AN EXPLORATION OF SEQUENCE A000975

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Abstract. Sequence A000975 in the Online Encyclopedia of Integer Sequences (OEIS) starts out 1, 2, 5, 10, 21, 42, 85, ... As of July 1, 2016, the description in the OEIS lists several characterizations of this sequence and numerous examples of instances where this sequence occurs. It also presents a "not yet proved" result, a conjecture, and an unanswered question concerning this sequence. In this paper we show that all of these proposed results are in fact true.

1. Characterizations of Sequence A000975

Sequence A000975 in the Online Encyclopedia of Integer Sequences (OEIS) [5] starts out 1, 2, 5, 10, 21, 42, 85, ... Throughout this paper we denote this sequence as sequence A and the nth term of this sequence as $A(n)$, written in function notation. The OEIS defines this sequence recursively as follows.

Characterization 1:

1. $A(1) = 1$,
2. $A(2n) = 2A(2n-1)$, and
3. $A(2n+1) = 2A(2n) + 1$.

Thus to get the next term we either double the preceding value or double it and add 1. But doubling is the same as appending a 0 in base 2, while doubling and adding 1 is the same as appending a 1 in base 2. This provides a second characterization.

Characterization 2: For $n > 0$, $A(n)$ is the number whose binary representation has length n and consists of alternating ones and zeros.

We illustrate this characterization in Table 1.

$$\begin{array}{rcrcr}
A(1) &=& 1 &=& 1_{(2)}\\
A(2) &=& 2 &=& 10_{(2)}\\
A(3) &=& 5 &=& 101_{(2)}\\
A(4) &=& 10 &=& 1010_{(2)}\\
A(5) &=& 21 &=& 10101_{(2)}\\
A(6) &=& 42 &=& 101010_{(2)}\\
A(7) &=& 85 &=& 1010101_{(2)}\\
A(8) &=& 170 &=& 10101010_{(2)}\\
A(9) &=& 341 &=& 101010101_{(2)}\\
A(10) &=& 682 &=& 1010101010_{(2)}
\end{array}$$

Table 1. The sequence A written in binary.

If we add $A(n)$ to $A(n-1)$, the binary representation of the sum is a string of n ones, with value 2^n-1. This gives us our third characterization.

Characterization 3:

1. $A(1) = 1$, and
2. $A(n) = (2^n - 1) - A(n-1)$ for $n > 1$.

Alternatively, if we subtract $A(n-2)$ from $A(n)$, most of the ones in the binary representations cancel, leaving 2^{n-1}. This gives us yet another characterization.

Characterization 4:

1. $A(1) = 1$,
2. $A(2) = 2$, and
3. $A(n) = A(n-2) + 2^{n-1}$ for $n > 2$.

There are several standard methods for obtaining a closed form expression for $A(n)$. We present a rather unusual derivation. Recall that the binary representation of the fraction $\frac{2}{3}$ is

$$\frac{2}{3} = 0.10101010\ldots_{\;(2)}.$$

Multiplying this value by 2^n moves the binary point n places to the right. Rounding down then truncates this expression, yielding $A(n)$, according to Characterization 2, and giving us our next characterization.

Characterization 5: $A(n) = \left\lfloor \frac{2}{3}(2^n) \right\rfloor$ for all $n \geq 1$.

Other expressions are

$$A(n) = \begin{cases} \dfrac{2^{n+1}-1}{3} & \text{if } n \text{ odd} \\[2ex] \dfrac{2^{n+1}-2}{3} & \text{if } n \text{ even} \end{cases}$$

$$= \frac{2^{n+2} - 3 - (-1)^n}{6}.$$

It is an easy exercise to prove that all these characterizations of $A(n)$ are equivalent. We observe that $A(n)$ is even or odd exactly when n is even or odd, respectively.

2. Occurrences of Sequence A000975

Why do we care about a particular sequence? Often it is because the sequence occurs as the answer to some counting problem. The OEIS lists several places where the values in sequence A000975 occur. Here are some of the more interesting.

Occurrence 1: $A(n)$ is the number of moves needed to solve the n-ring Chinese Rings puzzle (baguenaudier) if the rings are moved one at a time. See, for example, [2, Chapter 1].

Figure 1 shows the state graph for the 4-ring puzzle. The 16 vertices represent the possible states of the puzzle, and two vertices are joined by an edge if the two corresponding states are one move apart. The labels on the vertices denote the positions of the 4 rings in that state, with 0 representing a ring that is off the sliding bar and 1 a ring that is on.

