# GENERALIZATIONS OF DELANNOY AND CROSS POLYTOPE NUMBERS

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ABSTRACT. The Delannoy numbers and the figurate numbers for $n$-dimensional cross polytopes are doubly-recursive sequences that satisfy the same recursion formula. Using the expression $\sum_{j=0}^{n}(-1)^j\binom{m-j}{n+k}\binom{n}{j}$, we present an infinite collection of doubly recursive sequences that satisfy the same recursion formula. As a consequence, we prove some binomial identities. Some known integer sequences are subsequences of our sequences, and we give new connections between these sequences.

## 1. INTRODUCTION

Henri-Auguste Delannoy introduced the Delannoy numbers in 1895 to count the number of ways a queen can move from one square on a chessboard to another ([2]). Equivalently, the Delannoy number $D(m,n)$ counts the number of *lattice paths* from $(0,0)$ to $(m,n)$ if the allowed moves are one unit up, one unit right, or the diagonal from $(k,j)$ to $(k+1,j+1)$. Because there is only one way to move and stay on an axis, $D(m,0)=D(0,n)=1$. Otherwise, for $m,n\geq 1$, a path that arrives at $(m,n)$ comes from either $(m-1,n)$, $(m,n-1)$, or $(m-1,n-1)$, so the Delannoy numbers satisfy the recursion $D(m,n)=D(m-1,n)+D(m,n-1)+D(m-1,n-1)$. Delannoy observed that a move from $(0,0)$ to $(m,n)$ with $j$ diagonal steps corresponds to the number of words with $m-j$ letters $A$, $n-j$ letters $B$, and $j$ letters $C$, which is equal to

$$\frac{(m+n-j)!}{(m-j)!(n-j)!j!}=\binom{m+n-2j}{n-j}\binom{m+n-j}{j}=\binom{m+n-j}{n}\binom{n}{j}.$$

Thus, a formula for the Delannoy numbers comes from summing over the number of diagonal moves:

$$D(m,n)=\sum_{j=0}^{n}\binom{m+n-j}{n}\binom{n}{j}.$$

The Cross Polytope numbers $T(m,n)$ satisfy the same recurrence but with different seeds: $T(m,1)=1$, $T(1,n)=n$, and

$$T(m,n)=T(m-1,n)+T(m,n-1)+T(m-1,n-1)$$

for $m,n\geq 2$ [1, 4, 6]. The standard formula is $T(m,n)=\sum_{k=0}^{m-1}\binom{m-1}{k}\binom{n+k}{m}$. The Cross Polytope numbers are the number of vertices in a cross polytope, which is a regular, convex geometric figure. The two sets of numbers enjoy the relations $T(m,n)+T(m,n+1)=D(m,n)$ and $\sum_{k=0}^{n}D(m,k)=T(m+1,n+1)$. The primary goal of this paper is to introduce families of doubly-recursive sequences that can be generated using the expression $\sum_{j=0}^{n}(-1)^j\binom{m-j}{n+k}\binom{n}{j}$. The Delannoy and Cross Polytope numbers are members of the family. We will also establish some relations among and within these sequences. Some binomial identities and new results on known integer sequences follow as consequence.

## 2. Doubly-Recursive Sequences

Given integers $m$, $n$, and $k$ with $m \ge n \ge k \ge 1$, consider the following types of words of length $m$, where the allowed letters are $D$, $V$, and $H$. We allow $j$ letters to be $D$, but $D$ can only be used in the first $n$ letters. Of the remaining $m-j$ letters, $n-k$ are $V$, and the rest are $H$. Denote the total number of such words, for $0 \le j \le n$, by $S_k(m,n)$. Then

$$S_k(m,n) = \sum_{j=0}^{n} \binom{m-j}{n-k}\binom{n}{j}.$$

For any such word, if the first letter that is not $V$ is changed from $D$ to $H$ or from $H$ to $D$, then the parity of the number of $D$'s will change, and the new word will also be an allowable word. This shows that there is exactly the same number of words with an even number of $D$'s as with an odd number of $D$'s. Let $E_k(m,n)$ and $O_k(m,n)$ represent these numbers. Then

$$E_k(m,n) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{m-2j}{n-k}\binom{n}{2j} \quad \text{and} \quad O_k(m,n) = \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{m-(2j+1)}{n-k}\binom{n}{2j+1}.$$

We have shown that $E_k(m,n) = O_k(m,n)$, and in addition,

$$\sum_{j=0}^{n} (-1)^j \binom{m-j}{n-k}\binom{n}{j} = 0 \quad \text{for} \quad m \ge n \ge k \ge 1,$$

since the sum of the even-indexed terms equals the sum of the odd-indexed terms.

