol{P}^j\boldsymbol{T}-\boldsymbol{TP}^j$.*

Lemma 2.4. *Let d denote the period of $\boldsymbol{\lambda}$ and assume $1 < d \le K$. Let $m_1 = \lfloor d/2 \rfloor$ and $m_2 = \lceil d/2 \rceil$, where $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ are the floor and ceiling functions, respectively.*

- (a) *If $\boldsymbol{\lambda}$ is a type I palindromic string, then among the d distinct strings, $\boldsymbol{\lambda}$, $\boldsymbol{P\lambda}$, $\boldsymbol{P}^2\boldsymbol{\lambda}$, $\ldots$, $\boldsymbol{P}^{d-1}\boldsymbol{\lambda}$, only $\boldsymbol{\lambda}$ and $\boldsymbol{P}^{m_1}\boldsymbol{\lambda}$ are palindromic strings of either type.*
- (b) *If $\boldsymbol{\lambda}$ is a type II palindromic string, then among the d distinct strings, $\boldsymbol{\lambda}$, $\boldsymbol{P\lambda}$, $\boldsymbol{P}^2\boldsymbol{\lambda}$, $\ldots$, $\boldsymbol{P}^{d-1}\boldsymbol{\lambda}$, only $\boldsymbol{\lambda}$ and $\boldsymbol{P}^{m_2}\boldsymbol{\lambda}$ are palindromic strings of either type.*

2.2. **Results about generating functions.** Given a set $A \subseteq \mathbb{Z}_{>0}$, we denote by $c_A^L(N;K)$ and $c_A^R(N;K)$ the number of linear and cyclic compositions, respectively, of length K of positive integer N with parts in A. We also let

$$c_A^L(N) = \sum_{K=1}^{N} c_A^L(N;K) \quad \text{and} \quad c_A^R(N) = \sum_{K=1}^{N} c_A^R(N;K).$$

MacMahon [11], and probably others before him, proved that, for $1 \le K \le N$,

$$c_{\mathbb{Z}_{>0}}^L(N;K) = \binom{N-1}{K-1} \quad \text{and} \quad c_{\mathbb{Z}_{>0}}^L(N) = 2^{N-1}.$$

Regarding the number of cyclic compositions of N when $A = \mathbb{Z}_{>0}$, partial results were obtained by Sommerville [17]. His results were generalized more than seven decades later by Razen *et al.* [14]; see also [1], [3, p. 48], [8], [18, pp. 70-71], and [19]. In these references, it is proven that, for $1 \le K \le N$,

$$c_{\mathbb{Z}_{>0}}^R(N;K) = \frac{1}{N} \sum_{j \mid \gcd(N,K)} \phi(j) \binom{N/j}{K/j} \quad \text{and} \quad c_{\mathbb{Z}_{>0}}^R(N) = -1 + \frac{1}{N} \sum_{j \mid N} \phi(j) 2^{\frac{N}{j}},$$

where $\phi(j)$ is *Euler's totient function* at j, giving the number of positive integers that are less than or equal to j and co-prime to j. Note the summation ranges over all positive divisors j of $\gcd(N,K)$ in the first sum and all the positive divisors j of N in the second sum.

It is proven in Hoggatt and Lind [7] that the bivariate generating function for the number of linear compositions of N with K parts in the set $A \subseteq \mathbb{Z}_{>0}$ is

$$\sum_{N,K \ge 0} c_A^L(N;K) x^N y^K = \frac{1}{1 - y \sum_{m \in A} x^m}.$$

(Here we assume $c_A^L(N;0) = 0$ if $N > 0$, and $c_A^L(0;0) = 1$.) Setting $y = 1$ in the above equation, we get that the generating function of the total number of linear compositions of N with parts in A is

$$\sum_{N \ge 1} c_A^L(N) x^N = \frac{1}{1 - \sum_{m \in A} x^m}.$$

See also Beck and Robbins [2] and Heubach and Mansour [6].

It also follows from the results in Hadjicostas [5] that the bivariate generating function for the number of cyclic compositions of N with K parts in A is

$$\sum_{N,K \ge 0} c_A^R(N;K) x^N y^K = \sum_{N \ge 1} \frac{\phi(N)}{N} \log \frac{1}{1 - y^N \sum_{m \in A} x^{mN}}.$$

Setting again $y = 1$ in the above equation, we get that the generating function of the total number of cyclic compositions of N with parts in A is

$$\sum_{N \ge 1} c_A^R(N) x^N = \sum_{N \ge 1} \frac{\phi(N)}{N} \log \frac{1}{1 - \sum_{m \in A} x^{mN}}.$$

These generating functions can also be obtained using the theory in Flajolet and Sedgewick [3, pp. 27 and 729-730] and Flajolet and Soria [4]. This theory concerns the generating function of cycles of unlabeled combinatorial structures.

Now, we focus our attention on palindromic compositions. We let

$$P_A^{L_1}(N) = \sum_{K=1}^{N} P_A^{L_1}(N;K), \; P_A^{L_2}(N) = \sum_{K=1}^{N} P_A^{L_2}(N;K), \text{ and } P_A^R(N) = \sum_{K=1}^{N} P_A^R(N;K).$$

When the number of parts $K \in \mathbb{Z}_{>0}$ is fixed, it follows from Heubach and Mansour [6] that the generating function for the number of type I linear palindromic (or self-inverse) compositions of N with K parts in A is

$$\sum_{N\geq 1} P_A^{L_1}(N;K)x^N = \begin{cases} \left(\sum_{m\in A} x^m\right)\left(\sum_{m\in A} x^{2m}\right)^{(K-1)/2}, & \text{if } K \text{ is odd;} \\ \left(\sum_{m\in A} x^{2m}\right)^{K/2}, & \text{if } K \text{ is even.} \end{cases} \tag{2.2}$$

Summing over all $K \in \mathbb{Z}_{>0}$, we get that the generating function for the total number of type I linear palindromic (or self-inverse) compositions of N with parts in A is

$$\sum_{N\geq 1} P_A^{L_1}(N)x^N = \frac{1+\sum_{m\in A} x^m}{1-\sum_{m\in A} x^{2m}} - 1.$$

Using part (a) of Lemma 2.1, we prove in Section 3 the following lemma that gives various generating functions for the number of type II linear palindromic compositions:

Lemma 2.5. *Let $A \subseteq \mathbb{Z}_{>0}$.*

(a) *For each $K \in \mathbb{Z}_{>0}$,*

$$\sum_{N\geq 1} P_A^{L_2}(N;K)x^N = \begin{cases} \left(\sum_{m\in A} x^m\right)\left(\sum_{m\in A} x^{2m}\right)^{(K-1)/2}, & \textit{if } K \textit{ is odd;} \\ \left(\sum_{m\in A} x^m\right)^2\left(\sum_{m\in A} x^{2m}\right)^{(K/2)-1}, & \textit{if } K \textit{ is even.} \end{cases}$$

(b) *The bivariate generating function of the numbers $P_A^{L_2}(N;K)$ is given by*

$$\sum_{K\geq 1}\sum_{N\geq 1} P_A^{L_2}(N;K)x^N y^K = \frac{\left(\sum_{m\in A} x^m\right)y + \left(\sum_{m\in A} x^m\right)^2 y^2}{1-y^2\sum_{m\in A} x^{2m}}.$$