It is easy to confirm that the state graph for the n-ring puzzle is a path of length $2^n - 1$ from the state labeled 0^n to the state labeled 10^{n-1}. The sub-path from state 0^n to state 1^n constitutes an optimal solution of the n-ring puzzle, while the sub-path from state 1^n to state

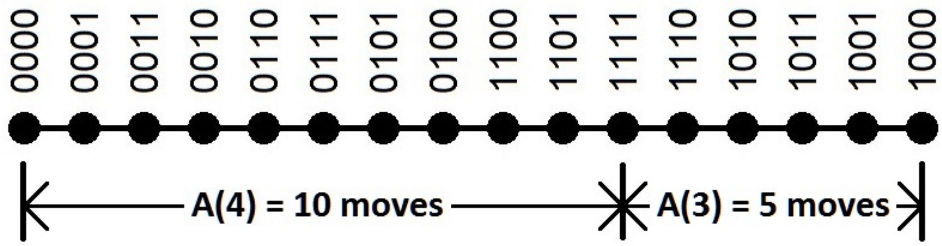

FIGURE 1. The state graph for the 4-ring Chinese Rings puzzle.

10^{n-1} represents an optimal solution of the $(n-1)$-ring puzzle. The number of moves in an optimal solution to this puzzle thus satisfies Characterization 3 of sequence A.

Figure 1 also illustrates a related occurrence of our sequence.

Occurrence 2: $A(n)$ is the distance between a string of n zeros and a string of n ones in the standard n-bit binary Gray code. Again, see, for example, [2, Chapter 1].

A special case of a problem posed by Donald Knuth [3] and solved by O. P. Lossers [4] provides our next occurrence

Occurrence 3: $A(n)$ is the number of ways to partition a set of $n+2$ people sitting around a circular table into three affinity groups with no two members of a group seated next to each other.

We illustrate this occurrence in Figure 2, showing the $A(4) = 10$ partitions of six people into three affinity groups. For concreteness we assign the names A and B to the affinity groups of the people at the bottom left and bottom right, respectively.

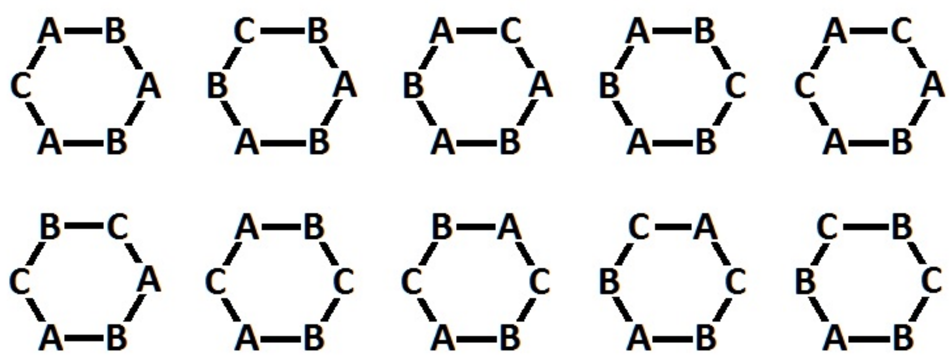

FIGURE 2. The ten partitions of 6 people into three affinity groups.

Following [4], we note that there are 2^n-1 strings of length $n+2$ that start with AB, contain all the letters A, B, and C, and have no two adjacent letters the same. If such a string ends in either B or C, it can be wrapped into a circle to represent an affinity partition of $n+2$ people. If such a string ends in A, the A can be deleted and the shortened string will represent an affinity partition of $n+1$ people. The number of affinity partitions of $n+2$ people is thus another problem that satisfies Characterization 3 of sequence A.

Equivalently, $A(n)$ is the number of different 3-colorings for the vertices of all triangulated $(n+2)$-gons if the colors of the two base vertices are fixed.

3. A "not yet proved" property proposed by Antti Karttunen

Antti Karttunen has suggested that our sequence A000975 serves as a link between sequences A0000217 (the triangular numbers) and A048702 (the even length binary palindromes divided by 3) in the OEIS. The triangular numbers, which we represent by $T(n)$, are well-known: $T(n) = n(n+1)/2$. The even length binary palindromes, denoted $P(n)$, (sequence A048701 in the OEIS), are less well known. We illustrate the sequence $P(n)/3$ in Table 2.