$E_k$ and $O_k$ are defined when $m < n$ and for negative values of $m$, although in these cases the expressions involve binomial coefficients with negative numbers. We do not focus on these values in this paper, but we need to establish that $E_k(m,n) = O_k(m,n)$ for all values of $m$ and $n$ where they are defined. To this end, we have a reflection formula.

| $E_1(m,n)$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | 1 | 0 | 3 | -16 | 85 | -356 |
| 2 | 1 | 2 | 1 | -4 | 25 | -146 |
| 3 | 1 | 4 | 3 | 0 | 5 | -36 |
| 4 | 1 | 6 | 9 | 4 | 1 | -6 |
| 5 | 1 | 8 | 19 | 16 | 5 | 0 |
| 6 | 1 | 10 | 33 | 44 | 25 | 6 |
| 7 | 1 | 12 | 51 | 96 | 85 | 36 |

| $E_2(m,n)$ | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|
| 1 | 2 | -2 | 12 | -60 | 310 | -1648 |
| 2 | 2 | 2 | 4 | -20 | 110 | -602 |
| 3 | 2 | 6 | 4 | -4 | 30 | -182 |
| 4 | 2 | 10 | 12 | 4 | 6 | -42 |
| 5 | 2 | 14 | 28 | 20 | 6 | -6 |
| 6 | 2 | 18 | 52 | 60 | 30 | 6 |
| 7 | 2 | 22 | 84 | 140 | 110 | 42 |

| $E_3(m,n)$ | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|
| 1 | 4 | -8 | 40 | -200 | 1036 | -5488 |
| 2 | 4 | 0 | 16 | -80 | 420 | -2240 |
| 3 | 4 | 8 | 8 | -24 | 140 | -784 |
| 4 | 4 | 16 | 16 | 0 | 36 | -224 |
| 5 | 4 | 24 | 40 | 24 | 12 | -48 |
| 6 | 4 | 32 | 80 | 80 | 36 | 0 |
| 7 | 4 | 40 | 136 | 200 | 140 | 48 |

| $E_4(m,n)$ | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|
| 1 | 8 | -24 | 120 | -616 | 3248 |
| 2 | 8 | -8 | 56 | -280 | 1456 |
| 3 | 8 | 8 | 24 | -104 | 560 |
| 4 | 8 | 24 | 24 | -24 | 176 |
| 5 | 8 | 40 | 56 | 24 | 48 |
| 6 | 8 | 56 | 120 | 104 | 48 |
| 7 | 8 | 72 | 216 | 280 | 176 |

Table 1. Values of $E_1$, $E_2$, $E_3$, $E_4$. $m$ is row, $n$ is column.

**Theorem 2.1** (Reflection Theorem)**.**

$$E_k(m,n) = (-1)^{k+n}E_k(-m+2n-k-1,n).$$

*Proof.* We use the identity $\binom{-p}{r} = (-1)^r\binom{p+r-1}{r}$.

$$(-1)^{k+n}E_k(-m+2n-k-1,n) = (-1)^{k+n}\sum_{j=0}^{\lfloor \frac{n}{2}\rfloor}\binom{-m+2n-k-1-2j}{n-k}\binom{n}{2j}$$

$$= (-1)^{k+n}(-1)^{n-k}\sum_{j=0}^{\lfloor \frac{n}{2}\rfloor}\binom{(m-2n+k+1+2j)+(n-k)-1}{n-k}\binom{n}{2j}$$

$$= \sum_{j=0}^{\lfloor \frac{n}{2}\rfloor}\binom{m-(n-2j)}{n-k}\binom{n}{n-2j} = \sum_{i=0}^{\lfloor \frac{n}{2}\rfloor}\binom{m-2i}{n-k}\binom{n}{2i} = E_k(m,n).$$

