

# A COLLECTION OF CENTRAL LIMIT TYPE RESULTS IN GENERALIZED ZECKENDORF DECOMPOSITIONS

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ABSTRACT. Zeckendorf's Theorem states that if the Fibonacci numbers are indexed as  $F_1 = 1$ ,  $F_2 = 2$ ,  $F_3 = 3$ ,  $F_4 = 5, \dots$ , then every positive integer can be written uniquely as the sum of non-adjacent Fibonacci numbers. This result can be generalized to certain classes of linear recurrence relations  $\{G_n\}$  with appropriate notions of decompositions. For many decompositions, the distribution of the number of summands in the decomposition of an  $M \in [G_n, G_{n+1})$  is known to converge to a Gaussian as  $n \rightarrow \infty$ . This work discusses a more general approach to proving this kind of asymptotic Gaussian behavior that also bypasses technical obstructions in previous approaches. The approach is motivated by the binomials  $a_{n,k} = \binom{n}{k}$ . The binomials satisfy the recursion  $a_{n,k} = a_{n-1,k} + a_{n-1,k-1}$  and are well known to have the property that the random variables  $\{X_n\}_{n=1}^\infty$  given by  $\Pr[X_n = k] = a_{n,k} / \sum_{i=0}^n a_{n,i}$  converge to a Gaussian as  $n \rightarrow \infty$ . This new approach proves that appropriate two-dimensional recurrences exhibit similar asymptotic Gaussian behavior. From this, we can prove that the number of summands in decompositions given by many linear recurrence relations is asymptotically Gaussian and additionally prove that for any non-negative integer  $g$ , the number of gaps of size  $g$  in the decomposition of an  $M \in [G_n, G_{n+1})$  also converges to a Gaussian as  $n \rightarrow \infty$ .

## 1. INTRODUCTION

**1.1. History.** The Fibonacci numbers is a fascinating sequence with many properties and interesting relationships; see for example [18]. Zeckendorf [30] proved that if the Fibonacci numbers are defined by  $F_1 = 1, F_2 = 2, F_3 = 3, F_4 = 5$ , and in general  $F_{n+1} = F_n + F_{n-1}$ , then every integer can be written as a sum of non-adjacent terms. The standard proof is by the greedy algorithm: to decompose an integer  $M$ , repeatedly subtract from  $M$  the largest Fibonacci number less than or equal to  $M$ . It is impossible that this process chooses two consecutive Fibonacci numbers  $F_{n-1}$  and  $F_n$ , as it would have chosen  $F_{n+1}$  instead, and for the same reason this process never chooses the same Fibonacci number twice.

Zeckendorf's theorem can be generalized to sequences other than the Fibonacci numbers. Consider for example the powers of 10 given by the recurrence  $G_n = 10G_{n-1}$  and having values  $G_1 = 1, G_2 = 10, G_3 = 100$ , and in general  $G_n = 10^{n-1}$ . For this sequence, a *legal decomposition* of a positive integer  $M$  is simply its base-10 representation. Note these decompositions disallow 10 or more copies of every distinct term in the decomposition, while Fibonacci decompositions disallow consecutive terms in decompositions. A general Zeckendorf's theorem can be stated for linear recurrences with nonnegative coefficients and appropriately defined initial conditions, and the proof has the same idea as the Fibonacci case (see Theorem 1.2). For even more examples of decompositions, see [1, 11] for signed decompositions, [10] for  $f$ -decomposition, and [6, 7, 8] for some recurrences where the leading term vanishes, which can lead to different limiting behavior.

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Many questions can be asked about decompositions. To begin, one must understand the average number of summands in a decomposition. Lekkerkerker [21] proved for Fibonacci numbers that the average number of summands of an  $M \in [F_n, F_{n+1})$  is  $n/(\varphi^2 + 1)$ , where  $\varphi$  is the golden mean. For many general sequences  $\{G_n\}$ , the average number of summands in the corresponding decomposition of an  $M \in [G_n, G_{n+1})$  is  $An + B + o(1)$  for constants  $A$  and  $B$ , meaning the quantity grows linearly [4, 9, 14, 15, 16, 17, 20] and lower order terms are well behaved [25]. Note that when  $\{G_n\}$  is powers of a fixed base  $b$ , the number of summands corresponds to the sum of digits function.