$$\begin{aligned}
P(1)/3 &= 11_{(2)}/3 = 3/3 = 1\\
P(2)/3 &= 1001_{(2)}/3 = 9/3 = 3\\
P(3)/3 &= 1111_{(2)}/3 = 15/3 = 5\\
P(4)/3 &= 100001_{(2)}/3 = 33/3 = 11\\
P(5)/3 &= 101101_{(2)}/3 = 45/3 = 15\\
P(6)/3 &= 110011_{(2)}/3 = 51/3 = 17\\
P(7)/3 &= 111111_{(2)}/3 = 63/3 = 21\\
P(8)/3 &= 10000001_{(2)}/3 = 129/3 = 43\\
P(9)/3 &= 10011001_{(2)}/3 = 153/3 = 51\\
P(10)/3 &= 10100101_{(2)}/3 = 165/3 = 55
\end{aligned}$$

Table 2. The sequence of even length binary palindromes divided by 3.

The proposal of Karttunen is that the values $A(k)$ serve as the indices n where $T(n)$ and $P(n)/3$ agree, as illustrated in Table 3.

n	**1**	**2**	3	4	**5**	6	7	8	9	**10**	11	12	13	14	15
$T(n)$	**1**	**3**	6	10	**15**	21	28	36	45	**55**	66	78	91	105	120
$P(n)/3$	**1**	**3**	5	11	**15**	17	21	43	51	**55**	63	65	73	77	85

Table 3. Common values of $T(n)$ and $P(n)/3$.

Lemma 3.1. *For any positive integer n let $k = \lfloor \log_2(n) \rfloor + 1$. Then the binary representation of n is k bits long, and $P(n)$ satisfies*

$$P(n) = n2^k + R(n),$$

where $R(n)$ is the binary reversal of n (sequence A030101 in the OEIS).

For example, when $n = 12 = 1100_{(2)}$ we have $k = 4$ and $R(12) = 0011_{(2)} = 3$, so $P(12) = 11000011_{(2)} = 12\left(2^4\right)+3 = 195$. This Lemma follows immediately from the definition of $P(n)$.

Theorem 3.2. *For all $n > 0$, we have $T(n) = P(n)/3$ if and only if $n = A(k)$ for some positive integer k.*

Proof. First suppose $n = A(k)$ where k is odd. Then we know that

$$n = \frac{2^{k+1}-1}{3},$$

n is k bits long, and $R(n) = n$. We have

$$\begin{aligned} P(n)/3 &= \left(n2^k + n\right)/3 \\ &= n\left(2^k + 1\right)/3 \\ &= \left(\frac{n}{2}\right)\left(2^{k+1} + 2\right)/3 \\ &= \frac{n(n+1)}{2} = T(n). \end{aligned}$$

Now suppose $n = A(k)$ where k is even. Then we know that

$$n = \left(2^{k+1} - 2\right)/3,$$

n is k bits long, and $R(n) = n/2$. We have

$$\begin{aligned} P(n)/3 &= \left(n2^k + n/2\right)/3 \\ &= \left(\frac{n}{2}\right)\left(2^{k+1} + 1\right)/3 \\ &= \frac{n(n+1)}{2} = T(n). \end{aligned}$$

Next suppose $2^{k-1} \leq n \leq A(k) - 1$. Then n is k bits long, and

$$\begin{aligned} P(n)/3 &= \left(n2^k + R(n)\right)/3 \\ &> \left(n2^k\right)/3 \\ &= \left(\frac{n}{2}\right)\left(\frac{2^{k+1}}{3}\right) \\ &> \left(\frac{n}{2}\right)A(k) \\ &\geq \frac{n(n+1)}{2} = T(n). \end{aligned}$$

Finally, suppose $A(k) + 1 \leq n \leq 2^k - 1$. Then n is k bits long, and

$$\begin{aligned} P(n)/3 &= \left(n2^k + R(n)\right)/3 \\ &< \left(n2^k + 2^k\right)/3 \\ &= \left(\frac{n+1}{2}\right)\left(\frac{2^{k+1}}{3}\right) \\ &< \left(\frac{n+1}{2}\right)\left(A(k) + 1\right) \\ &\leq \frac{(n+1)n}{2} = T(n). \end{aligned}$$

□

4. The conjectures of Reinhard Zumkeller

Reinhard Zumkeller has conjectured that our sequence A is related to sequence A265158 in the OEIS. We denote this sequence as B and its nth term as $B(n)$. The sequence is defined by

1. $B(1) = 1$,
2. $B(2n) = 2B(n)$ for $n \geq 1$, and
3. $B(2n+1) = \lfloor B(n)/2) \rfloor$ for $n \geq 1$.