□

Clearly, $O_k(m,n)$ satisfies the same reflection formula as does $S_k(m,n)$. A consequence of this theorem is that in column $n$ of $E_k(m,n)$, the reflection is at row $n-\frac{k+1}{2}$. When $k$ is odd, the reflection point is at an entry, but when $k$ is even, the reflection point is between rows $n-\frac{k+2}{2}$ and $n-\frac{k}{2}$. We next prove that in the first non-zero column of $E_k$, every entry is $2^{k-1}$.

**Lemma 2.2.** *For positive integers $k$, $E_k(m,k) = 2^{k-1}$.*

*Proof.*

$$E_k(m,k) = \sum_{j=0}^{\lfloor \frac{k}{2}\rfloor}\binom{m-2j}{0}\binom{k}{2j} = \sum_{j=0}^{\lfloor \frac{k}{2}\rfloor}\binom{k}{2j} = 2^{k-1}.$$

□

**Theorem 2.3.** *For all integers $m$ and for $n \geq k$, we have recursion formulas*

$$E_k(m,n) = E_k(m-1,n) + E_k(m-2,n-1) + E_k(m-1,n-1), \quad \textit{and}$$
$$O_k(m,n) = O_k(m-1,n) + O_k(m-2,n-1) + O_k(m-1,n-1).$$

*Proof.* Let $n$ be even. Since Pascal's identity, $\binom{p}{r} = \binom{p-1}{r} + \binom{p-1}{r-1}$, is valid for all integer values of $p$,

$$E_k(m,n) = \sum_{i=0}^{\frac{n}{2}}\binom{m-2i}{n-k}\binom{n}{2i} = \sum_{i=0}^{\frac{n}{2}}\left[\binom{m-2i-1}{n-k} + \binom{m-2i-1}{n-k-1}\right]\binom{n}{2i}$$

$$= \sum_{i=0}^{\frac{n}{2}}\binom{(m-1)-2i}{n-k}\binom{n}{2i} + \sum_{i=0}^{\frac{n}{2}}\binom{m-2i-1}{n-k-1}\binom{n}{2i}$$

$$= E_k(m-1,n) + \sum_{i=0}^{\frac{n}{2}}\binom{m-2i-1}{n-k-1}\left[\binom{n-1}{2i-1} + \binom{n-1}{2i}\right]$$

$$= E_k(m-1,n) + \sum_{i=0}^{\frac{n}{2}}\binom{m-2i-1}{n-k-1}\binom{n-1}{2i-1} + \sum_{i=0}^{\frac{n}{2}}\binom{(m-1)-2i}{n-k-1}\binom{n-1}{2i}.$$

Re-index the first sum with $j = i-1$, and note that the first term of the first sum and the last term of the second are both zero, to get

$$E_k(m-1,n) + \sum_{j=0}^{\lfloor \frac{(n-1)-1}{2} \rfloor} \binom{(m-2)-(2j+1)}{(n-1)-k}\binom{n-1}{2j} + \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{(m-1)-2i}{(n-1)-k}\binom{n-1}{2i}$$
$$= E_k(m-1,n) + O_k(m-2,n-1) + E_k(m-1,n-1).$$

The cases for $n$ odd and for $O_k(m,n)$ are similar. The recursion for $E_k$ follows because $E_k = O_k$ for small values of $m$. Specifically, for each $n \geq k$, $O_k(n,n) = E_k(n,n) = (-1)^{k+n} E_k(n-k-1,n) = (-1)^{k+n} O_k(n-k-1,n)$. □

The next corollary has already been shown for $m \geq n$. The general result follows from the reflection formula, the first column values, and the recursion formula.