(c) *The generating function of the numbers $P_A^{L_2}(N)$ is given by*

$$\sum_{N\geq 1} P_A^{L_2}(N)x^N = \frac{\left(\sum_{m\in A} x^m\right) + \left(\sum_{m\in A} x^m\right)^2}{1-\sum_{m\in A} x^{2m}}.$$

Theorem 2.6. *Let $A \subseteq \mathbb{Z}_{>0}$. For $K \in \mathbb{Z}_{>0}$,*

$$\sum_{N\geq 1} P_A^{R}(N;K)\,x^N = \begin{cases} \left(\sum_{m\in A} x^m\right)\left(\sum_{m\in A} x^{2m}\right)^{(K-1)/2}, & \textit{if } K \textit{ is odd;} \\ \frac{1}{2}\left[\left(\sum_{m\in A} x^m\right)^2 + \left(\sum_{m\in A} x^{2m}\right)\right]\left(\sum_{m\in A} x^{2m}\right)^{(K/2)-1}, & \textit{if } K \textit{ is even.} \end{cases}$$

Next, we state another main result of the paper regarding the generating functions for the four cases of Sommerville's symmetrical cyclic compositions of N with K parts in A. For this theorem, we use the following notation:

$$A_o = A \cap \{2j-1 \mid j \in \mathbb{Z}_{>0}\} \quad \text{and} \quad A_e = A \cap \{2j \mid j \in \mathbb{Z}_{>0}\}.$$

Theorem 2.7. *Let $A \subseteq \mathbb{Z}_{>0}$ and k be a nonnegative integer. Then*

(1) $\sum_{n=0}^{\infty} P_A^R(2n+1;2k+1)\,x^{2n+1} = \left(\sum_{m\in A_o} x^m\right)\left(\sum_{m\in A} x^{2m}\right)^k$.

(2) $\sum_{n=0}^{\infty} P_A^R(2n+2;2k+1)\,x^{2n+2} = \left(\sum_{m\in A_e} x^m\right)\left(\sum_{m\in A} x^{2m}\right)^k$.

(3) $\sum_{n=0}^{\infty} P_A^R(2n+1;2k)\,x^{2n+1} = \left(\sum_{m\in A_o} x^m\right)\left(\sum_{m\in A_e} x^m\right)\left(\sum_{m\in A} x^{2m}\right)^{k-1}$.

(4) $\sum_{n=1}^{\infty} P_A^R(2n;2k)\,x^{2n} = \frac{1}{2}h_A(x)\left(\sum_{m\in A} x^{2m}\right)^{k-1}$, *where*

$$h_A(x) := \left(\sum_{m\in A_o} x^m\right)^2 + \left(\sum_{m\in A_e} x^m\right)^2 + \left(\sum_{m\in A} x^{2m}\right). \tag{2.3}$$

(*For cases* (3) *and* (4) *above we assume $k \geq 1$.*)

Corollary 2.8. *For $K = 2k+1$ with $k \geq 0$,*

$$P_A^{L_1}(N, K = 2k+1) = P_A^{L_2}(N, K = 2k+1) = P_A^R(N, K = 2k+1) \quad \text{for all } N \in \mathbb{Z}_{>0}.$$

Remark 2: Corollary 2.8 reveals that, when K is a fixed odd positive integer, the numbers of type I linear palindromic compositions, type II linear palindromic compositions, and Sommerville's symmetrical cyclic compositions of a positive integer N with K parts in A are all equal.

Corollary 2.9. *Let $A \subseteq \mathbb{Z}_{>0}$. Then the generating function for the total number of Sommerville's symmetrical cyclic compositions of N with parts in A is*

$$\sum_{N \geq 1} P_A^R(N)\ x^N = \frac{\left(1 + \sum_{m \in A} x^m\right)^2}{2\left(1 - \sum_{m \in A} x^{2m}\right)} - \frac{1}{2}.$$

In addition,

$$\sum_{N \geq 1} P_A^R(N; odd)\ x^N = \frac{\sum_{m \in A} x^m}{1 - \sum_{m \in A} x^{2m}},$$

and

$$\sum_{N \geq 1} P_A^R(N; even)\ x^N = \frac{\left(\sum_{m \in A} x^m\right)^2 + \left(\sum_{m \in A} x^{2m}\right)}{2\left(1 - \sum_{m \in A} x^{2m}\right)}.$$

Remark 3: Multiplying both sides of the equation in Theorem 2.6 by y^K and summing from $K = 1$ to $K = \infty$, it is easy to derive the following bivariate generating function for $P_A^R(N; K)$:

$$\sum_{N, K \geq 1} P_A^R(N; K)\, x^N y^K = \frac{\left(1 + y \sum_{m \in A} x^m\right)^2}{2\left(1 - y^2 \sum_{m \in A} x^{2m}\right)} - \frac{1}{2}.$$

2.3. **Results about some general recurrences.** It follows from (1.2) that

$$P_{\mathbb{Z}_{>0}}^R(2n+1; \text{even}) = \sum_{k=1}^{n} P_{\mathbb{Z}_{>0}}^R(2n+1; 2k) = 2^n - 1;$$
$$P_{\mathbb{Z}_{>0}}^R(2n+1; \text{odd}) = \sum_{k=0}^{n} P_{\mathbb{Z}_{>0}}^R(2n+1; 2k+1) = 2^n;$$
$$P_{\mathbb{Z}_{>0}}^R(2n; \text{even}) = \sum_{k=1}^{n} P_{\mathbb{Z}_{>0}}^R(2n; 2k) = 2^n - 1;$$
$$P_{\mathbb{Z}_{>0}}^R(2n; \text{odd}) = \sum_{k=0}^{n-1} P_{\mathbb{Z}_{>0}}^R(2n; 2k+1) = 2^{n-1}.$$

In addition, $P_{\mathbb{Z}_{>0}}^R(2n+1) = 2^{n+1} - 1$ and $P_{\mathbb{Z}_{>0}}^R(2n) = 3 \cdot 2^{n-1} - 1$. In this subsection we state some general recurrence relations about the numbers

$$P_A^R(2n+1; \text{odd}),\ P_A^R(2n+2; \text{odd}),\ P_A^R(2n+1; \text{even}), \text{ and } P_A^R(2n; \text{even}) \qquad (2.4)$$

for a general set $A \subseteq \mathbb{Z}_{>0}$. These recurrences are useful in proving generalized Fibonacci-type recurrence equations in next subsection when A is a set of positive integers that avoid multiples of a fixed integer.