The first few values of this sequence are displayed in Table 4.

n	$B(n)$	n	$B(n)$	n	$B(n)$	n	$B(n)$
1	1	9	2	17	4	25	0
2	2	10	2	18	4	26	0
3	0	11	0	19	1	27	0
4	4	12	0	20	4	28	0
5	1	13	0	21	1	29	0
6	0	14	0	22	0	30	0
7	0	15	0	23	0	31	0
8	8	16	16	24	0	32	32

Table 4. The sequence B.

A notable property of this sequence is that it contains long runs of zeros. Zumkeller conjectures that the lengths of the record runs of zeros in sequence B are exactly the values found in sequence A. The run lengths themselves form sequence A264784, with values **1**, **2**, **5**, **10**, 1, **21**, 2, **42**, 1, 1, 5, 1, **85**, 2, 2, 10, 2, **170**, 1, 1, 1, 5, 1, 1, 5, 1, 21, 1, 1, 5, 1, **341**, 2, 2, 2, 10, 2, 2, 10, 2, 42, 2, 2, 10, 2, **682**, 1, 1, 1, 1, 5, 1, 1, 1, 5, 1, 1, 5, 1, 21, 1, 1, 1, 5, 1, 1, 5, 1, 21, 1, 1, 5, 1, 85, 1, 1, 1, 5, 1, 1, 5, 1, 21, 1, 1, 5, 1, **1365**, ... We will refer to this sequence of run lengths as sequence L, with entries $L(n)$.

Lemma 4.1.

$$B(A(k)) = \begin{cases} 1 & \textit{if } k \textit{ is odd} \\ 2 & \textit{if } k \textit{ is even} \end{cases}$$

Proof. The proof is by induction on k. For $k = 1$ we have $B(A(1)) = B(1) = 1$. Now let k be an integer greater than 1, and suppose that the claim is true for smaller values. For k even we have

$$\begin{aligned} B(A(k)) &= B\left(2A(k-1)\right) \\ &= 2B\left(A(k-1)\right) \\ &= 2(1) = 2. \end{aligned}$$

For $k \geq 3$ odd we have

$$\begin{aligned} B(A(k)) &= B\left(2A(k-1)+1\right) \\ &= \left\lfloor B\left(A(k-1)\right)/2 \right\rfloor \\ &= \lfloor 2/2 \rfloor = 1. \end{aligned}$$

□

Lemma 4.2. *$B(2^k) = 2^k$ for all $k \geq 0$.*

Proof. The proof is by induction on k. For $k = 0$ we have $B(2^0) = B(1) = 1 = 2^0$. Now assuming the lemma is true for all values smaller than some $k > 0$, we have

$$B(2^k) = 2B(2^{k-1}) = 2(2^{k-1}) = 2^k.$$

□

Lemma 4.3. *For every $k \geq 2$ we have $B(n) = 0$ for all n satisfying $A(k) + 1 \leq n \leq 2^k - 1$.*

Proof. We again us induction on k. For $k = 2$ we must consider n satisfying $A(2) + 1 \leq n \leq 2^2 - 1$, or $3 \leq n \leq 3$. We know $B(3) = 0$ so the claim is true here.

We now consider the claim for some $k \geq 3$, assuming that it is true for smaller values of k. For n even, the lower bound $n \geq A(k) + 1$ implies

$$\frac{n}{2} \geq \left\lceil \frac{A(k)+1}{2} \right\rceil \geq \left\lceil \frac{2A(k-1)+1}{2} \right\rceil = A(k-1) + 1,$$

while the upper bound $n \leq 2^k - 1$ implies

$$\frac{n}{2} \leq \left\lfloor \frac{2^k - 1}{2} \right\rfloor = 2^{k-1} - 1.$$

The induction hypothesis then yields

$$B(n) = 2B(n/2) = 2 \times 0 = 0.$$

For n odd, we consider first the case $n = A(k) + 1$. Here we have

$$\begin{aligned} B(n) &= B\left(A(k) + 1\right) \\ &= B\left(2A(k-1) + 1\right) \\ &= \left\lfloor B\left(A(k-1)\right)/2 \right\rfloor \\ &= \lfloor 1/2 \rfloor = 0. \end{aligned}$$

Finally, we consider the case where n is odd and $A(k) + 2 \leq n \leq 2^k - 1$. The lower bound $n \geq A(k) + 2$ implies

$$\frac{n-1}{2} \geq \left\lceil \frac{A(k)+1}{2} \right\rceil \geq \left\lceil \frac{2A(k-1)+1}{2} \right\rceil = A(k-1) + 1,$$

while the upper bound $n \leq 2^k - 1$ implies

$$\frac{n-1}{2} \leq \left\lfloor \frac{2^k - 2}{2} \right\rfloor = 2^{k-1} - 1.$$

The induction hypothesis then yields

$$B(n) = \left\lfloor \frac{B((n-1)/2)}{2} \right\rfloor = \left\lfloor \frac{0}{2} \right\rfloor = 0.$$

□

We can now prove the first form of Zumkeller's conjecture.