After determining the mean, it remains to determine the variance, or in general, the distribution of the number of summands. For many decompositions, fluctuations about the mean have been shown to converge to a Gaussian [3, 8, 13, 15, 16, 17, 20, 23, 24, 25, 26, 27, 28, 29]. Koloğlu, Kopp, Miller and Wang [19] adopt a more combinatorial approach to prove that the number of summands in the decomposition for Fibonacci numbers converges to a Gaussian. They explicitly count with Stirling's formula the number of  $M$  with exactly  $k$  summands in the decomposition, which they prove is a binomial coefficient. Using this approach they also exactly determine the mean and variance of the number of summands over  $M \in [F_n, F_{n+1})$ . Miller and Wang [25] extend these results to general linear recursive sequences with positive integer coefficients; the method from [19] cannot be carried over directly as there is not a tractable closed form expression for the number of  $M$  with exactly  $k$  summands. Their approach uses appropriately selected generating functions to compute the moments of the number of summands and show that such moments, appropriately normalized, converge to the moments of the standard normal.

Additionally we can analyze the gaps of decompositions. One can ask the same questions about the mean, variance, and general distribution of the gaps of decomposition. Beckwith et al. [2] and Bower et al. [5] (see also [12]) proved results on the distribution of gaps in many generalized decompositions arising from linear recurrences. In particular, they proved that the average number of size- $g$  gaps in an  $M \in [G_n, G_{n+1})$  decays exponentially as  $g$  grows and determined that the distribution of the longest gap between summands behaves similarly to the distribution of the longest run of heads in tossing a biased coin. Li and Miller [22] prove the analogue of Miller and Wang's [25] results for gaps, proving linearity of mean and variance as well as asymptotic normality for size- $g$  gaps.

We survey results on generalized Zeckendorf decompositions in §1.2 and §1.3 and outline proofs of asymptotic Gaussianity in §2 and §3. Though these results can be established in general [22], we focus on the case of Fibonacci numbers to highlight the ideas and techniques.

**1.2. Decompositions.** Before our main discussion, we introduce some notation and basic facts about Zeckendorf Decompositions.

**Definition 1.1.** A positive linear recurrence sequence (PLRS) is a sequence  $\{G_n\}$  satisfying

$$G_n = c_1 G_{n-1} + \cdots + c_L G_{n-L} \quad (1.1)$$

with non-negative integer coefficients  $c_i$  with  $c_1, c_L, L \geq 1$  and initial conditions  $G_1 = 1$  and  $G_n = c_1 G_{n-1} + c_2 G_{n-2} + \cdots + c_{n-1} G_1 + 1$  for  $1 \leq n \leq L$ .

We can generalize Zeckendorf's Theorem to PLRS. Essentially, the notion of a legal decomposition means that when we write an  $M$  as a sum of terms of the sequence that we cannot use the recurrence relation to replace consecutive terms with another term in the sequence.

**Theorem 1.2** (Generalized Zeckendorf Theorem). *Let  $\{G_n\}$  be a positive linear recurrence sequence. For each integer  $M > 0$ , there exists a unique **legal** decomposition*

$$M = \sum_{i=1}^N a_i G_{N+1-i} \quad (1.2)$$

with  $a_1 > 0$  and the other  $a_i \geq 0$ , and one of the following two conditions, which define a legal decomposition, holds.

- (1) We have  $N < L$  and  $a_i = c_i$  for  $1 \leq i \leq N$ .
- (2) There exists an  $s \in \{1, \dots, L\}$  such that  $a_1 = c_1, a_2 = c_2, \dots, a_{s-1} = c_{s-1}$  and  $a_s < c_s, a_{s+1}, \dots, a_{s+\ell} = 0$  for some  $\ell \geq 0$ , and  $\{b_i\}_{i=1}^{N-s-\ell}$  (with  $b_i = a_{s+\ell+i}$ ) is either legal or empty.