Theorem 2.10. *Let $A \subseteq \mathbb{Z}_{>0}$ and n be a non-negative integer. Then*

$$P_A^R(2n+1; odd) = \sum_{s=0}^{n-1} P_A^R(2s+1; odd)\, I(n-s \in A) + I(2n+1 \in A); \tag{2.5}$$

$$P_A^R(2n+2; odd) = \sum_{s=0}^{n-1} P_A^R(2s+2; odd)\, I(n-s \in A) + I(2n+2 \in A); \tag{2.6}$$

$$P_A^R(2n+1; even) = \sum_{s=0}^{n-1} P_A^R(2s+1; even)\, I(n-s \in A) + \sum_{s=0}^{n-1} I(2s+1 \in A)\, I(2(n-s) \in A); \tag{2.7}$$

$$P_A^R(2n; even) = \sum_{s=1}^{n-1} P_A^R(2s; even)\, I(n-s \in A) + \frac{1}{2}\sum_{s=0}^{n-1} I(2s+1 \in A)\, I[2(n-s)-1 \in A] + \frac{1}{2}\sum_{s=1}^{n-1} I(2s \in A)\, I(2(n-s) \in A) + \frac{1}{2} I(n \in A). \tag{2.8}$$

(*For equation (2.8) we assume $n \geq 1$.*)

2.4. **Fibonacci-type recurrence equations.** When A is a set of positive integers that avoid all multiples of a fixed positive integer $r \geq 2$, the four sequences of numbers in (2.4) satisfy Fibonacci-like recurrence equations similar to those in [5, 15, 16, 20].

Theorem 2.11. *Let r be a fixed positive integer ($r \geq 2$) and A be the set all of positive integers that are not multiples of r.*

(1) *If $f_n = P_A^R(2n+1; odd)$ or $f_n = P_A^R(2n+2; odd)$, then the sequence $(f_n : n \in \mathbb{Z}_{\geq 0})$ satisfies*

$$f_n = f_{n-1} + f_{n-2} + \cdots + f_{n-r} \quad \text{for } n \geq r.$$

In addition, we have:

(a) *If $f_n = P_A^R(2n+1; odd)$ and r is even, then $f_n = 2^n$ for $0 \leq n \leq r-1$.*

(b) *If $f_n = P_A^R(2n+1; odd)$ and r is odd, then*

$$f_n = \begin{cases} 2^n, & \text{if } 0 \leq n \leq \frac{r-3}{2}, \\ 2^{\frac{r-1}{2}} - 1, & \text{if } n = \frac{r-1}{2}, \\ 2^n - 2^{n-\frac{r+1}{2}}, & \text{if } \frac{r+1}{2} \leq n \leq r-1. \end{cases}$$

(c) *If $f_n = P_A^R(2n+2; odd)$ and r is even, then*

$$f_n = \begin{cases} 2^n, & \text{if } 0 \leq n \leq \frac{r}{2} - 2, \\ 2^{\frac{r}{2}-1} - 1, & \text{if } n = \frac{r}{2} - 1, \\ 2^n - 2^{n-\frac{r}{2}}, & \text{if } \frac{r}{2} \leq n \leq r-2, \\ 2^{r-1} - 2^{\frac{r}{2}-1} - 1 & \text{if } n = r-1. \end{cases}$$

(d) *If $f_n = P_A^R(2n+2; odd)$ and r is odd, then $f_n = 2^n$ for $0 \leq n \leq r-2$ and $f_{r-1} = 2^{r-1} - 1$.*

(2) *If $f_n = P_A^R(2n+1; even)$, then the sequence $(f_n : n \in \mathbb{Z}_{\geq 0})$ satisfies*

$$f_n = f_{n-1} + f_{n-2} + \cdots + f_{n-r} + \alpha(n) \quad \text{for } n \geq r,$$

where $\alpha(n) = r-2$ *if* r *is even, and*

$$\alpha(n) = \begin{cases} r-1, & \text{if } n \equiv \frac{r-1}{2} \pmod{r}, \\ r-2, & \text{if } n \not\equiv \frac{r-1}{2} \pmod{r}, \end{cases}$$

if r *is odd. In addition, for* $0 \le n \le r-1$*, we have*

$$f_n = \begin{cases} 2^n - 1, & \text{if } 0 \le n \le \lceil \frac{r}{2} \rceil - 1, \\ 2^n - 2^{n - \lceil \frac{r}{2} \rceil} - 1, & \text{if } \lceil \frac{r}{2} \rceil \le n \le r-1. \end{cases}$$

(3) *If* $f_n = P_A^R(2n; even)$*, then the sequence* $(f_n : n \in \mathbb{Z}_{\ge 1})$ *satisfies*

$$f_n = f_{n-1} + f_{n-2} + \cdots + f_{n-r} + \beta(n) \quad \text{for } n \ge r+1,$$

where

$$\beta(n) = \begin{cases} 1, & \text{if } r = 2, \\ r-1, & \text{if } n \equiv 0 \left(\mathrm{mod}\ \frac{r}{2}\right) \text{ and } r \text{ is even} \ge 4, \\ r-2, & \text{if } n \not\equiv 0 \left(\mathrm{mod}\ \frac{r}{2}\right) \text{ and } r \text{ is even} \ge 4, \\ r-1, & \text{if } n \equiv 0 \pmod{r} \text{ and } r \text{ is odd}, \\ r-2, & \text{if } n \not\equiv 0 \pmod{r} \text{ and } r \text{ is odd}. \end{cases}$$

In addition, for $1 \le n \le r$*, we have*

$$f_n = \begin{cases} 2^n - 1, & \text{if } 1 \le n \le \lfloor \frac{r}{2} \rfloor, \\ 2^n - 2^{n-1-\lfloor \frac{r}{2} \rfloor} - 1, & \text{if } \lfloor \frac{r}{2} \rfloor + 1 \le n \le r. \end{cases}$$

3. Proofs

Proof of Lemma 2.1. (a) Assume $N, K \in \mathbb{Z}_{>0}$ with $2 \le K \le N$. Let $\mathcal{P}_A^{L_1}(N;K)$ and $\mathcal{P}_A^{L_2}(N;K)$ be the collections of all linear palindromic compositions of N with K parts in A of types I and II, respectively. Let

$$B = \{1, \ldots, N-K+1\} \cap A, \quad \mathcal{C}_1 = \mathcal{P}_A^{L_2}(N;K), \quad \text{and} \quad \mathcal{C}_2 = \bigcup_{\ell \in B} \mathcal{P}_A^{L_1}(N-\ell; K-1),$$

and define the function $g : \mathcal{C}_1 \to \mathcal{C}_2$ by

$$g((\lambda_1, \ldots, \lambda_K)) = (\lambda_2, \ldots, \lambda_K) \quad \text{for all } (\lambda_1, \ldots, \lambda_K) \in \mathcal{C}_1.$$

One can easily show that g is well-defined and is a bijection between the sets $\mathcal{C}_1$ and $\mathcal{C}_2$ with inverse function $g^{-1} : \mathcal{C}_2 \to \mathcal{C}_1$ given by

$$g^{-1}((\lambda_2, \ldots, \lambda_K)) = \left(N - \sum_{s=2}^{K} \lambda_s, \lambda_2, \ldots, \lambda_K \right) \quad \text{for all } (\lambda_2, \ldots, \lambda_K) \in \mathcal{C}_2.$$

This implies that

$$P_A^{L_2}(N;K) = \#\mathcal{P}_A^{L_2}(N;K) = \# \bigcup_{\ell \in B} \mathcal{P}_A^{L_1}(N-\ell; K-1) = \sum_{\ell=1}^{N-K+1} P_A^{L_1}(N-\ell; K-1)\, I(\ell \in A),$$

which proves the first part of the lemma.

