SOME NEW IDENTITIES FOR DERANGEMENT NUMBERS

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In loving memory of my dear wife, Anke (1967–2018)

Abstract. In this paper, we present several generalizations of identities for derangement numbers by Bhatnagar [2] and by Deutsch and Elizalde [3]. The study is motivated by the recent paper [4].

1. Introduction

Let D_n denote the number of permutations of $\{1, \dots, n\}$ with no fixed points, the so-called derangements. If we define $D_0 = 1$, the two well-known recursive formulas

$$D_{n+1} = nD_n + nD_{n-1}, \qquad D_0 = 1, \qquad D_1 = 0, \tag{1.1}$$

and

$$D_n = nD_{n-1} + (-1)^n, \qquad D_0 = 1, \tag{1.2}$$

are valid for $n \in \mathbb{N}$. From the second formula, one easily derives the closed expression

$$D_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}. \tag{1.3}$$

More material on this sequence can be found in the On-Line Encyclopedia of Integer Sequences (OEIS) [5, Sequence A000166].

Deutsch and Elizalde [3, Eq. (11)] gave two proofs of the identity

$$D_n = \sum_{j=2}^{n} (j-1) \binom{n}{j} D_{n-j} \tag{1.4}$$

by combinatorial arguments and analytically by using the exponential generating function

$$D(z) = \sum_{n=0}^{\infty} D_n \frac{z^n}{n!} = \frac{e^{-z}}{1-z} \qquad (|z| < 1) \tag{1.5}$$

of the sequence $(D_n)_{n=0}^{\infty}$. Bhatnagar presented families of identities for some sequences including the shifted derangement numbers [1, 2], deriving them using Euler's identity [1, Eq. (2.1)]. Recently, Martinjak and Dajana Stanić [4, Theorem 1] demonstrated for the nice derangement identity

$$1 + \sum_{k=1}^{n} \frac{D_k}{k!} = \frac{D_{n+2}}{(n+1)!} \qquad (n \in \mathbb{N}) \tag{1.6}$$

[1, Eq. (10.14)] an interesting combinatorial proof.

In what follows, we present a short combinatorial proof of Eq. (1.4) and generalize it in the form

$$D_n = \sum_{j=p}^{n} f_p(j) \binom{j-1}{p-1} \binom{n}{j} D_{n-j} \qquad (n = p, p+1, \dots),$$

for arbitrary $p \in \mathbb{N}$. The functions $f_p(j)$ will be determined by explicit formulas. Furthermore, we generalize the formula (1.6) by presenting closed expressions of

$$\sum_{k=0}^{n} \frac{D_{k+r}}{k!} \qquad \text{and} \qquad \sum_{k=0}^{n} \frac{D_k}{(k+r+1)!} \qquad (n \in \mathbb{N}),$$

for arbitrary positive integers r.

2. The identity by Deutsch and Elizalde

We start with a combinatorial argument for demonstrating Eq. (1.4). Let $D_n(j)$ denote the number of permutations of $\{1, \dots, n\}$ having exactly j fixed points. Thus, $D_n(0) = D_n$. Obviously, we have $D_n(j) = \binom{n}{j} D_{n-j}$ and $\sum_{j=0}^{n} D_n(j) = n!$. Hence,

$$\begin{aligned} \sum_{j=1}^{n} (j-1) \binom{n}{j} D_{n-j} &= n \sum_{j=0}^{n-1} \binom{n-1}{j} D_{n-1-j} - \sum_{j=1}^{n} \binom{n}{j} D_{n-j} \\ &= n(n-1)! - (n! - D_n) = D_n. \end{aligned}$$

The next theorem generalizes the result by Deutsch and Elizalde [3, Eq. (11)] by presenting a recursive formula for D_n in terms of $D_0, D_1, \dots, D_{n-p}$.

Theorem 2.1. *For each positive integer p, the recursive formula*

$$D_n = \sum_{j=p}^{n} f_p(j) \binom{j-1}{p-1} \binom{n}{j} D_{n-j} \qquad (n = p, p+1, \dots), \tag{2.1}$$

is valid with

$$f_p(j) = jD_{p-1} + (-1)^p,$$

for $p \le j \le n$.

Remark 1. *In the special cases $p = 1$ and $p = 2$, Theorem 2.1 reduces to formula (1.4) since $f_1(j) = j - 1$ and $f_2(j) = 1$. The special case $p = n$, i.e.,*

$$D_n = f_n(n) = nD_{n-1} + (-1)^n$$

is equivalent to the recursive formula (1.2).