**Corollary 2.4.** *For all integers $m$ and for all integers $n \geq k \geq 1$, $E_k(m,n) = O_k(m,n)$, or*

$$\sum_{j=0}^{n} (-1)^j \binom{m-j}{n-k}\binom{n}{j} = 0.$$

The numbers $E_1$ and $O_1$ were introduced in [3], where it was also shown that $E_1(m,n) = T(m+1-n,n)$. So, $E_1$ and $O_1$ are the same numbers as the Cross Polytope numbers, and they satisfy the same recurrence relation, after re-indexing, but $E_1$ and $O_1$ give different formulas for the Cross Polytope numbers.

## 3. Basic Properties of $E_k(m,n)$

We next outline some basic structure of $E_k$ to reach a major goal of this section, which is to show that for all values of $m$ and $n$, $E_k(m,n)$ is divisible by $2^{k-1}$. Our first result is that in the second column, for $k$ odd, the reflection occurs at a value of 0, and for $k$ even, the number right below the reflection point is $2^{k-1}$.

**Lemma 3.1.** *For $i \geq 0$, $E_{2i+1}(i+1, 2i+2) = 0$, and for $i \geq 1$, $E_{2i}(i+1, 2i+1) = 2^{2i-1}$.*

*Proof.* If $i+1$ is even, then the term $j = \frac{1}{2}(i+1)$ in the summation for $E_{2i+1}$ is 0, so we have

$$E_{2i+1}(i+1,2i+2) = \sum_{j=0}^{i+1} \binom{i+1-2j}{1}\binom{2i+2}{2j}$$
$$= \sum_{j=0}^{\frac{i-1}{2}} (i+1-2j)\binom{2i+2}{2j} + \sum_{j=\frac{i+3}{2}}^{i+1} (i+1-2j)\binom{2i+2}{2j}.$$

If we reverse the order of summation for the second sum by letting $s = i+1-j$, the second sum becomes

$$\sum_{s=0}^{\frac{i-1}{2}} (-i-1+2s)\binom{2i+2}{2i+2-2s} = -\sum_{s=0}^{\frac{i-1}{2}} (i+1-2s)\binom{2i+2}{2s}.$$

It follows that $E_{2i+1}(i+1,2i+2) = 0$ in this case. The case for $i+1$ odd is similar. Now for $E_{2i}$, we consider $S_{2i} = E_{2i} + O_{2i}$. Here the $j = i+1$ term is 0, so

$$S_{2i}(i+1,2i+1) = \sum_{j=0}^{2i+1} \binom{i+1-j}{1}\binom{2i+1}{j} = \sum_{j=0}^{i} (i+1-j)\binom{2i+1}{j} + \sum_{j=i+2}^{2i+1} (i+1-j)\binom{2i+1}{j}.$$

Pulling off the last term from the first sum, and re-indexing the second sum by replacing $2i+1-j$ by $j$, this equals

$$\begin{aligned}&\sum_{j=0}^{i-1}(i+1-j)\binom{2i+1}{j}+\binom{2i+1}{i}+\sum_{j=0}^{i-1}(j-i)\binom{2i+1}{2i+1-j}\\&=\sum_{j=0}^{i-1}(i+1-j)\binom{2i+1}{j}+\sum_{j=0}^{i-1}(j-i)\binom{2i+1}{j}+\binom{2i+1}{i}\\&=\sum_{j=0}^{i-1}\binom{2i+1}{j}+\binom{2i+1}{i}=\sum_{j=0}^{i}\binom{2i+1}{j}=2^{2i}.\end{aligned}$$

It follows that $E_{2i}(i+1,2i+1)=2^{2i-1}$. □

We next show that the second column of $E_k$ is $2^{k-1}$ times consecutive even or odd integers, as $k$ is odd or even.

**Theorem 3.2.** *For all integers $m$ and for all positive integers $k$,*

$$E_k(m,k+1)=2^{k-1}[2m-(k+1)].$$

*Proof.* The proof begins with induction on $m$, using the recurrence for $E_k$. We proved the base case in the previous lemma.

$$\begin{aligned}E_k(m+1,k+1)&=E_k(m,k+1)+E_k(m-1,k)+E_k(m,k)\\&=2^{k-1}(2m-(k+1))+2^{k-1}+2^{k-1}=2^{k-1}[2(m+1)-(k+1)].\end{aligned}$$

The general result follows from the Reflection Theorem. □

We now show that the "main diagonal" numbers $E_k(n,n)$ have a simple form.

