FIBONACCI MATRICES

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Abstract. We show that a result about Fibonacci matrices actually holds for all real symmetric matrices; then we discuss this in the context of general constant coefficient linear recurrence relations.

1. Introduction

In [3], the author introduces the Fibonacci numbers F_n, (defined, as usual, by $F_0 = 0$, $F_1 = 1$, and $F_{n+2} = F_n + F_{n+1}$), and the matrix

$$M = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix},$$

and then observes that (because of known relations between the Fibonacci numbers)

$$\lim_{n\to\infty} \frac{1}{F_{2n-1}} M^n = \begin{pmatrix} 1 & \tau \\ \tau & \tau^2 \end{pmatrix}, \text{ where } \tau = (1+\sqrt{5})/2. \tag{1.1}$$

It is perhaps easier to see (1.1) if we let

$$S = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}. \tag{1.2}$$

Then $S^2 = M$ and (by induction),

$$S^n = \begin{pmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{pmatrix}, \quad \text{and} \quad M^n = S^{2n} = \begin{pmatrix} F_{2n-1} & F_{2n} \\ F_{2n} & F_{2n+1} \end{pmatrix}.$$

The author of [3] then asks if

$$X = \begin{pmatrix} 1 & 1 \\ 1 & 1+x \end{pmatrix} \text{ and } X^n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix},$$

then

$$\lim_{n\to\infty} a_n^{-1} X^n = \begin{pmatrix} 1 & \varphi \\ \varphi & \varphi^2 \end{pmatrix}, \text{ where } \varphi = \tfrac{1}{2}\big(x + \sqrt{x^2+4}\big). \tag{1.3}$$

We shall show that much more than this is true, and that it has little to do with Fibonacci numbers. For a brief history of the use of the matrix S given in (1.2), see [2].

2. Real Symmetric Matrices

It is a standard result in linear algebra that any real square symmetric matrix A has real eigenvalues, and there is an orthogonal matrix P (that is, its transpose P^t is its inverse P^{-1}) such that PAP^t is a diagonal matrix (whose diagonal elements are the eigenvalues of A). Thus, given a real, symmetric, 2×2 matrix A, with eigenvalues λ and μ, there is an orthogonal matrix P such that

$$PAP^t = D, \quad D = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}, \quad P = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad P^{-1} = P^t = \begin{pmatrix} a & c \\ b & d \end{pmatrix}.$$

As $A^n = P^t(PA^nP^t)P = P^t(PAP^t)^nP = P^tD^nP$, this shows that

$$A^n = \begin{pmatrix} a^2\lambda^n + c^2\mu^n & ab\lambda^n + cd\mu^n \\ ab\lambda^n + cd\mu^n & b^2\lambda^n + d^2\mu^n \end{pmatrix}.$$

If we now suppose that $|\lambda| > |\mu|$, then we find that

$$\lim_{n\to\infty} \frac{1}{a^2\lambda^n} A^n = \begin{pmatrix} 1 & y \\ y & y^2 \end{pmatrix}, \quad y = b/a. \tag{2.1}$$

If $A = X$, then (2.1) must agree with (1.3) so that we must have $b/a = \frac{1}{2}(x + \sqrt{x^2+4})$. However, it is easy to show directly that this is so. As $AP^t = P^tD$, we have

$$A\begin{pmatrix} a \\ b \end{pmatrix} = \lambda \begin{pmatrix} a \\ b \end{pmatrix}, \quad A\begin{pmatrix} c \\ d \end{pmatrix} = \mu \begin{pmatrix} c \\ d \end{pmatrix};$$

thus, y in (2.1) is the slope of any eigenvector associated with the eigenvalue λ (of largest modulus). Now if $A = X$, then the eigenvalues of A are

$$\lambda = 1 + \tfrac{1}{2}\big(x + \sqrt{x^2+4}\big), \quad \mu = 1 + \tfrac{1}{2}\big(x - \sqrt{x^2+4}\big),$$

where $|\lambda| > |\mu|$. As

$$\begin{pmatrix} 1 & 1 \\ 1 & 1+x \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \lambda \begin{pmatrix} a \\ b \end{pmatrix},$$

we see that $a + b = \lambda a$ so that $b/a = \lambda - 1 = \frac{1}{2}\big(x + \sqrt{x^2+4}\big)$, as required.

We leave the reader to investigate the three remaining cases, namely when (i) $|\mu| > |\lambda|$, (ii) $\lambda = \mu$, and (iii) $\lambda = -\mu$.

3. A Closing Remark

The analysis above prompts a remark that might be of greater interest than the answer to the question posed in [3]. A question about Fibonacci numbers has led to a result that holds in a general situation, and we suggest that this is a more general phenomenon than is generally realized. In particular, the so-called *primary solution* x_n, defined by $x_0 = 0$ and $x_1 = 1$, of *any* (real or complex) second order constant coefficient recurrence relation

$$x_{n+2} = ax_{n+1} + bx_n, \tag{3.1}$$

can, by a change of variable, be converted into the Fibonacci sequence, and it follows from this that for every identity that is satisfied by the F_n, *there is a corresponding identity that is satisfied by the solution of* (3.1). For example, the three well-known identities

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n,$$
$$F_{p+2}F_{q+1} + F_{p+1}F_q = F_{p+q+2},$$
$$F_{n+1}^2 + F_n^2 = F_{2n+1}.$$

are special cases of the general identities

$$x_{n+1}x_{n-1} - x_n^2 = (-1)^n b^{n-1},$$
$$x_{p+2}x_{q+1} + bx_{p+1}x_q = x_{p+q+2},$$
$$x_{n+1}^2 + bx_n^2 = x_{2n+1},$$

which are satisfied by the primary solution of (3.1) (see [1] for more details). Surely then, we should, wherever possible, focus on the recurrence relation (3.1) instead of so often considering the Fibonacci sequence. The link between [3] and recurrence relations is simply that every second order constant coefficient recurrence relation is given by a matrix; for example, (3.1) is

$$\begin{pmatrix} x_{n+1} \\ x_{n+2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ b & a \end{pmatrix} \begin{pmatrix} x_n \\ x_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ b & a \end{pmatrix}^{n+1} \begin{pmatrix} x_0 \\ x_1 \end{pmatrix},$$

and the sequence F_n is the special case of $a = b = 1$.

References

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