(b) Using the identity

$$\sum_{m=s}^{n} \binom{m}{s} = \binom{n+1}{s+1}, \tag{3.1}$$

the first part of the lemma, and MacMahon's [11] results (see equations (2.1)), we can easily prove the second part of the lemma. We only show the proof for the formula for $P^{L_2}_{\mathbb{Z}_{>0}}(2n;2k)$. We have

$$\begin{aligned} P^{L_2}_{\mathbb{Z}_{>0}}(2n;2k) &= \sum_{\ell=1}^{2n-2k+1} P^{L_1}_{\mathbb{Z}_{>0}}(2n-\ell;2k-1) \\ &= \sum_{t=1}^{n-k} P^{L_1}_{\mathbb{Z}_{>0}}(2n-2t;2k-1) + \sum_{t=0}^{n-k} P^{L_1}_{\mathbb{Z}_{>0}}(2n-2t-1;2k-1) \\ &= \sum_{t=1}^{n-k} \binom{n-t-1}{k-1} + \sum_{t=0}^{n-k} \binom{n-t-1}{k-1} \\ &= \binom{n-1}{k} + \binom{n}{k}, \end{aligned}$$

where we have applied the identity (3.1) twice. □

Proof of Lemma 2.2. The proof is easy and the details are omitted here. □

Proof of Lemma 2.3. We only prove Case 1 when $d = 2m$ with $m \in \mathbb{Z}_{>0}$. The proof of Case 2 is similar and hence is omitted.

(1)(a) If $\boldsymbol{S\lambda} = \boldsymbol{\lambda}$, then, for $1 \le j \le d-1$,

$$\boldsymbol{P}^j\boldsymbol{S\lambda} = \boldsymbol{P}^j\boldsymbol{\lambda} = (\lambda_{j+1}, \ldots, \lambda_{2m}, \overrightarrow{\boxed{\cdots}}, \lambda_1, \ldots, \lambda_j)',$$

where $\overrightarrow{\boxed{\cdots}}$ comprises $(K/d) - 1$ replicates of $(\lambda_1, \ldots, \lambda_d)$. Using Lemma 2.2 and the fact that $\boldsymbol{\lambda}$ is a type I palindromic string, we then have

$$\begin{aligned} \boldsymbol{SP}^j\boldsymbol{\lambda} &= \boldsymbol{S}(\boldsymbol{P}^j\boldsymbol{\lambda}) \\ &= (\lambda_j, \ldots, \lambda_1, \overleftarrow{\boxed{\cdots}}, \lambda_{2m}, \ldots, \lambda_{j+1})' \\ &= (\lambda_j, \ldots, \lambda_1, \overrightarrow{\boxed{\cdots}}, \lambda_{2m}, \ldots, \lambda_{j+1})' \\ &= (\lambda_{2m+1-j}, \ldots, \lambda_{2m}, \overrightarrow{\boxed{\cdots}}, \lambda_1, \ldots, \lambda_{2m-j})' \\ &= \boldsymbol{P}^{2m-j}\boldsymbol{\lambda} = \boldsymbol{P}^{d-j}\boldsymbol{\lambda}, \end{aligned}$$

where $\overleftarrow{\boxed{\cdots}}$ comprises $(K/d) - 1$ replicates of $(\lambda_d, \ldots, \lambda_1)$. Since d is the period of $\boldsymbol{\lambda}$, the strings $\boldsymbol{\lambda}$, $\boldsymbol{P\lambda}$, ..., $\boldsymbol{P}^{d-1}\boldsymbol{\lambda}$ are all different. Thus,

$$\boldsymbol{P}^j\boldsymbol{S\lambda} = \boldsymbol{SP}^j\boldsymbol{\lambda} \Leftrightarrow \boldsymbol{P}^j\boldsymbol{\lambda} = \boldsymbol{P}^{d-j}\boldsymbol{\lambda} \Leftrightarrow j = d-j \Leftrightarrow j = \frac{d}{2} = m.$$

This proves part (1)(a) of Case 1 in the lemma.

(1)(b) If $\boldsymbol{S\lambda} = \boldsymbol{\lambda}$, then, for $1 \le j \le d-1$,

$$\boldsymbol{P}^j\boldsymbol{S\lambda} = \boldsymbol{P}^j\boldsymbol{\lambda} = (\lambda_{j+1}, \ldots, \lambda_{2m}, \overrightarrow{\boxed{\cdots}}, \lambda_1, \ldots, \lambda_j)'.$$

Using Lemma 2.2 and the fact that $\boldsymbol{\lambda}$ is a type I palindromic string, we then have

$$\begin{aligned}\boldsymbol{TP}^j\boldsymbol{\lambda} &= \boldsymbol{T}(\boldsymbol{P}^j\boldsymbol{\lambda})\\ &= (\lambda_{j+1}, \lambda_j, \ldots, \lambda_1, \overleftarrow{\boxed{\cdots}}, \lambda_{2m}, \ldots, \lambda_{j+2})'\\ &= (\lambda_{j+1}, \lambda_j, \ldots, \lambda_1, \overrightarrow{\boxed{\cdots}}, \lambda_{2m}, \ldots, \lambda_{j+2})'\\ &= (\lambda_{2m-j}, \lambda_{2m+1-j}, \ldots, \lambda_{2m}, \overrightarrow{\boxed{\cdots}}, \lambda_1, \ldots, \lambda_{2m-j-1})'\\ &= \boldsymbol{P}^{2m-1-j}\boldsymbol{\lambda}.\end{aligned}$$

Because $j \neq 2m-1-j$,

$$\boldsymbol{P}^j\boldsymbol{\lambda} \neq \boldsymbol{P}^{2m-1-j}\boldsymbol{\lambda}, \quad \text{and thus,} \quad \boldsymbol{P}^j\boldsymbol{S}\boldsymbol{\lambda} \neq \boldsymbol{TP}^j\boldsymbol{\lambda}.$$

At this point, the proof of (1) in Case 1 is complete.

(2)(a) If $\boldsymbol{T\lambda} = \boldsymbol{\lambda}$, then, for $1 \leq j \leq d-1$,

$$\boldsymbol{P}^j\boldsymbol{T\lambda} = \boldsymbol{P}^j\boldsymbol{\lambda} = (\lambda_{j+1}, \ldots, \lambda_{2m}, \overrightarrow{\boxed{\cdots}}, \lambda_1, \ldots, \lambda_j)'.$$

Because $\boldsymbol{\lambda}$ is a type II palindromic string, we then have

$$\begin{aligned}\boldsymbol{TP}^j\boldsymbol{\lambda} &= \boldsymbol{T}(\boldsymbol{P}^j\boldsymbol{\lambda})\\ &= (\lambda_{j+1}, \lambda_j, \ldots, \lambda_1, \overleftarrow{\boxed{\cdots}}, \lambda_{2m}, \ldots, \lambda_{j+2})'\\ &= (\lambda_{j+1}, \lambda_j, \ldots, \lambda_2, \overrightarrow{\boxed{\cdots}}, \lambda_1, \lambda_{2m}, \ldots, \lambda_{j+2})'\\ &= (\lambda_{2m+1-j}, \lambda_{2m+2-j}, \ldots, \lambda_{2m}, \overrightarrow{\boxed{\cdots}}, \lambda_1, \lambda_2, \ldots, \lambda_{2m-j})'\\ &= \boldsymbol{P}^{2m-j}\boldsymbol{\lambda} = \boldsymbol{P}^{d-j}\boldsymbol{\lambda}.\end{aligned}$$

Thus,

$$\boldsymbol{P}^j\boldsymbol{T\lambda} = \boldsymbol{TP}^j\boldsymbol{\lambda} \Leftrightarrow \boldsymbol{P}^j\boldsymbol{\lambda} = \boldsymbol{P}^{d-j}\boldsymbol{\lambda} \Leftrightarrow j = d-j \Leftrightarrow j = \frac{d}{2} = m.$$

This proves part (2)(a) in Case 1 of the lemma.