Given  $\{G_n\}$  a PLRS, and positive integer  $M$ , we can rewrite the legal decomposition given by Theorem 1.2 as

$$M = \sum_{i=1}^N a_i G_{N+1-i} = G_{i_1} + G_{i_2} + \dots + G_{i_k}. \quad (1.3)$$

for some positive integer  $k = a_1 + a_2 + \dots + a_N$  and  $i_1 \geq i_2 \geq \dots \geq i_k$ . With this representation, we say  $M$  has  $k$  summands in the decomposition (or simply,  $M$  has  $k$  summands). The gaps in the decomposition of  $M$  are the numbers  $i_1 - i_2, i_2 - i_3, \dots, i_{k-1} - i_k$  (for example,  $101 = F_{10} + F_5 + F_3 + F_1$ , and thus has gaps 5, 2, and 2). We often refer to the gaps in the decomposition of  $M$  as simply the *gaps of  $M$* . Let  $k_\Sigma(M)$  denote the number of summands of  $M$  and  $k_g(M)$  the number of gaps of size  $g$  in  $M$ 's decomposition. Note that if  $M$  has  $k$  summands, then  $M$  has  $k - 1$  gaps. In this sense,  $k_g(M)$  is a decomposition of  $k_\Sigma(M)$ , as

$$k_\Sigma(M) = 1 + \sum_{g=0}^{\infty} k_g(M). \quad (1.4)$$

Throughout this paper we let  $K_{\Sigma,n}$  denote the random variable equal to  $k_\Sigma(M)$  for an  $M$  chosen uniformly from  $[G_n, G_{n+1})$  and let  $K_{g,n}$  denote a random variable equal to  $k_g(M)$  for an  $M$  chosen uniformly from  $[G_n, G_{n+1})$ .

**1.3. Asymptotic normality theorems.** Versions of the next result are known for many sequences; see for example [13, 15, 16, 17, 20, 23, 25, 26, 27, 28, 29] (we especially follow below the approach in [25], as the authors there work with PLRS). Note that the first part of the theorem regarding  $\mu_n$  generalizes Lekkerkerker's [21] work for Fibonacci numbers.

**Theorem 1.3.** *Let  $\{G_n\}$  be a PLRS. Let  $K_{\Sigma,n}$  be the random variable defined above and suppose it has mean  $\mu_n$  and variance  $\sigma_n^2$ . There exists positive constants  $A$  and  $C$  and real constants  $B$  and  $D$  such that*

$$\begin{aligned} \mu_n &= An + B + o(1) \\ \sigma_n^2 &= Cn + D + o(1). \end{aligned} \quad (1.5)$$

Furthermore  $(K_{\Sigma,n} - \mu_n)/\sigma_n$  converges weakly to the standard normal  $N(0, 1)$  as  $n \rightarrow \infty$ .

Li and Miller [22] prove the following result on gaps of decompositions. The computation of  $\mu_{g,n}$  was known by Bower et al. [2], except for the lower order terms. Bower et al. further computed the leading coefficient  $A$ . Li and Miller provide formulas for explicitly computing  $A$  and  $C$  from the recurrence relation, though they do not follow through the computation as it is not necessary for their main result on asymptotic Gaussianity. To the authors' knowledge,

the rest of Theorem 1.4 is new. Note that in the case of the Fibonacci numbers each  $M$  has no gaps of size 0 or 1 so the random variable  $K_{g,n}$  is always 0. We therefore must be careful to exclude such cases from the result.

**Theorem 1.4** (Gaussian Behavior for Gaps of Decompositions). *Let  $g \geq 0$  be a fixed positive integer and let  $\{G_n\}$  be a PLS with the additional constraint that all  $c_i$ s are positive. Let  $K_{g,n}$  be the random variable defined above and suppose it has mean  $\mu_{g,n}$  and variance  $\sigma_{g,n}^2$ . Suppose there exists  $n_0 \in \mathbb{N}$  such that  $K_{g,n}$  is non-trivial for  $n \geq n_0$ . There exists positive constants  $A$  and  $C$  and real constants  $B$  and  $D$  such that*

$$\begin{aligned}\mu_{g,n} &= An + B + o(1) \\ \sigma_{g,n}^2 &= Cn + D + o(1).\end{aligned}\tag{1.6}$$

Furthermore  $(K_{g,n} - \mu_{g,n})/\sigma_{g,n}$  converges weakly to the standard normal  $N(0, 1)$  as  $n \rightarrow \infty$ .