**Theorem 3.3.** *For positive integers $k$ and $n\ge k$, $E_k(n,n)=2^{k-1}\binom{n}{k}$.*

*Proof.* For $k$ even,

$$E_k(n,n)=\sum_{j=0}^{n/2}\binom{n-2j}{n-k}\binom{n}{2j}=\sum_{j=0}^{k/2}\binom{n-2j}{n-k}\binom{n}{2j}=\sum_{j=0}^{k/2}\binom{n-2j}{k-2j}\binom{n}{2j},$$

using $\binom{p}{r}=\binom{p}{p-r}$. Now using the identity $\binom{n-i}{r}\binom{n}{i}=\binom{r+i}{r}\binom{n}{r+i}$, this equals

$$\sum_{j=0}^{k/2}\binom{k}{k-2j}\binom{n}{k}=\binom{n}{k}\sum_{j=0}^{k/2}\binom{k}{2j}=\binom{n}{k}2^{k-1}.$$

The proof for $k$ odd is similar. □

Since $S_k=2E_k$, we can state an equivalent result.

**Corollary 3.4.** *For any positive integers $n$ and $k$ with $n\ge k$,*

$$\sum_{j=0}^{n}\binom{n-j}{n-k}\binom{n}{j}=2^k\binom{n}{k}.$$

When $n=k$, this is the well-known result that the sum of a row from Pascal's triangle is $2^k$. The theorems above, with the recursion formula and the Reflection Theorem, give the following result.

**Corollary 3.5.** *For positive integers $k$ and $n \geq k$, $E_k(m,n)$ is divisible by $2^{k-1}$.*

We next give the values immediately below the "main diagonal".

**Theorem 3.6.** *For positive integers $k$, and $m \geq k+1$, $E_k(m,m-1) = 2^{k-1}\left(\binom{m}{k+1} + \binom{m-1}{k+1}\right)$.*

*Proof.* We have

$$E_k(k+1,k) = \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} \binom{k+1-2j}{0}\binom{k}{2j} = \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} \binom{k}{2j} = 2^{k-1} = 2^{k-1}\left[\binom{k+1}{k+1} + \binom{k}{k+1}\right].$$

Proceeding by induction on $m$,

$$\begin{aligned} E_k(m+1,m) &= E_k(m,m) + E_k(m-1,m-1) + E_k(m,m-1) \\ &= 2^{k-1}\left(\binom{m}{k} + \binom{m-1}{k} + \left[\binom{m}{k+1} + \binom{m-1}{k+1}\right]\right) \\ &= 2^{k-1}\left(\left[\binom{m}{k} + \binom{m}{k+1}\right] + \left[\binom{m-1}{k} + \binom{m-1}{k+1}\right]\right) \\ &= 2^{k-1}\left(\binom{m+1}{k+1} + \binom{m}{k+1}\right). \end{aligned}$$

□

We next show that the third column of $E_k$ consists of numbers that are a product of a power of 2 with a number that is $k+1$ larger than a square.

**Theorem 3.7.** *For positive integers $k$,*

$$E_k(m,k+2) = 2^{k-2}\left([2m-(k+3)]^2 + (k+1)\right).$$

*Proof.* The proof begins with induction on $m$, using the recurrence for $E_k$. We use $m = k+2$ as the base case. Using Theorem 3.3, we have

$$\begin{aligned} E_k(k+2,k+2) &= 2^{k-1}\binom{k+2}{k} = 2^{k-1}\binom{k+2}{2} \\ &= 2^{k-2}(k^2+3k+2) = 2^{k-2}\left[(2(k+2)-(k+3))^2 + (k+1)\right]. \end{aligned}$$