Theorem 4.4. *For every $k \geq 2$ there is a run of zeros in $B(n)$ for n in the interval $A(k)+1 \leq n \leq 2^k - 1$. This run has length $A(k-1)$, and together these runs are the runs of record length.*

Proof. The previous three lemmas tell us that each of these intervals forms a run of zeros in sequence B. The length of the kth interval is $\left(2^k - 1\right) - A(k) = A(k-1)$. That these are precisely the runs of record length follows from the fact that $B(2^k) > 0$ for all $k \geq 0$ and that each of these runs covers two-thirds of the entries between consecutive powers of 2. □

We can say more about the number and lengths of runs of zeros in B if we write the index n in binary notation, which we do for the remainder of this section. Using ω to denote an arbitrary word over the alphabet $\mathcal{A} = \{0, 1\}$, we can restate the definition of sequence B as follows:

1. $B(1) = 1$,
2. $B(1\omega 0) = 2B(1\omega)$, and
3. $B(1\omega 1) = \lfloor B(1\omega)/2 \rfloor$.

Lemma 4.5. *Suppose that ω is a word consisting of r zeros and s ones in any order. Then $B(1\omega)$ is either 0 or 2^{r-s}.*

Proof. We use induction on the length of ω. If ω is the empty word we have $r = s = 0$ and $B(1) = 1 = 2^0$ as desired. Now consider a non-empty word ω and suppose that the claim is true for all shorter words. If $\omega = \widehat{\omega}0$ then $\widehat{\omega}$ has $r-1$ zeros and s ones, and by hypothesis $B(1\widehat{\omega})$ is either 0 or 2^{r-1-s}. Then $B(1\omega) = B(1\widehat{\omega}0) = 2B(1\widehat{\omega})$ which is either 0 or $2 \cdot 2^{r-1-s} = 2^{r-s}$ as claimed. If $\omega = \widehat{\omega}1$ then $\widehat{\omega}$ has r zeros and $s-1$ ones, and by hypothesis $B(1\widehat{\omega})$ is either 0 or 2^{r-s+1}. Then $B(1\omega) = B(1\widehat{\omega}1) = \lfloor B(1\widehat{\omega})/2 \rfloor$. If $B(1\widehat{\omega})$ is either 0 or 1 then $B(1\omega) = 0$. Otherwise $B(1\omega) = 2^{r-s+1}/2 = 2^{r-s}$ as claimed. □

Lemma 4.6. *$B(1\omega) = 0$ if and only if ω can be written as $\omega = \omega_1\omega_2$ where ω_1 consists of more ones than zeros.*

Proof. Suppose that $B(1\omega) = 0$. Let ω_1 be the shortest initial subword of ω such that $B(1\omega_1) = 0$, and let ω_2 be the rest of the word ω. Now ω_1 cannot be the empty word, as $B(1) = 1 \neq 0$. Likewise ω_1 cannot be of the form $\widehat{\omega_1}0$, or we would have $0 = B(1\omega) = B(1\widehat{\omega_1}0) = 2B(1\widehat{\omega_1})$ and $\widehat{\omega_1}$ would be a shorter initial subword. So $\omega_1 = \widehat{\omega_1}1$ and $0 = B(1\omega_1) = B(1\widehat{\omega_1}1) = \lfloor B(1\widehat{\omega_1})/2 \rfloor$. This implies $B(1\widehat{\omega_1}) = 1$, and by Lemma 4.5 we have that $\widehat{\omega_1}$ consists of an equal number of zeros and ones. Then ω_1 contains more ones than zeros, as claimed.

Conversely, suppose that ω can be written as $\omega_1\omega_2$ where ω_1 contains more ones than zeros. We can assume that ω_1 is the shortest such initial subword, so that $\omega_1 = \widehat{\omega_1}1$ and $\widehat{\omega_1}$ consists of an equal number of zeros and ones. By Lemma 4.5 we know that $B(\widehat{\omega_1})$ is either 0 or 1. Then $B(1\omega_1) = B(1\widehat{\omega_1}1) = \lfloor B(1\widehat{\omega_1})/2 \rfloor = 0$, which implies $B(1\omega) = 0$. □

We will call a word ω of even length a *Catalan* word if
1. ω consists of an equal number of zeros and ones; and
2. no initial subword of ω contains more ones than zeros.