In the next section, we outline two proofs of Theorem 1.3 when  $\{G_n\}$  is the Fibonacci numbers. The first is given by Miller and Wang [25] and the second is given by Li and Miller [22]. We then show in the following section how the later proof extends to proving Theorem 1.4.

## 2. GAUSSIAN NUMBER OF SUMMANDS

**2.1. Generating Function Approach.** We sketch the proof by Miller and Wang [25, 26] of Theorem 1.3 in this subsection. Though the theorem holds in general, we restrict our discussion here to the Fibonacci numbers to highlight the main ideas, and we focus on the proof of asymptotic normality, as the linearity of mean and variance follow as intermediate results.

Miller and Wang use the Method of Moments to prove convergence to a Gaussian. The Method of Moments states that if the moments of a sequence of random variances converges to the moments of a Gaussian distribution, the sequence converges in distribution to that Gaussian. Recall that the odd moments of the standard normal  $N(0, 1)$  are 0 and that the even moments are  $(2m - 1)!! = (2m - 1) \cdot (2m - 3) \cdots 1$ .

**Lemma 2.1** (Method of Moments). *Suppose  $X_1, X_2, \dots$  are random variables such that for all integers  $m \geq 0$ , we have*

$$\lim_{n \rightarrow \infty} \mathbf{E}[X_n^{2m}] = (2m - 1)!! \quad \text{and} \quad \lim_{n \rightarrow \infty} \mathbf{E}[X_n^{2m+1}] = 0.\tag{2.1}$$

Then the sequence  $X_1, X_2, \dots$  converges weakly in distribution to the standard normal  $N(0, 1)$ .

Thus, setting

$$\tilde{\mu}_n(m) = \mathbf{E}[(K_{\Sigma,n} - \mu_n)^m],\tag{2.2}$$

we have

$$\mathbf{E}\left[\left(\frac{K_{\Sigma,n} - \mu_n}{\sigma_n}\right)^m\right] = \frac{\tilde{\mu}_n(2m)}{\tilde{\mu}_n(2)^m}\tag{2.3}$$

where  $\mu_n = \mathbf{E}[K_{\Sigma,n}]$  and  $\sigma_n^2 = \mathbf{Var}[K_{\Sigma,n}]$ . It thus suffices to prove for all  $m$

$$\lim_{n \rightarrow \infty} \frac{\tilde{\mu}_n(2m)}{\tilde{\mu}_n(2)^m} = (2m - 1)!! \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{\tilde{\mu}_n(2m + 1)}{\tilde{\mu}_n(2)^{m+\frac{1}{2}}} = 0.\tag{2.4}$$

Our goal therefore is to compute  $\tilde{\mu}_n(m)$  for all nonnegative integers  $m$ .

For  $n, k \geq 0$ , let  $p_{n,k}$  be the number of  $M \in [F_n, F_{n+1})$  with exactly  $k$  summands in its Zeckendorf decomposition. Then  $\Pr[K_{\Sigma,n} = k] = \frac{p_{n,k}}{\sum_{k=0}^{\infty} p_{n,k}}$ . Note that if  $M \in [F_n, F_{n+1})$ , then the decomposition of  $M$  begins with  $F_n$ . Furthermore

$$M - F_n \in [0, F_{n-1}) = \{0\} \bigcup_{i=1}^{n-2} [F_i, F_{i+1}), \quad (2.5)$$

from which we establish

$$\begin{aligned} p_{n,k} &= p_{n-2,k-1} + p_{n-3,k-1} + \cdots \\ p_{n-1,k} &= p_{n-3,k-1} + p_{n-4,k-1} + \cdots \end{aligned} \quad (2.6)$$