For the inductive step, we have

$$\begin{aligned} &E_k(m+1,k+2) = E_k(m,k+2) + E_k(m,k+1) + E_k(m-1,k+1) \\ &= 2^{k-2}\left([2m-(k+3)]^2 + (k+1)\right) + 2^{k-1}\left(2(m-1)-(k+1)\right) + 2^{k-1}\left(2m-(k+1)\right) \\ &= 2^{k-2}\left([2m-(k+3)]^2 + (k+1) + 2\left(2(m-1)-(k+1)\right) + 2\left(2m-(k+1)\right)\right) \\ &= 2^{k-2}\left(4m^2 - 4m(k+3) + (k+3)^2 + (k+1) + 8m - 4k - 8\right) \\ &= 2^{k-2}\left(4m^2 + 8m + 4 - 4(m+1)(k+3) + (k+3)^2 + (k+1)\right) \\ &= 2^{k-2}\left(4(m+1)^2 - 4(m+1)(k+3) + (k+3)^2 + (k+1)\right) \\ &= 2^{k-2}\left([2(m+1)-(k+3)]^2 + (k+1)\right). \end{aligned}$$

Once again, the Reflection Theorem completes the proof for all values of $m$. □

The next theorem provides a relationship between any $n$ consecutive entries in the $n$th non-zero column of $E_k$.

**Theorem 3.8.**

$$\sum_{i=0}^{n}(-1)^i\binom{n}{i}E_k(m+i,n+k-1)=0.$$

*Proof.*

$$\sum_{i=0}^{n}(-1)^i\binom{n}{i}E_k(m+i,n+k-1)=\sum_{i=0}^{n}(-1)^i\binom{n}{i}\sum_{j=0}^{\lfloor\frac{n+k-1}{2}\rfloor}\binom{m+i-2j}{n-1}\binom{n+k-1}{2j}$$

$$=\sum_{i=0}^{n}\sum_{j=0}^{\lfloor\frac{n+k-1}{2}\rfloor}(-1)^i\binom{n}{i}\binom{m+i-2j}{n-1}\binom{n+k-1}{2j}$$

$$=\sum_{j=0}^{\lfloor\frac{n+k-1}{2}\rfloor}\binom{n+k-1}{2j}\sum_{i=0}^{n}(-1)^i\binom{m-2j+i}{n-1}\binom{n}{i}.$$

But,

$$\sum_{i=0}^{n}(-1)^i\binom{m-2j+i}{n-1}\binom{n}{i}=0$$

by Corollary 2.2. □

The final result of this section shows that for each positive $k$, the $n$th non-zero column of $E_k$ is a sequence of numbers that satisfy an $n$th-order recursion formula.

**Corollary 3.9.** *Let $a_m=E_k(m,n+k-1)$. Then*

$$a_m=\sum_{i=1}^{n}(-1)^{i+1}\binom{n}{i}a_{m-i}.$$

| $\frac{1}{2^{k-1}}E_k(m,n)$ | $n=k$ | $n=k+1$ | $n=k+2$ | $n=k+3$ | $n=k+4$ | $n=k+5$ |
|---|---|---|---|---|---|---|
| $m=k$ | 1 | | | | | |
| $m=k+1$ | 1 | $\binom{k+1}{1}$ | | | | |
| $m=k+2$ | 1 | k+3 | $\binom{k+2}{2}$ | | | |
| $m=k+3$ | 1 | k+5 | $\binom{k+3}{2}+\binom{k+2}{1}$ | $\binom{k+3}{3}$ | | |
| $m=k+4$ | 1 | k+7 | $\frac{1}{2}[(k+5)^2+k+1]$ | $\binom{k+4}{3}+\binom{k+3}{2}$ | $\binom{k+4}{4}$ | |
| $m=k+5$ | 1 | k+9 | $\frac{1}{2}[(k+7)^2+k+1]$ | | $\binom{k+5}{4}+\binom{k+4}{3}$ | $\binom{k+5}{5}$ |

TABLE 2. Basic structure of $\frac{1}{2^{k-1}}E_k$

## 4. Other Values of $k$

If we were going to use notation consistent with what's above, then for $k\le 1$ the story is that

$$\sum_{j=0}^{n}(-1)^j\binom{m-j}{n-k}\binom{n}{j}=\binom{m-n}{-k}.$$

At this point, let's replace $k$ by $-k$ in this formula. For consistency, we continue to use $E_k$ and $O_k$ exactly as before. Then we have the following theorem.