The first few Catalan words are the empty word, 01, 0011, 0101, 000111, 001011, 001101, 010011, and 010101. It is well known that the number of Catalan words of length $2n$ is the nth Catalan number $\binom{2n}{n}/(n+1)$ for all $n \geq 0$. These Catalan numbers, which we denote $C(n)$, form sequence A000108 in the OEIS.

Lemma 4.7. *A binary word ω is a Catalan word if and only if $B(1\omega) = 1$.*

Proof. This follows immediately from Lemmas 4.5 and 4.6. □

The first few of these indices where the B sequence takes the value 1 are 1, 5, 19, 21, 71, 75, 77, 83, and 85. This sequence is not currently in the OEIS.

Lemma 4.8. *There is a run of zeros in sequence B beginning at index n if any only if the binary representation of n has the form $1\omega 1_{(2)}$ (when n is odd), or $1\omega 10_{(2)}$ (when n is even), where ω is a Catalan word.*

According to this lemma, the first few indices where runs of zeros begin are $11_{(2)} = 3$, $110_{(2)} = 6$, $1011_{(2)} = 11$, $10110_{(2)} = 22$, $100111_{(2)} = 39$, and $1001110_{(2)} = 78$. This sequence is also not currently in the OEIS.

Proof. Suppose $n = 1\omega 1_{(2)}$, with ω a Catalan word. Then $n-1 = 1\omega 0_{(2)}$. Lemma 4.7 implies that $B(1\omega 1) = \lfloor B(1\omega)\rfloor/2 = \lfloor 1/2\rfloor = 0$ and $B(1\omega 0) = 2B(1\omega) = 2$. Likewise, if $n = 1\omega 10_{(2)}$ then $n-1 = 1\omega 01_{(2)}$, and we have $B(1\omega 10) = 2\lfloor B(1\omega)/2\rfloor = 0$ and $B(1\omega 01) = \lfloor (2B(1\omega)/2\rfloor = 1$. In both cases, then, a run of zeros starts at index n.

Now suppose n is an index where $B(n) = 0$ but $B(n-1) \neq 0$. If $n \geq 3$ is odd, then $n = 1\omega 1_{(1)}$ for some arbitrary word ω, and $n-1 = 1\omega 0_{(2)}$. Now $B(1\omega 0) = 2B(1\omega) \neq 0$ and $B(1\omega 1) = \lfloor B(1\omega)/2\rfloor = 0$ together imply that $B(1\omega) = 1$. By Lemma 4.6, ω is a Catalan word, and n is of the claimed form. If $n \geq 6$ is even, then $n = 1\omega 10^k_{(2)}$ for some integer $k \geq 1$ and some arbitrary word ω, and $n-1 = 1\omega 01^k_{(2)}$. Then $0 = B(1\omega 10^k) = 2^k\lfloor B(1\omega)/2\rfloor$ implies $B(1\omega)$ is either 0 or 1. Also $0 \neq B(1\omega 01^k) = B(1\omega)2^{1-k}$, so $B(1\omega) = 1$, ω is a Catalan word, and $k = 1$. We conclude that $n = 1\omega 10_{(2)}$ as claimed in this case. □

We conclude that for $2^{2k-1} \leq n < 2^{2k}$, where the binary representation of n contains $2k$ bits, there are exactly $C(k-1)$ runs of zeros in sequence B, corresponding to the $C(k-1)$ Catalan sequences of length $2k-2$. The longest run of zeros in this range is the last one, of length $A(k)$. There are an additional $C(k-1)$ runs when $2^{2k} \leq n < 2^{2k+1}$ and n is $2k+1$ bits long. The longest run here is also the last one, of length $A(k+1)$.

The record-length runs of zeros in B are thus runs number $C(0) = 1$, $2C(0) = 2$, $2C(0) + C(1) = 3$, $2C(0) + 2C(1) = 4$, $2C(0) + 2C(1) + C(2) = 6$, $2C(0) + 2C(1) + 2C(2) = 8$, $2C(0) + 2C(1) + 2C(2) + C(3) = 13$, $2C(0) + 2C(1) + 2C(3) + 2C(4) = 18$, and so on. These numbers form sequence A155051 in the OEIS, with the index off by 1.

This confirms the second form of Zumkeller's conjecture.