Subtracting the second line from the first gives the two-dimensional recursive formula

$$p_{n,k} = p_{n-1,k} + p_{n-2,k-1}. \quad (2.7)$$

Let

$$\begin{aligned} \mathcal{G}(x, y) &:= \sum_{n,k \geq 0} p_{n,k} x^k y^n \\ P_n(x) &:= \sum_{k=0}^{\infty} p_{n,k} x^k \\ \Omega_n &:= P_n(1) = \sum_{k=0}^{\infty} p_{n,k} = F_{n+1} - F_n \end{aligned} \quad (2.8)$$

so that

$$\mathcal{G}(x, y) = \sum_{n \geq 0} P_n(x) y^n. \quad (2.9)$$

To finish the problem it suffices to compute  $P_n(x)$ . Indeed, we know  $P_n(1) = \sum_{k \geq 0} p_{n,k} = F_{n+1} - F_n$  and if we know  $P_n(x)$ , we can determine

$$\mu_n := \frac{P'_n(1)}{P_n(1)}. \quad (2.10)$$

Taking appropriate derivatives of  $P_n(x)/x^{\mu_n}$ , we have

$$\begin{aligned} \tilde{\mu}_n(1) &= \mathbf{E}[(K_{\Sigma,n} - \mu_n)^0] = 1 \\ \tilde{\mu}_n(1) &= \mathbf{E}[K_{\Sigma,n} - \mu_n] = 0 \\ \tilde{\mu}_n(2) &= \mathbf{E}[(K_{\Sigma,n} - \mu_n)^2] = \frac{1}{P_n(1)} \cdot x \left( x \left( \frac{P_n(x)}{x^{\mu_n}} \right)' \right)' \Big|_{x=1} \\ \tilde{\mu}_n(3) &= \mathbf{E}[(K_{\Sigma,n} - \mu_n)^3] = \frac{1}{P_n(1)} \cdot x \left( x \left( x \left( \frac{P_n(x)}{x^{\mu_n}} \right)' \right)' \right)' \Big|_{x=1} \end{aligned} \quad (2.11)$$

and so on, which allows us to compute the moments  $\tilde{\mu}_n(m)$  of  $K_{\Sigma,n} - \mu_n$ .

Miller and Wang's technique for computing  $P_n(x)$  is the following. Using (2.7) and the initial conditions of the recursion, we have

$$\mathcal{G}(x, y) = \frac{xy}{1 - y - xy^2}. \quad (2.12)$$

Decomposing this with partial fractions, we write

$$-\frac{y}{y_1(x) - y_2(x)} \left( \frac{1}{y - y_1(x)} - \frac{1}{y - y_2(x)} \right) \tag{2.13}$$

where  $y_1(x)$  and  $y_2(x)$  are the roots of  $1 - y - xy^2$ . Rewriting  $\frac{1}{y - y_1(x)}$  as  $-\left(1 - \frac{y}{y_1(x)}\right)^{-1}$  and using power series expansion, we can compute  $P_n(x)$ .

This concludes the sketch of Miller and Wang’s proof of Theorem 1.3 when  $\{G_n\}$  is the Fibonacci numbers. For general recursions  $\{G_n\}$ , the proof is similar, but the more complicated generating functions lead to significantly more involved computations.

**2.2. Recursive Generating Function Approach.** Li and Miller [22] present a new approach for obtaining Central Limit type results like Theorem 1.3, and while their main result is proving the asymptotic normality of the number of gaps, we first illustrate the approach by discussing its application to the number of summands.

While the previous approach uses partial fractions to compute  $P_n(x)$ , this new approach computes  $P_n(x)$  recursively, using the nice recursive behavior of the coefficients in (2.7). This has several benefits. First, we don’t need to worry about initial conditions of the recurrence. Not only does this save tedious calculations, but it shows that the recurrence relation of  $p_{n,k}$  is the only thing on which Gaussian behavior depends. Additionally, this approach gives a general framework for characterizing Gaussian behavior arising in two-dimensional recursions, from which we can also prove that the number of gaps approaches a Gaussian.