**Theorem 4.1.** *For any integer $k$, and for integers $m \geq n$,*

$$\sum_{j=0}^{n}(-1)^j\binom{m-j}{n+k}\binom{n}{j}=\binom{m-n}{k}.$$

*Proof.* We have already proved this for negative $k$, as the binomial coefficient on the right is 0 in this case. As before, consider words of length $m$ made from the letters $D$, $V$, and $H$ where $D$ is only allowed in the first $n$ letters. Select $j$ to be $D$. From the remaining $m-j$, select $n+k$ to be $V$, and make the rest $H$. Once again, match words by changing the first letter that is not $V$ from $D$ to $H$ or vice versa. This gives a correspondence between words with even numbers of $D$'s and those with odd. But, unmatched are words with no $D$'s ($j=0$) and that also have no $H$ in the first $n$ spots. These words have $m-(n+k)=m-n-k$ $H$'s, so the number of unmatched words is $\binom{m-n}{m-n-k}=\binom{m-n}{k}$. □

We now define, for non-negative integers $k$,

$$E^k(m,n)=\sum_{j=0}^{\lfloor\frac{n}{2}\rfloor}\binom{m-2j}{n+k}\binom{n}{2j}\quad\text{and}\quad O^k(m,n)=\sum_{j=0}^{\lfloor\frac{n-1}{2}\rfloor}\binom{m-(2j+1)}{n+k}\binom{n}{2j+1}.$$

If we use negative values for $k$, these formulas agree with the previous definitions for $E_k$ and $O_k$. This time we see that $E^k$ does not equal $O^k$, since Theorem 4.1 says that $E^k(n,m)-O^k(n,m)=\binom{n-m}{k}$. This time, we have two different recursion formulas that depend on $k$.

**Theorem 4.2.** *For $k \geq 0$,*

$$E^k(m,n)=E^k(m-1,n)+E^k(m-2,n-1)+E^k(m-1,n-1)-\binom{m-n-1}{k}$$

*and*

$$O^k(m,n)=O^k(m-1,n)+O^k(m-2,n-1)+O^k(m-1,n-1)+\binom{m-n-1}{k}.$$

*Proof.* For $E^k(m,n)$, we consider four types of words that can be denoted XXH, VXX, HXV, and DXV. In the first type, the last letter is $H$. The number of potential $D$'s is still $n$, and there are still $n+k$ of the $V$'s, which makes $E^k(m-1,n)$ words with the last letter $H$.

Consider next when the first letter is $V$. Then $(n-1)+k$ more $V$'s are needed, and the first $V$ means only $n-1$ of the $D$'s are possible. This gives $E^k(m-1,n-1)$ words with the first letter $V$.

We have now counted words where the first letter is $V$ and the last letter is $H$ twice, but the number of such words is exactly the same as the number with the first letter $H$ and the last letter $V$.

Finally, when the first letter is $D$ and the last is $V$, $m-2$ letters remain, and of these $n-1$ can be $D$, but the remaining number of $D$'s is odd, so this is counted by $O^k(m-2,n-1)$. Fortunately, $O^k(m-2,n-1)=E^k(m-2,n-1)-\binom{m-2-(n-1)}{k}=E^k(m-2,n-1)-\binom{m-n-1}{k}$. The proof for $O^k(m,n)$ is similar. □

**Theorem 4.3.** *For positive integers $k$,*

$$2^{k-1}(E^k(m,n)+O^k(m,n)) = E_k(m,m-n) = O_k(m,m-n), \quad \text{or}$$

$$2^{k-1}\sum_{j=0}^{n}\binom{m-j}{n+k}\binom{n}{j} = \sum_{j=0}^{\lfloor\frac{m-n}{2}\rfloor}\binom{m-2j}{m-n-k}\binom{m-n}{2j} = \sum_{j=0}^{\lfloor\frac{m-n-1}{2}\rfloor}\binom{m-(2j+1)}{m-n-k}\binom{m-n}{2j+1}.$$