Theorem 4.9. *The nth record-length run of zeros in sequence B has length $A(n)$ and is run number $A155051(n-1)$. More concisely, $A(n) = L(A155051(n-1))$.*

5. The Question of N. J. A. Sloane

Continuing in the OEIS listing for sequence A (sequence A0000975), there is a link to an electronic paper Enveloping Operads and Bicoloured Noncrossing Configuration [1] by F. Chapoton and S. Giraudo. In Table 2 of that paper the sequence 1, 2, 5, 10, 21, 42, 85 is listed twice. In the OEIS Sloane asks "Is the sequence in Table 2 this sequence [i.e., sequence A]?". In this section we demonstrate that the answer to this question is "yes."

In the somewhat peculiar language of [1], these numbers are the first few coefficients of the colored Hilbert series giving the number of bubbles of increasing arity, first based and then nonbased, of the 2-colored suboperad $\langle\langle \boldsymbol{\Delta}, \boldsymbol{\Delta}\rangle\rangle$ of the 2-colored operad Bubble generated by these two generators of arity 2.

In more simple language, a *bubble* is a polygon consisting of a base edge and two or more nonbase edges, with each edge either colored blue or left uncolored. The bubble is *based* if and only if the base edge is blue. The *border* of a bubble is its set of nonbase edges. The *arity* of a bubble is the number of edges in its border.

The based bubbles in the 2-colored suboperad of interest start with a triangle of blue edges. New bubbles are generated by either replacing a blue border edge with two consecutive uncolored edges, or by replacing an uncolored border edge with two consecutive blue edges. Unbased bubbles in this suboperad are the same, but with blue and uncolored interchanged. See Figure 3 for the based bubbles of arity 2, 3, and 4 in this suboperad.

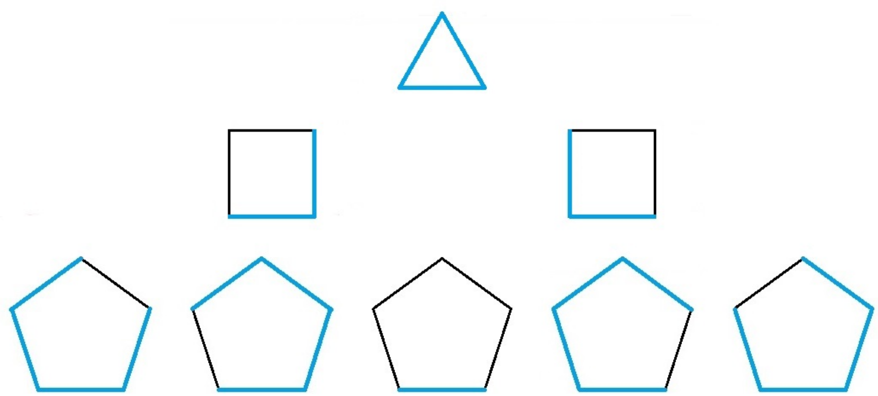

FIGURE 3. Based bubbles of arity 2, 3, and 4.

These based bubbles of arity n are characterized in [1] as those with the following properties.

(1) the number of blue edges in the border is congruent to $2n+1 \pmod 3$ (and the number of uncolored edges in the border is thus congruent to $2n-1 \pmod 3$); and
(2) there exist 2 consecutive edges in the border that are either both blue or both uncolored.

In addition, the paper [1] gives 2-variable generating functions (or colored Hilbert series) for based and unbased bubbles in this suboperad. Unfortunately these generating functions are flawed: the numerator of the second term in the first generating function should be z_1, not z_2, and the numerator of the second term in the second generating function should be z_2, not z_1. We now produce our own count of these bubbles.

Lemma 5.1. *Let $S(n)$ denote the number of strings of length n over the alphabet $\{b, u\}$ such that the number of occurrences of the character b is congruent to $2n+1 \pmod 3$. Then $S(n) + S(n+1) = 2^n$. Exactly one of these strings consists of alternating b characters and u characters.*

Proof. Clearly the number of strings of length n with k occurrences of the character b is $\binom{n}{k}$. So we have

$$\begin{aligned} S(n)+S(n{+}1) &= \sum_i \binom{n}{3i+2n+1} + \sum_i \binom{n+1}{3i+2(n+1)+1} \\ &= \sum_i \binom{n}{3i+2n+1} + \sum_i \left(\binom{n}{3i+2n+2} + \binom{n}{3i+2n+3} \right) \\ &= \sum_i \binom{n}{i} = 2^n. \end{aligned}$$

The restrictions on the number of b characters and u characters in a string imply that these counts cannot be equal, so an alternating string must be of odd length. If $n = 2k{+}1$ the number of b characters must be congruent to $2(2k{+}1)+1 \equiv k \pmod 3$ and the number of u characters must be congruent to $2(2k{+}1)-1 \equiv k+1 \pmod 3$, so there is a unique alternating string of length n, consisting of k occurrences of the character b and $k{+}1$ occurrences of the character u. □

Theorem 5.2. *The number of based* (*or, equivalently, unbased*) *bubbles of arity* n *in the indicated suboperad of bubbles is* $A(n{-}1)$.