Remark 2. *For $p = 1, 2, 3, \dots$, formula (2.1) can be rewritten in the form*

$$D_{n+p} = \binom{n+p}{p} \sum_{j=0}^{n} \frac{p}{j+p} \binom{n}{j} f_p(j+p) D_{n-j} \qquad (n = 0, 1, 2, \dots). \tag{2.2}$$

This follows by elementary manipulations and application of the binomial identity

$$\binom{j+p-1}{p-1} \binom{n+p}{j+p} = \frac{p}{j+p} \binom{n+p}{p} \binom{n}{j}.$$

We emphasize that the proof given below is not merely a verification of the formula in Theorem 2.1, but it yields the explicit formula for $f_p(j)$.

Proof of Theorem 2.1. Let p be a positive integer. The representation

$$D_n = \sum_{j=p}^{n} f_p(j) \binom{j-1}{p-1} \binom{n}{j} D_{n-j} \qquad (n = p, p+1, \dots),$$

is equivalent to

$$D(z)=\sum_{n=0}^{p-1} D_n \frac{z^n}{n!}+\sum_{n=p}^{\infty}\sum_{j=p}^{n} f_p(j)\binom{j-1}{p-1}\binom{n}{j} D_{n-j}\frac{z^n}{n!}, \tag{2.3}$$

where $D(z)=(1-z)^{-1}e^{-z}$ is the exponential generating function (1.5) of the sequence $(D_n)_{n=0}^{\infty}$. The double sum is equal to

$$\sum_{j=p}^{\infty}\binom{j-1}{p-1}\frac{f_p(j)}{j!}\sum_{n=j}^{\infty} D_{n-j}\frac{z^n}{(n-j)!}=F_p(z)\,D(z),$$

where

$$F_p(z)=\sum_{j=p}^{\infty}\binom{j-1}{p-1} f_p(j)\frac{z^j}{j!}. \tag{2.4}$$

Therefore, (2.3) is equivalent to

$$F_p(z)=(1-z)\,e^z\sum_{n=p}^{\infty} D_n\frac{z^n}{n!}. \tag{2.5}$$

Binomial convolution yields

$$e^z\sum_{n=p}^{\infty} D_n\frac{z^n}{n!}=\sum_{n=p}^{\infty}\frac{z^n}{n!}\sum_{j=p}^{n}\binom{n}{j}D_j.$$

Using

$$\sum_{j=p}^{n}\binom{n}{j}D_j=\sum_{j=p}^{n} D_n(n-j)=n!-\sum_{j=0}^{p-1}\binom{n}{j}D_j,$$

we arrive at

$$\begin{aligned} F_p(z) &= (1-z)\sum_{n=p}^{\infty}\frac{z^n}{n!}\left(n!-\sum_{j=0}^{p-1}\binom{n}{j}D_j\right)\\ &= z^p-\sum_{n=p}^{\infty}\frac{z^n}{n!}\sum_{j=0}^{p-1}\binom{n}{j}D_j+\sum_{n=p+1}^{\infty}\frac{nz^n}{n!}\sum_{j=0}^{p-1}\binom{n-1}{j}D_j. \end{aligned}$$

Since $\sum_{j=0}^{p-1}\binom{p}{j}D_j=p!-D_p$, we obtain

$$F_p(z)=D_p\frac{z^p}{p!}+\sum_{n=p+1}^{\infty}\frac{z^n}{n!}\sum_{j=0}^{p-1}(n-j-1)\binom{n}{j}D_j.$$

Comparison with Eq. (2.4) yields, for $n\geq p+1$,

$$\begin{aligned} f_p(n) &= \sum_{j=0}^{p-1}(n-j-1)\binom{n}{j}D_j=\sum_{i=0}^{p-1}\frac{(-1)^i}{i!}\sum_{j=i}^{p-1}\left((j+1)!\binom{n}{j+1}-j!\binom{n}{j}\right)\\ &= \sum_{i=0}^{p-1}\frac{(-1)^i}{i!}\left(p!\binom{n}{p}-i!\binom{n}{i}\right)\\ &= p\binom{n}{p}D_{p-1}-(-1)^{p-1}\binom{n-1}{p-1} \end{aligned}$$

which implies

$$f_p(n) = nD_{p-1} + (-1)^p .$$

By the recursive equation (1.2), the latter formula is valid also for $n = p$. This completes the proof of Theorem 2.1. □

3. The identity by Bhatnagar

We prove the following formulas that generalize identity (1.6) by Bhatnagar in two directions.