(2)(b) If $\boldsymbol{T\lambda} = \boldsymbol{\lambda}$, then, for $1 \leq j \leq d-1$,

$$\boldsymbol{P}^j\boldsymbol{T\lambda} = \boldsymbol{P}^j\boldsymbol{\lambda} = (\lambda_{j+1}, \ldots, \lambda_{2m}, \overrightarrow{\boxed{\cdots}}, \lambda_1, \ldots, \lambda_j)'.$$

We have

$$\begin{aligned}\boldsymbol{SP}^j\boldsymbol{\lambda} &= \boldsymbol{S}(\boldsymbol{P}^j\boldsymbol{\lambda})\\ &= (\lambda_j, \ldots, \lambda_1, \overleftarrow{\boxed{\cdots}}, \lambda_{2m}, \ldots, \lambda_{j+1})'\\ &= \begin{cases} (\overrightarrow{\boxed{\cdots}}, \lambda_1, \lambda_{2m}, \ldots, \lambda_2)' = \boldsymbol{\lambda}, & \text{if } j = 1,\\ (\lambda_j, \ldots, \lambda_2, \overrightarrow{\boxed{\cdots}}, \lambda_1, \lambda_{2m}, \ldots, \lambda_{j+1})', & \text{if } 2 \leq j \leq d-1,\end{cases}\\ &= \begin{cases} \boldsymbol{\lambda}, & \text{if } j = 1,\\ (\lambda_{2m+2-j}, \ldots, \lambda_{2m}, \overrightarrow{\boxed{\cdots}}, \lambda_1, \lambda_2, \ldots, \lambda_{2m+1-j})', & \text{if } 2 \leq j \leq d-1,\end{cases}\\ &= \begin{cases} \boldsymbol{\lambda}, & \text{if } j = 1,\\ \boldsymbol{P}^{2m+1-j}\boldsymbol{\lambda}, & \text{if } 2 \leq j \leq d-1.\end{cases}\end{aligned}$$

Since $j \neq 2m+1-j$, we have proved that

$$\boldsymbol{P}^j\boldsymbol{T\lambda} \neq \boldsymbol{SP}^j\boldsymbol{\lambda}.$$

At this point, the proof of (2) in Case 1 is complete. □

Proof of Lemma 2.4. It follows immediately from Lemma 2.3. □

Proof of Theorem 1.1. Part (a) of the theorem follows from Lemma 2.4. To prove part (b), note that part (a) implies that the equivalence class of every symmetric cyclic composition of N with K parts in A (at least two of which are distinct) contains exactly two linear palindromic compositions of N with K parts in A (of either type). In addition, every cyclic composition of N with K parts in A that are all equal contains exactly one linear composition (with all parts in A) that is palindromic of both types. Also, every linear palindromic composition (with parts in A) of type I or II belongs to exactly one symmetrical cyclic composition of N with K parts in A. Hence,

$$2P_A^R(N;K) = P_A^{L_1}(N;K) + P_A^{L_2}(N;K),$$

and this completes the proof of the theorem. □

Proof of Lemma 2.5. (a) For $K = 1$, we have $P_A^{L_2}(N; K = 1) = I(N \in A)$, so

$$\sum_{N\geq 1} P_A^{L_2}(N;K)\, x^N = \left(\sum_{m\in A} x^m\right)\left(\sum_{m\in A} x^{2m}\right)^{(1-1)/2}.$$

Assume now $K \geq 2$. By part (a) in Lemma 2.1,

$$\sum_{N\geq K} P_A^{L_2}(N;K)\, x^N = \sum_{N\geq K} \sum_{i=1}^{N-K+1} P_A^{L_1}(N-i;K-1)\, I(i\in A)\, x^N.$$

Since $i \leq N - K + 1$ if and only if $N \geq i + K - 1$, changing the order of summation in the above double sum, we get

$$\sum_{N\geq K} P_A^{L_2}(N;K)\, x^N = \sum_{i\geq 1} I(i\in A)\, x^i \sum_{N\geq i+K-1} P_A^{L_1}(N-i;K-1)\, x^{N-i}.$$

Defining $P_A^{L_1}(N;K) = 0 = P_A^{L_2}(N;K)$ when $N < K$ and using the change of variables $M = N - i$, we get

$$\begin{aligned}\sum_{N\geq 1} P_A^{L_2}(N;K)\, x^N &= \sum_{m\in A} x^m \sum_{M\geq K-1} P_A^{L_1}(M;K-1)\, x^M \\ &= \left(\sum_{m\in A} x^m\right)\left(\sum_{M\geq 1} P_A^{L_1}(M;K-1) x^M\right).\end{aligned}$$

Using equations (2.2), we get

$$\sum_{N\geq 1} P_A^{L_2}(N;K)\, x^N = \left(\sum_{m\in A} x^m\right)\begin{cases} \left(\sum_{m\in A} x^m\right)\left(\sum_{m\in A} x^{2m}\right)^{(K-2)/2}, & \text{if } K-1 \text{ is odd;} \\ \left(\sum_{m\in A} x^{2m}\right)^{(K-1)/2}, & \text{if } K-1 \text{ is even,}\end{cases}$$

from which we can easily prove part (a) of the lemma.

(b) We have from part (a) of the lemma that

$$\begin{aligned}\sum_{K\geq 1}\sum_{N\geq 1} P_A^{L_2}(N;K)\,x^N y^K &= \left(\sum_{m\in A} x^m\right)\sum_{\ell=0}^{\infty}\left(\sum_{m\in A} x^{2m}\right)^{\frac{(2\ell+1)-1}{2}} y^{2\ell+1}\\ &\quad+\left(\sum_{m\in A} x^m\right)^2\sum_{\ell=1}^{\infty}\left(\sum_{m\in A} x^{2m}\right)^{\frac{2\ell}{2}-1} y^{2\ell}\\ &= \left(\sum_{m\in A} x^m\right) y\sum_{\ell=0}^{\infty}\left(y^2\sum_{m\in A} x^{2m}\right)^{\ell}\\ &\quad+\left(\sum_{m\in A} x^m\right)^2 y^2\sum_{\ell=1}^{\infty}\left(y^2\sum_{m\in A} x^{2m}\right)^{\ell-1},\end{aligned}$$

from which part (b) of the lemma follows easily.

(c) This part of the lemma follows from part (b) by setting $y = 1$. □

Proof of Theorem 2.6. The theorem follows directly from part (b) of Theorem 1.1, equation (2.2), and part (a) of Lemma 2.5. □

Proof of Theorem 2.7. We indicate how to prove parts (1) and (4). The proofs of parts (2) and (3) are similar, and hence are omitted.