Proof. Every based bubble in our suboperad can be constructed from a blue base and a border colored according to a string described in Lemma 4. There is one based bubble of arity 2, and the number of based bubbles of arity n plus the number of based bubbles of arity $n{+}1$ is $2^n - 1$. Thus the number of based bubbles satisfies Characterization 3 from Section 1.

The result for unbased bubbles follows by interchanging blue and uncolored in all the proofs. □

We conclude by illustrating an alternative approach to this counting problem: a one-to-one correspondence between the number of partitions into affinity groups described in Section 2 and the based bubbles described here. To construct a bubble from an affinity group partition, we traverse the diagram of the affinity group partition counter-clockwise. If an A is followed by a B, a B by a C, or a C by an A, we color the corresponding edge of the bubble blue. If an A is followed by a C, or a B by an A, or a C by a B, we leave the corresponding edge of the bubble uncolored. The correspondence between the 10 affinity group partitions for 6 people and the 10 bubbles of arity 5 is illustrated in Figure 4.

The confirmation that this is indeed always a one-to-one correspondence is left to the reader.

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References

[1] F. Chapoton, and S. Giraudo, Enveloping Operads and Bicoloured Noncrossing Configuration, arXiv preprint arXiv:1310.4521, 2013. Downloaded July 1, 2016.

[2] A. M. Hinz, S. Klavžar, U. Milutinović, and C. Petr, *The Tower of Hanoi—Myths and Maths*. Springer, Basel, 2013.

[3] D. E. Knuth, "Problem 11151," *American Mathematical Monthly* **112** (2005), 367.

[4] O. P. Lossers, "Solution to Problem 11151: Partitions of a Circular Set," *American Mathematical Monthly* **114** (2007), 265–266.

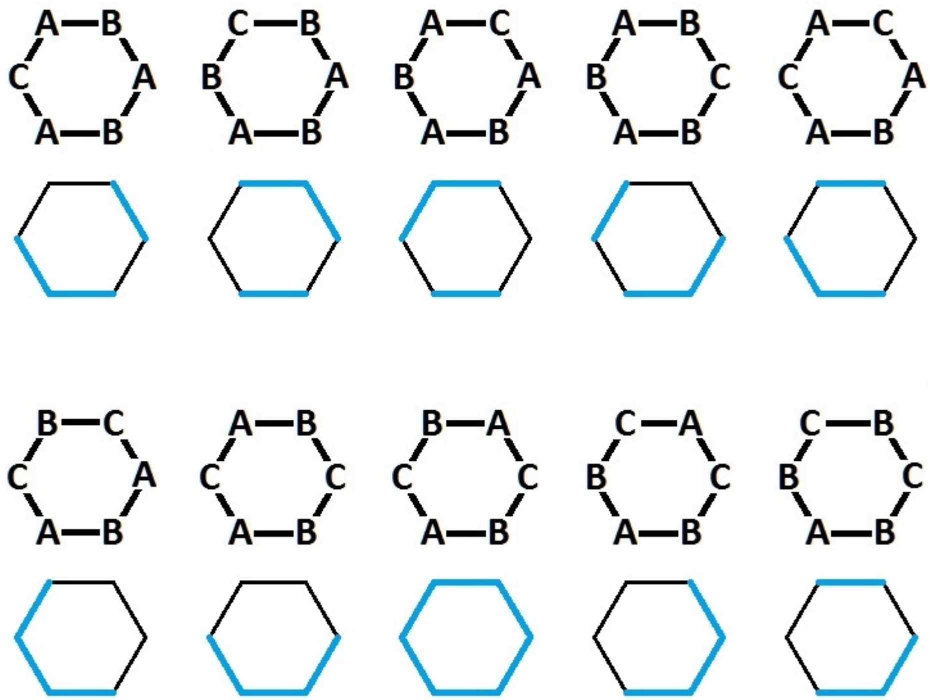

FIGURE 4. A one-to-one correspondence.

[5] OEIS Foundation Inc., The On-Line Encyclopedia of Integer Sequences, https://oeis.org. Downloads are current as of July 1, 2016.

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