Theorem 3.1. *The derangement numbers D_n satisfy, for $r = 1, 2, 3, \ldots$, the identities*

$$rn! \sum_{k=0}^{n} \frac{D_{k+r-1}}{k!} = D_{n+r} - (-1)^r D_n \tag{3.1}$$

and

$$r \sum_{k=0}^{n-1} \frac{D_k}{(k+r+1)!} = (-1)^r \left(\frac{D_{n+r}}{(n+r)!} - \frac{D_{r-1}}{(r-1)!} \right) - \frac{D_n}{(n+r)!}. \tag{3.2}$$

Remark 3. *In the special case $r = 1$, identity (3.1) reduces to*

$$n! \sum_{k=0}^{n} \frac{D_k}{k!} = D_{n+1} + D_n,$$

which becomes Bhatnagar's result (1.6) after an application of the recursive formula (1.1).

Remark 4. *For $n = 0$, (3.1) is the recursive equation (1.2) when recalling that $D_0 = 1$.*

Remark 5. *It would be interesting to find a closed expression of the finite sum*

$$\sum_{k=0}^{n} \frac{D_k}{(k+1)!}.$$

Please note that the similar looking identity [1, Eq. (10.14)] *is equivalent to (1.6) because therein, D_n denote the shifted derangement numbers D_{n+1} in our notation.*

Proof of Theorem 3.1. Let n be a positive integer. The explicit representation (1.3) and interchanging the order of summation leads to

$$\sum_{k=0}^{n+r} \frac{D_k}{k!} z^k = \sum_{j=0}^{n+r} \frac{(-1)^j}{j!} \sum_{k=j}^{n+r} z^k \qquad (z \in \mathbb{C}) .$$

Differentiating $(r-1)$ times with respect to z und putting $z = 1$ yields

$$\sum_{k=r-1}^{n+r} \frac{D_k}{(k-r+1)!} = \sum_{j=0}^{n+r} \frac{(-1)^j}{j!} \sum_{k=j}^{n+r} (r-1)! \binom{k}{r-1}.$$

Inserting $\sum_{k=j}^{n+r} \binom{k}{r-1} = \binom{n+r+1}{r} - \binom{j}{r}$, we obtain

$$\begin{aligned} \frac{1}{(r-1)!} \sum_{k=0}^{n+1} \frac{D_{k+r-1}}{k!} &= \sum_{j=0}^{n+r} \frac{(-1)^j}{j!} \left[\binom{n+r+1}{r} - \binom{j}{r} \right] \\ &= \binom{n+r+1}{r} \frac{D_{n+r}}{(n+r)!} - \frac{1}{r!} \sum_{j=0}^{n} \frac{(-1)^{j+r}}{j!}, \end{aligned}$$

which implies

$$r\sum_{k=0}^{n+1}\frac{D_{k+r-1}}{k!}=(n+r+1)\,\frac{D_{n+r}}{(n+1)!}-(-1)^r\,\frac{D_n}{n!}.$$

Subtracting $rD_{n+r}/\,(n+1)!$ from both sides of the equation and multiplying by $n!$ completes the proof of Eq. (3.1).
To prove (3.2), we integrate

$$\sum_{k=0}^{n}\frac{D_k}{k!}z^k=\sum_{j=0}^{n}\frac{(-1)^j}{j!}\sum_{k=j}^{n}z^k \qquad (z\in\mathbb{C})$$

$r+1$ times and put $z=1$ to obtain

$$\sum_{k=0}^{n}\frac{D_k}{(k+r+1)!}=\sum_{j=0}^{n}\frac{(-1)^j}{j!}\sum_{k=j}^{n}\frac{k!}{(k+r+1)!}.$$

Telescoping

$$\sum_{k=j}^{n}\frac{k!}{(k+r+1)!}=\frac{-1}{r}\sum_{k=j}^{n}\left(\frac{(k+1)!}{(k+r+1)!}-\frac{k!}{(k+r)!}\right)=\frac{-1}{r}\left(\frac{(n+1)!}{(n+r+1)!}-\frac{j!}{(j+r)!}\right),$$

we obtain

$$-r\sum_{k=0}^{n}\frac{D_k}{(k+r+1)!}=\frac{n+1}{(n+r+1)!}D_n-\sum_{j=0}^{n}\frac{(-1)^j}{(j+r)!}.$$

Adding $rD_n/\,(n+r+1)!$ to both sides of the equation yields

$$-r\sum_{k=0}^{n-1}\frac{D_k}{(k+r+1)!}=\frac{D_n}{(n+r)!}-\sum_{j=0}^{n}\frac{(-1)^j}{(j+r)!}.$$

Observing that

$$\sum_{j=0}^{n}\frac{(-1)^j}{(j+r)!}=(-1)^r\sum_{j=r}^{n+r}\frac{(-1)^j}{j!}=(-1)^r\left(\frac{D_{n+r}}{(n+r)!}-\frac{D_{r-1}}{(r-1)!}\right)$$

completes the proof. □

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