Part (1). It follows from Theorem 2.6 that, for $k \geq 0$,

$$\begin{aligned}\sum_{n=0}^{\infty} P_A^{R}(2n+1;2k+1)\,x^{2n+1} &= \frac{1}{2}\sum_{N\geq 1} P_A^{R}(N;2k+1)\left[x^N-(-x)^N\right]\\ &= \frac{1}{2}\left(\sum_{m\in A} x^m\right)\left(\sum_{m\in A} x^{2m}\right)^k - \frac{1}{2}\left[\sum_{m\in A}(-x)^m\right]\left[\sum_{m\in A}(-x)^{2m}\right]^k\\ &= \left(\sum_{m\in A_o} x^m\right)\left(\sum_{m\in A} x^{2m}\right)^k.\end{aligned}$$

Part (4). It follows again from Theorem 2.6 that, for $k \geq 0$,

$$\begin{aligned}
\sum_{n=1}^{\infty} P_A^R(2n; 2k)\, x^{2n} &= \frac{1}{2} \sum_{N \geq 1} P_A^R(N; 2k)[x^N + (-x)^N] \\
&= \frac{1}{4} \left[\left(\sum_{m \in A} x^m \right)^2 + \sum_{m \in A} x^{2m} \right] \left(\sum_{m \in A} x^{2m} \right)^{k-1} \\
&\quad + \frac{1}{4} \left[\left(\sum_{m \in A} (-x)^m \right)^2 + \sum_{m \in A} (-x)^{2m} \right] \left(\sum_{m \in A} (-x)^{2m} \right)^{k-1} \\
&= \frac{1}{4} \left[\left(\sum_{m \in A_e} x^m + \sum_{m \in A_o} x^m \right)^2 + \left(\sum_{m \in A_e} x^m - \sum_{m \in A_o} x^m \right)^2 \right. \\
&\quad \left. + 2 \sum_{m \in A} x^{2m} \right] \left(\sum_{m \in A} x^{2m} \right)^{k-1} \\
&= \frac{1}{2} \left[\left(\sum_{m \in A_e} x^m \right)^2 + \left(\sum_{m \in A_o} x^m \right)^2 + \sum_{m \in A} x^{2m} \right] \left(\sum_{m \in A} x^{2m} \right)^{k-1}.
\end{aligned}$$

This completes the proof of Theorem 2.7. □

Proof of Corollary 2.8. It follows from equation (2.2), part (a) of Lemma 2.5, and Theorem 2.6. □

Proof of Corollary 2.9. If we let

$$P_A^R(N; \mathrm{odd}) = \sum_{k=0}^{\infty} P_A^R(N; 2k+1) \quad \text{and} \quad P_A^R(N; \mathrm{even}) = \sum_{k=1}^{\infty} P_A^R(N; 2k),$$

then the result below follows immediately from Lemma 2.6. □

Proof of Theorem 2.10. We prove the recursive formulas for $P_A^R(2n+1; \mathrm{odd})$ and $P_A^R(2n; \mathrm{even})$. The proofs of the other two recursive formulas, about $P_A^R(2n+2; \mathrm{odd})$ and $P_A^R(2n+1; \mathrm{even})$, are similar and are thus omitted. It follows from Theorem 2.7 that

$$\begin{aligned}
\sum_{n=0}^{\infty} P_A^R(2n+1; \mathrm{odd})\, x^{2n+1} &= \sum_{n=0}^{\infty} \sum_{\ell=0}^{n} P_A^R(2n+1; 2\ell+1)\, x^{2n+1} \\
&= \sum_{\ell=0}^{\infty} \sum_{n=0}^{\infty} P_A^R(2n+1; 2\ell+1)\, x^{2n+1} \\
&= \sum_{\ell=0}^{\infty} \left(\sum_{m \in A_o} x^m \right) \left(\sum_{m \in A} x^{2m} \right)^{\ell} \\
&= \frac{\sum_{m \in A_o} x^m}{1 - \sum_{m \in A} x^{2m}}.
\end{aligned}$$

Therefore,

$$\begin{aligned}\sum_{n=0}^{\infty} P_A^R(2n+1;\text{odd})\,x^{2n+1} &= \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} I(m\in A)\,x^{2m}\,P_A^R(2n+1;\text{odd})\,x^{2n+1} \\ &\quad + \sum_{m=1}^{\infty} I(m\in A_o)\,x^m \\ &= \sum_{n=0}^{\infty}\left[\sum_{s=0}^{n-1} P_A^R(2s+1;\text{odd})\,I(n-s\in A)\right] x^{2n+1} \\ &\quad + \sum_{n=0}^{\infty} I(2n+1\in A)\,x^{2n+1},\end{aligned}$$

from which the recursive relation about $P_A^R(2n+1;\text{odd})$ follows easily.

Again, it follows from Theorem 2.7 that

$$\begin{aligned}\sum_{n=1}^{\infty} P_A^R(2n;\text{even})\,x^{2n} &= \sum_{n=1}^{\infty}\sum_{\ell=1}^{n} P_A^R(2n;2\ell)\,x^{2n} \\ &= \sum_{\ell=1}^{\infty}\sum_{n=1}^{\infty} P_A^R(2n;2\ell)\,x^{2n} \\ &= \frac{h_A(x)}{2}\sum_{\ell=1}^{\infty}\left(\sum_{m\in A} x^{2m}\right)^{\ell-1} \\ &= \frac{h_A(x)}{2\left(1-\sum_{m\in A} x^{2m}\right)},\end{aligned}$$

where $h_A(x)$ is defined by equation (2.3). We then have

$$\begin{aligned}2\sum_{n=1}^{\infty} P_A^R(2n;\text{even})\,x^{2n} &= 2\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} P_A^R(2n;\text{even})\,I(m\in A)\,x^{2(n+m)} \\ &\quad + \sum_{s=0}^{\infty}\sum_{\ell=0}^{\infty} I(2s+1\in A)\,I(2\ell+1\in A)\,x^{2(s+\ell+1)} \\ &\quad + \sum_{s=1}^{\infty}\sum_{\ell=1}^{\infty} I(2s\in A)\,I(2\ell\in A)\,x^{2(s+\ell)} + \sum_{n=1}^{\infty} I(n\in A)\,x^{2n} \\ &= 2\sum_{n=1}^{\infty}\left[\sum_{s=1}^{n-1} P_A^R(2s;\text{even})\,I(n-s\in A)\right] x^{2n} + \\ &\quad + \sum_{n=1}^{\infty}\left[\sum_{s=0}^{n-1} I(2s+1\in A)\,I[2(n-s)-1\in A]\right] x^{2n} \\ &\quad + \sum_{n=1}^{\infty}\left[\sum_{s=1}^{n-1} I(2s\in A)\,I(2(n-s)\in A)\right] x^{2n} + \sum_{n=1}^{\infty} I(n\in A)\,x^{2n}.\end{aligned}$$

Equating coefficients of x^{2n}, we can easily prove the recursive relationship for $P_A^R(2n;\text{even})$ in Theorem 2.10. □

Proof of Theorem 2.11. Assume $r \in \mathbb{Z}_{\geq 2}$ and A is the set of all positive integers that are not multiples of r. We only prove parts (1)(a,b) and (3). The proofs of the other parts are similar and hence are omitted.