To begin, define

$$\begin{aligned} P_n(x) &:= \sum_{k=0}^{\infty} p_{n,k} x^k \\ \Omega_n &:= P_n(1) = \sum_{k=0}^{\infty} p_{n,k} = F_{n+1} - F_n \end{aligned} \tag{2.14}$$

as before and additionally define

$$\begin{aligned} \tilde{P}_{n,0}(x) &:= \frac{P_n(x)}{x^{\mu+1}} \\ \tilde{P}_{n,m}(x) &:= (x\tilde{P}_{n,m-1}(x))' \end{aligned} \tag{2.15}$$

so that

$$\mathbf{E}[(K_{\Sigma,n} - \mu_n)^m] = \tilde{\mu}_n(m) = \frac{\tilde{P}_{n,m}(1)}{\Omega_n}. \tag{2.16}$$

Using (2.7) we deduce recursive relationships for  $P_n(x)$ ,  $\Omega_n$  and  $\mu_n$ :

$$\begin{aligned} P_n(x) &= \sum_{k=0}^{\infty} p_{n,k} x^k = \sum_{k=0}^{\infty} (p_{n-1,k} + p_{n-2,k-1}) x^k = P_{n-1}(x) + xP_{n-2}(x) \\ \Omega_n &= P_n(1) = P_{n-1}(1) + 1 \cdot P_{n-2}(1) = \Omega_{n-1} + \Omega_{n-2} \\ \mu_n &= \frac{P'_n(1)}{\Omega_n} = \frac{P'_{n-1}(1) + 1 \cdot P'_{n-2}(1) + P_{n-2}(1)}{\Omega_n} = \frac{\Omega_{n-1}}{\Omega_n} \mu_{n-1} + \frac{\Omega_{n-2}}{\Omega_n} (\mu_{n-2} + 1). \end{aligned} \tag{2.17}$$

Induction also gives recursive formulas for  $\tilde{P}_{n,m}$  and  $\tilde{\mu}_n(m)$

$$\begin{aligned}
 \tilde{P}_{n,m} &= \sum_{\ell=0}^m \binom{m}{\ell} \left( (\mu_{n-1} - \mu_n)^\ell \tilde{P}_{n-1,m-\ell}(x) \cdot x^{\mu_{n-1}-\mu_n} \right. \\
 &\quad \left. + (1 + \mu_{n-2} - \mu_n)^\ell \tilde{P}_{n-2,m-\ell}(x) \cdot x^{1+\mu_{n-2}-\mu_n} \right) \\
 \tilde{\mu}_n(m) &= \sum_{\ell=0}^m \binom{m}{\ell} \left( \frac{\Omega_{n-1}}{\Omega_n} (\mu_{n-1} - \mu_n)^\ell \tilde{\mu}_{n-1}(m-\ell) + \frac{\Omega_{n-2}}{\Omega_n} (1 + \mu_{n-2} - \mu_n)^\ell \tilde{\mu}_{n-2}(m-\ell) \right).
 \end{aligned} \tag{2.18}$$

The recursive formula for  $\mu_n$  lets us prove  $\mu_n$  is linear. Finally, (2.18) allows us to compute the moments. By our earlier discussion, the following lemma implies Theorem 1.3.

**Lemma 2.2.** *For each integer  $m \geq 0$ , there exist polynomials  $Q_{2m}$  of degree exactly  $m$  and  $Q_{2m+1}$  of degree at most  $m$  such that*

$$\begin{aligned}
 \tilde{\mu}_n(2m) &= Q_{2m}(n) + o(1) \\
 \tilde{\mu}_n(2m+1) &= Q_{2m+1}(n) + o(1).
 \end{aligned} \tag{2.19}$$

Furthermore, there exists a constant  $\alpha$  such that the leading coefficient of  $Q_{2m}$  is  $(2m-1)!! \cdot \alpha^m$ .