*Proof.* Let us define $S^k = E^k + O^k$. Then $S^k$ satisfies the same recursion as $E_k$ and $O_k$. By Lemma 2.2,

$$2^{k-1}S^k(m,m-k) = 2^{k-1}\sum_{j=0}^{m-k}\binom{m-j}{m}\binom{m-k}{j} = 2^{k-1}\binom{m}{m}\binom{m-k}{0} = 2^{k-1} = E_k(m,k).$$

Also, by Theorem 3.6,

$$2^{k-1}S^k(m,1) = 2^{k-1}\sum_{j=0}^{1}\binom{m-j}{k+1}\binom{1}{j} = 2^{k-1}\left[\binom{m}{k+1}+\binom{m-1}{k+1}\right] = E_k(m,m-1).$$

The result now follows since $E_k$ and $S^k$ satisfy the same recursion. □

An equivalent result is

$$2^k\sum_{j=0}^{n}\binom{m-j}{n+k}\binom{n}{j} = \sum_{j=0}^{m-n}\binom{m-j}{m-n-k}\binom{m-n}{j}.$$

## 5. Delannoy Numbers and Known Integer Sequences

Up to this point, we have neglected $k=0$. We have

$$S^0(m+n,n) = \sum_{j=0}^{n}\binom{m+n-j}{n}\binom{n}{j} = D(m,n),$$

or $S^0(m,n) = D(m-n,n)$, so $S^0(m,n)$ counts the number of lattice paths from $(0,0)$ to $(m-n,n)$. We can now show that more generally $S^k(m,n)$ counts the number of lattice paths from $(0,0)$ to $(m-(n+k),n+k)$, where the number of diagonal moves is restricted by the rule that no diagonals are allowed after $n$ vertical moves have been made. $E^k$ and $O^k$ then count such paths with even or odd numbers of diagonals.

As before, consider words of length $m$ made from the letters $D$, $V$, and $H$, where $D$ is allowed only in the first $n$ letters. Select $j$ letters to be $D$. From the remaining $m-j$, select $n+k$ to be $V$, and then the rest are $H$. We construct a lattice path from such a word as follows. Each $H$ is a horizontal step. If the $i$th letter is $D$, then the $i$th $V$ is a diagonal step; otherwise a $V$ is a vertical step. Then, because the $j$ diagonal steps produce the same result as $j$ vertical and $j$ horizontal, a lattice path that starts at $(0,0)$ will end at $(m-(n+k),n+k)$.

We conclude with a selection of subsequences of our numbers in the Online Encyclopedia of Integer Sequences (OEIS). For each $k \geq 1$, a consequence of Theorem 3.7 is that $\frac{1}{2^{k-2}}E_k(m,k+2)$, scaled entries in the third column of $E_k$, are numbers $s$ such that $s-(k+1)$ is a perfect square. Among these, $\frac{1}{8}E_4(m,6)$ contains $3, 7, 15, 27, 43, 63, \ldots$, entry A097080, which are the sum of the pairwise averages of five consecutive triangular numbers. Also, $\frac{1}{16}E_5(m,7)$ contains $3, 5, 11, 21, 35, 53, \ldots$, entry A093328, which count the number of 132-avoiding two-stack sortable permutations that also avoid 4321. In addition, $\frac{1}{32}E_4(m,m+2)$ contains $1, 3, 8, 16, 33, 50, 80, \ldots$, entry A002624, which count the number of partitions of $n$

into parts with three kinds of 1 and two kinds of 2. As a final example, $\frac{1}{16}E_5(m, m+2)$ contains $1, 2, 5, 8, 14, 20, 30, \ldots$, entry A006918, which count the maximum number of squares that can be formed from $n$ lines. All of these and more are unified as members of this family of doubly recursive sequences.


## Acknowledgements

The first author thanks Yutong Yang for many useful conversations about the content of this paper. The authors also thank the referee for a careful reading of the manuscript, and for providing helpful suggestions for improvements.


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MSC2010: 11B37, 05A19, 05A10, 05A05


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