Part (1)(a,b). Assume $f_n = P_A^R(2n+1; \text{odd})$ for $n \in \mathbb{Z}_{\geq 0}$. When $n \geq r$, equation (2.5) in Theorem 2.10 gives

$$f_n = \sum_{s=n-r}^{n-1} f_s\, I(n-s \in A) + \sum_{s=0}^{n-r-1} f_s\, I(n-r-s \in A) + I(2(n-r)+1 \in A) \tag{3.2}$$

because $x \in A$ if and only if $x - r \in A$ for $x > r$. By applying equation (2.5) again for $n-r$ rather than n, we get

$$f_{n-r} = \sum_{s=0}^{n-r-1} f_s\, I(n-r-s \in A) + I(2(n-r)+1 \in A). \tag{3.3}$$

Subtracting equation (3.3) from equation (3.2), we get

$$f_n = \sum_{s=n-r}^{n-1} f_s\, I(n-s \in A) + f_{n-r} = \sum_{s=1}^{r} f_{n-s}$$

because $I(r \in A) = 0$.

If r is even, then for $0 \leq n \leq r-1$ we have $1 \leq 2n+1 \leq 2r-1$, i.e., $I(2n+1 \in A) = 1$; in addition, for $0 \leq s \leq n-1$ we have $1 \leq n-s \leq n \leq r-1$ and so $I(n-s \in A) = 1$. In such a case, it follows from equation (2.5) that $f_0 = 1$ and $f_n = \sum_{s=0}^{n-1} f_s + 1$, from which we can easily prove that $f_n = 2^n$ for $0 \leq n \leq r-1$. This completes the proof of Part (1)(a).

If r is odd, then for $0 \leq n \leq \frac{r-3}{2}$ we have $1 \leq 2n+1 \leq r-2$. In this case, using equation (2.5) we get $f_n = \sum_{s=0}^{n-1} f_s + 1$, and thus, $f_n = 2^n$ for $0 \leq n \leq \frac{r-3}{2}$.

If r is odd and $n = \frac{r-1}{2}$, then $2n+1 = r$, and so, equation (2.5) implies

$$f_n = \sum_{s=0}^{n-1} f_s = \sum_{s=0}^{n-1} 2^s = 2^n - 1.$$

If r is odd and $\frac{r+1}{2} \leq n \leq r-1$, then $r+2 \leq 2n+1 \leq 2r-1$ and $1 \leq n-s \leq n \leq r-1$ for $0 \leq s \leq n-1$. It follows from equation (2.5) that

$$f_n = \sum_{s=0}^{n-1} f_s + 1.$$

This implies

$$f_{\frac{r+1}{2}} = \sum_{s=0}^{\frac{r-3}{2}} 2^s + \left(2^{\frac{r-1}{2}} - 1\right) + 1 = 2^{\frac{r+1}{2}} - 1.$$

In addition, $f_{n+1} = 2f_n$, and thus we can easily prove by induction that $f_n = 2^n - 2^{n-\frac{r+1}{2}}$ for $\frac{r+1}{2} \leq n \leq r-1$. This completes the proof of Part (1)(b).

Part (3). Assume first $n \geq r+1$. Applying equation (2.8) twice, once for n and once for $n-r$, we get

$$f_n = \sum_{s=n-r}^{n-1} f_s + \beta(n),$$

where

$$\beta(n) := \frac{1}{2}\sum_{s=n-r}^{n-1} I(2s+1 \in A)\, I[2(n-s)-1 \in A] + \frac{1}{2}\sum_{s=n-r}^{n-1} I(2s \in A)\, I(2(n-s) \in A). \quad (3.4)$$

We shall prove that $\beta(n)$ is given by the formula in Part (3) of Theorem 2.11. When $r = 2$, we get

$$\beta(n) = \frac{1}{2}[I(2n-3 \in A)\, I(3 \in A) + I(2n-1 \in A)\, I(1 \in A) + I(2n-4 \in A)\, I(4 \in A) + I(2n-2 \in A)\, I(2 \in A)] = 1.$$

Next assume r is even ≥ 4 and $n \equiv 0 \pmod{\frac{r}{2}}$. It is clear in this case that

$$I(2s+1 \in A)\, I(2(n-s)-1 \in A) = 1 \text{ for } n-r \leq s \leq n-1. \quad (3.5)$$

Also, $1 \leq n-s \leq r$ for $n-r \leq s \leq n-1$; in such a case, $I(2(n-s) \in A) = 0$ if and only if $s \in \{n-\frac{r}{2}, n-r\}$. In addition, there is $\ell \in \mathbb{Z}_{\geq 1}$ such that $n = \ell\frac{r}{2}$. It follows that

$$r(\ell-2) \leq 2s \leq r\ell - 2 \quad \text{for } n-r \leq s \leq n-1.$$

In such a case, $I(2s \in A) = 0$ if and only if $s \in \{\frac{r(\ell-2)}{2}, \frac{r(\ell-1)}{2}\} = \{n-r, n-\frac{r}{2}\}$. It follows from equation (3.4) that $\beta(n) = (r+r-2)/2 = r-1$.

Next assume r is even ≥ 4 and $n \not\equiv 0 \pmod{\frac{r}{2}}$. Again, equation (3.5) holds. Furthermore, for $n-r \leq s \leq n-1$, $I(2(n-s) \in A) = 0$ if and only if $s \in \{n-\frac{r}{2}, n-r\}$. Also, there is $\ell \in \mathbb{Z}_{\geq 0}$ such that $n = \ell\frac{r}{2} + a$, where $a \in \{1, \ldots, \frac{r}{2}-1\}$. Then

$$r(\ell-2) + 2a \leq 2s \leq r\ell + 2a - 2 \quad \text{for } n-r \leq s \leq n-1.$$

It follows that $I(2s \in A) = 0$ if and only if $s \in \{\frac{r(\ell-1)}{2}, \frac{r\ell}{2}\}$. Since $1 \leq a \leq \frac{r}{2}-1$, we have

$$\{n-r, n-\frac{r}{2}\} \cap \{\frac{r(\ell-1)}{2}, \frac{r\ell}{2}\} = \{\frac{r(\ell-2)}{2} + a, \frac{r(\ell-1)}{2} + a\} \cap \{\frac{r(\ell-1)}{2}, \frac{r\ell}{2}\} = \emptyset,$$

and therefore $\beta(n) = (r+r-4)/2 = r-2$.

Assume next that r is odd ≥ 3 and $n \equiv 0 \pmod{r}$. Then there is $\ell \in \mathbb{Z}_{\geq 1}$ such that $n = \ell r$. Since $2 \leq 2(n-s) \leq 2r$ for $n-r \leq s \leq n-1$, we have $I(2(n-s) \in A) = 0$ if and only if $s = n-r = r(\ell-1)$. Since also $1 \leq 2(n-s)-1 \leq 2r-1$ for $n-r \leq s \leq n-1$, we have $I(2(n-s)-1 \in A) = 0$ if and only if $2(n-s)-1 = r$ if and only if $s = n - \frac{r+1}{2} = \frac{r(2\ell-1)-1}{2}$. In addition, we have

$$2r(\ell-1) = 2n-2r \leq 2s \leq 2\ell r - 2 = 2n-2 \quad \text{for } n-r \leq s \leq n-1;$$

whence $I(2s \in A) = 0$ if and only if $s = r(\ell-1) = n-r$. Finally, $I(2s+1 \in A) = 0$ if and only if $2s+1 = r(2\ell-1)$ if and only if $s = \frac{r(2\ell-1)-1}{2}$. It follows from equation (3.4) that $\beta(n) = (r-1+r-1)/2 = r-1$.