The idea for the proof is as follows. First, the lemma is true for  $m = 0$  as  $\tilde{\mu}_n(0) = 1$  and  $\tilde{\mu}_n(1) = 0$  for all  $n$ . For higher moments, note in the calculation of  $\mu_n(m)$  in (2.18) that the coefficients  $\mu_{n-i}(m)$  of the  $m^{\text{th}}$  moments sum to 1, the coefficients of  $\tilde{\mu}_{n-i}(m-1)$ , the  $(m-1)^{\text{th}}$  moments, sum to 0, and the coefficients of  $\tilde{\mu}_{n-i}(m-2)$ , the  $(m-2)^{\text{th}}$  moments, sum to  $\binom{m}{2} \cdot (\text{constant})$ . This allows us to pin down the polynomial behavior of  $\mu_n(m)$ . The idea is that if  $A$  is a degree  $d$  polynomial, then  $A(1) + A(2) + \dots + A(n)$  is a degree  $d+1$  polynomial in  $n$ . For example, by (2.18), each second moment is the weighted average of previous second moments plus a constant, so the second moments should be linear in  $n$ . Similarly, assuming the lemma is true for  $m = 0$  and  $m = 1$ , each fourth moment is the weighted average of previous fourth moments plus a linear in  $n$ , so the fourth moments grow quadratically in  $n$ . Because the coefficients of the  $(m-1)^{\text{th}}$  moments in (2.18) sum to 0, the degrees of the polynomials increase by one with every two values of  $m$  as opposed to every one.

The actual proof of this lemma is more involved as the coefficients for our recursion (2.18) are not fixed. For example, the coefficients for  $\tilde{\mu}_{n-1}(m)$  and  $\tilde{\mu}_{n-2}(m)$  are  $\Omega_{n-1}/\Omega_n$  and  $\Omega_{n-2}/\Omega_n$ , respectively, which vary with  $n$ . However these coefficients converge quickly to  $1/\varphi$  and  $1/\varphi^2$ , respectively, where  $\varphi$  is the golden mean, so the moments  $\tilde{\mu}_n(m)$  still behave as we expect. For a full proof, see Section 2.3 of [22], particularly Lemma 2.12.

### 3. GAUSSIAN NUMBER OF GAPS

**3.1. General Two Dimensional Recursions.** The technique in §2.2 generalizes to two-dimensional recursions.

**Theorem 3.1** (Central Limit Theorem in 2D Recursions). *Let  $i_0$  and  $j_0$  be positive integers. Let  $t_{i,j}$  be real numbers for  $1 \leq i \leq i_0, 0 \leq j \leq j_0$  such that for all  $i$ ,  $\hat{t}_i := \sum_{j=0}^{j_0} t_{i,j} \geq 0$ . Suppose that the polynomial  $T(x) = x^{i_0} - \sum_{i=1}^{i_0} \hat{t}_i x^{i_0-i}$  has a unique, multiplicity 1, maximum magnitude root  $\lambda_1 > 0$ . Suppose  $p_{n,k}$  is a two-dimensional recurrence sequence satisfying, for*

$n \geq n_0$ ,

$$p_{n,k} = \sum_{i=1}^{i_0} \sum_{j=0}^{j_0} t_{i,j} p_{n-i,k-j}. \tag{3.1}$$

Suppose further that  $p_{n,k} \geq 0$  for all  $n$  and  $k$ ,  $p_{n,k} = 0$  when  $n < 0$  or  $k < 0$ , finitely many  $p_{n,k}$  are nonzero for  $n < n_0$ , and  $\sum_{i=0}^{\infty} p_{n,i} = \Theta(\lambda_1^n)$ . Let  $X_n$  be the random variable with mean  $\mu_n$  and variance  $\sigma_n^2$  whose mass function is proportional to  $p_{n,k}$  over varying  $k$  so that

$$\Pr[X_n = k] = \frac{p_{n,k}}{\sum_{i=0}^{\infty} p_{n,i}}. \tag{3.2}$$

There exist constants  $A, B, C$  and  $D$  such that  $\mu_n = An + B + o(1)$ ,  $\sigma_n^2 = Cn + D + o(1)$ , and  $A$  and  $C$  are explicitly computable from the  $t_{i,j}$ s. Furthermore, if  $C$  is positive,  $(X_n - \mu_n)/\sigma_n$  converges weakly to the standard normal  $N(0, 1)$  as  $n \rightarrow \infty$ .

Outside of the technical requirement that  $\sum_{i=0}^{\infty} p_{n,i} = \Theta(\lambda_1^n)$ , there is no constraint on the initial conditions of  $p_{n,k}$ . Note that in general the asymptotic behavior of recursive sequences is not independent of the initial conditions. For example, the recursion  $b_n = 5b_{n-1} - 6b_{n-2}$  has the general solution  $b_n = \alpha \cdot 3^n + \beta \cdot 2^n$ , but if we choose initial conditions  $b_1 = 2, b_2 = 4$ , then we have  $b_n = 2^n$  and the  $3^n$  term of the general solution vanishes. For this reason the technical constraint is required to ensure the largest term of  $\sum_{i=0}^{\infty} p_{n,i}$  does not vanish.

For intuition on the theorem, consider the specific case of the two-dimensional recurrence  $a_{n,k} = a_{n-1,k} + a_{n-1,k-1}$  with initial condition  $a_{0,0} = 1$ . This recurrence produces the binomials  $a_{n,k} = \binom{n}{k}$ , and the random variables  $\{X_n\}_{n=1}^{\infty}$  given by  $\Pr[X_n = k] = a_{n,k} / \sum_{i=0}^{\infty} a_{n,i}$  are well known to converge to a Gaussian as  $n \rightarrow \infty$ . Additionally, for any discrete random variable  $Y_n$  taking on finitely many integer values  $q_1, \dots, q_b \geq 0$  with probabilities  $r_1, \dots, r_b$  summing to 1, the Theorem applied to the sequence  $a_{n,k} = \sum_{i=1}^b r_i a_{n-1,k-q_i}$  gives the classical Central Limit Theorem for  $Y_n$ .

We now show that Theorem 3.1 applies to gaps in Fibonacci numbers. For discussion on general recursions, see [22].

Fix an integer  $g \geq 2$ . Let  $p_{g,n,k}$  denote the number of  $M \in [F_n, F_{n+1})$  with exactly  $k$  gaps of size  $g$  in its decomposition. The decomposition of an  $M \in [F_n, F_{n+1})$  begins with  $F_n$  and  $M - F_n$  is in  $[0, F_{n-1})$ . The term  $F_n$  is part of a gap of size  $g$  if and only if  $M - F_n \in [F_{n-g}, F_{n-g+1})$ . Thus we may write

$$\begin{aligned} p_{g,n,k} &= \left( \sum_{i=1}^{n-g-1} p_{g,i,k} \right) + p_{g,n-g,k-1} + \left( \sum_{i=n-g+1}^{n-2} p_{g,i,k} \right) \\ &= \left( \sum_{i=1}^{n-2} p_{g,i,k} \right) + p_{g,n-g,k-1} - p_{g,n-g,k}. \end{aligned} \tag{3.3}$$

Shifting indices gives

$$p_{g,n-1,k} = \left( \sum_{i=1}^{n-3} p_{g,i,k} \right) + p_{g,n-g-1,k-1} - p_{g,n-g-1,k}. \tag{3.4}$$

Subtracting (3.4) from (3.3) and simplifying gives

$$p_{g,n,k} = p_{g,n-1,k} + p_{g,n-2,k} + (p_{g,n-g,k-1} - p_{g,n-g,k} - p_{g,n-g-1,k-1} + p_{g,n-g-1,k}). \tag{3.5}$$

We can check (3.5) satisfies the requirements for Theorem 3.1, implying Theorem 1.4. As a technical detail, we must check that the  $C$  given by Theorem 1.4 is positive. For sake of



brevity, we do not include the formula for  $C$  in this article, but the interested reader may see [22] for discussion.

#### 4. FURTHER WORK AND OPEN QUESTIONS

We end with a few natural questions for future work. The first is to see how far Theorem 3.1 can be generalized. Can we loosen any technical conditions? What about three-dimensional relations? What about infinite sized recursions? For example, if we proved a similar theorem with  $t_{i,j}$  for bounded  $i$  and unbounded  $j$ , we might generalize the standard Central Limit Theorem for any integer valued random variable. This contrasts with the current formulation, which generalizes CLT only on integer random variables with finite support.

In Theorem 1.4, can one remove the additional constraint on the PLRS that every coefficient  $c_i$  must be positive and obtain the same results (that is, if some of  $c_i$  are allowed to be zero)? In some previous problems this constraint on the  $c_i$ 's was to simplify the algebra, but for others it was essential as otherwise very different behavior emerges. What about arbitrary linear recursions where some coefficients might be negative?

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