Finally, assume r is odd ≥ 3 and $n \not\equiv 0 \pmod{r}$. Thus, there is $\ell \in \mathbb{Z}_{\geq 0}$ and $a \in \{1, \ldots, r-1\}$ such that $n = r\ell + a$. Since $2 \leq 2(n-s) \leq 2r$ for $n-r \leq s \leq n-1$, we have $I(2(n-s) \in A) = 0$ if and only if $s = n-r = r(\ell-1)+a$; also, $I(2(n-s)-1 \in A) = 0$ if and only if $2(n-s)-1 = r$ if and only if $n-s = \frac{r+1}{2}$ if and only if $s = \frac{r(2\ell-1)-1}{2} + a$. In addition,

$$2r(\ell-1) + 2a = 2n-2r \leq 2s \leq 2n-2 = 2r\ell + 2a - 2 \quad \text{for } n-r \leq s \leq n-1.$$

It follows that $I(2s \in A) = 0$ if and only if $2s = 2r\ell$ if and only if $s = r\ell$. Also, $I(2s+1 \in A) = 0$ if and only if $2s+1 = r(2\ell-1)$ if and only if $s = \frac{r(2\ell-1)-1}{2}$. Since the numbers $\frac{r(2\ell-1)-1}{2}$ and $\frac{r(2\ell-1)-1}{2} + a$ are distinct, and so are the numbers $r(\ell-1)+a$ and $r\ell$, it follows from equation (3.4) that $\beta(n) = (r-2+r-2)/2 = r-2$.

We finish the proof of Part (3) of the theorem by verifying the formulae for the initial conditions. For $1 \le n \le r$, we have $I(n-s \in A) = 1$ when $1 \le s \le n-1$ because $n-1 < r$. Also, $I(n \in A) = 1$ when $1 \le n \le r-1$ and $I(n \in A) = 0$ when $n = r$.

Assume first $1 \le n \le \left\lfloor \frac{r}{2} \right\rfloor$. For $0 \le s \le n-1$, we have

$$1 \le 2s+1 \le 2n-1 \le 2\left\lfloor \frac{r}{2} \right\rfloor - 1 \le r-1 < r \quad \text{and} \quad 1 \le 2(n-s)-1 \le 2n-1 < r.$$

Thus, in this case, $I(2s+1 \in A) = 1 = I(2(n-s)-1 \in A)$. In addition, for $1 \le s \le n-1$, we have

$$2 \le 2s \le 2n-2 < r \quad \text{and} \quad 2 \le 2(n-s) \le 2n-2 < r.$$

In such a case, $I(2s \in A) = 1 = I(2(n-s) \in A)$. Using equation (2.8), we can prove that, for $1 \le n \le \left\lfloor \frac{r}{2} \right\rfloor$,

$$f_n = \sum_{s=1}^{n-1} f_s + \frac{n+n-1+1}{2} = \sum_{s=1}^{n-1} f_s + n. \tag{3.6}$$

It is then easy to prove by finite induction that $f_n = 2^n - 1$.

Finally, assume $\left\lfloor \frac{r}{2} \right\rfloor + 1 \le n \le r$. Then, for $0 \le s \le n-1$, we have

$$1 \le 2s+1 \le 2n-1 \le 2r-1 \quad \text{and} \quad 1 \le 2(n-s)-1 \le 2n-1 \le 2r-1.$$

Thus, in this case, $I(2s+1 \in A) = 0$ if and only if r is odd and $s = \frac{r-1}{2}$; and $I(2(n-s)-1 \in A) = 0$ if and only if r is odd and $s = n - \frac{r+1}{2}$. (In particular, $I(2s+1 \in A) = 0 = I(2(n-s)-1 \in A)$ if and only if r is odd, $n = r$, and $s = \frac{r-1}{2}$.) On the other hand, for $1 \le s \le n-1$,

$$2 \le 2s \le 2n-2 \le 2r-2 \quad \text{and} \quad 2 \le 2(n-s) \le 2n-2 \le 2r-2.$$

It follows that (in this case) $I(2s \in A) = 0$ if and only if r is even and $s = \frac{r}{2}$; also, $I(2(n-s) \in A) = 0$ if and only if r is even and $s = n - \frac{r}{2}$. (In particular, $I(2s \in A) = 0 = I(2(n-s) \in A)$ if and only if r is even, $n = r$, and $s = \frac{r}{2}$.) Note that when r is odd, $\frac{r-1}{2} = \left\lfloor \frac{r}{2} \right\rfloor \le n-1$, while when r is even, we have $\frac{r}{2} = \left\lfloor \frac{r}{2} \right\rfloor \le n-1$.

If $\left\lfloor \frac{r}{2} \right\rfloor + 1 \le n < r$ and r is odd, then

$$f_n = \sum_{s=1}^{n-1} f_s + \frac{n-2+n-1+1}{2} = \sum_{s=1}^{n-1} f_s + n - 1. \tag{3.7}$$

On the other hand, if $\left\lfloor \frac{r}{2} \right\rfloor + 1 \le n < r$ and r is even, then

$$f_n = \sum_{s=1}^{n-1} f_s + \frac{n+n-3+1}{2} = \sum_{s=1}^{n-1} f_s + n - 1.$$

One can easily prove that, if $n = r$ and r is odd, or $n = r$ and r is even, the formula $f_n = \sum_{s=1}^{n-1} f_s + n - 1$ is still true.

It follows from equations (3.6) and (3.7) that

$$f_{\left\lfloor \frac{r}{2} \right\rfloor + 1} = 2 f_{\left\lfloor \frac{r}{2} \right\rfloor} = 2^{\left\lfloor \frac{r}{2} \right\rfloor + 1} - 2.$$

Equation (3.7) also implies $f_n = 2f_{n-1} + 1$ for $\left\lfloor \frac{r}{2} \right\rfloor + 1 < n \le r$. We can then prove by finite induction that

$$f_n = 2^n - 2^{n-1-\left\lfloor \frac{r}{2} \right\rfloor} - 1 \quad \text{for } \left\lfloor \frac{r}{2} \right\rfloor + 1 \le n \le r.$$

This completes the proof of Part (3) of Theorem 2.11. □

4. Concluding Remarks

In Remark 1, we mentioned that our results are useful for studying dihedral compositions. Dihedral compositions of N of length K are defined as equivalence classes on the set of all linear compositions of N of length K. Here, two linear compositions of N with length K are said to be equivalent if and only if they differ by a cyclic shift or a reversal of order; see Knopfmacher and Robbins [9]. Given a set $A \subseteq \mathbb{Z}_{>0}$, we denote by $c_A^D(N;K)$ the number of dihedral compositions of N with length K and parts in A. With insights gained from this study, we have

$$\begin{aligned} c_A^D(N;K) &= \frac{c_A^R(N;K) - P_A^R(N;K)}{2} + P_A^R(N;K) \\ &= \frac{c_A^R(N;K) + P_A^R(N;K)}{2} \\ &= \frac{2c_A^R(N;K) + P_A^{L_1}(N;K) + P_A^{L_2}(N;K)}{4}, \end{aligned}$$

which generalizes Theorem 1 in [9].

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