THE FIBONACCI NUMBERS OF THE FORM $2^a \pm 2^b + 1$

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Abstract. Let $(F_n)_{n\geqslant 0}$ be the *Fibonacci sequence* given by $F_0 = 0$, $F_1 = 1$, and the recurrence formula $F_{n+2} = F_{n+1} + F_n$ for all $n \geqslant 0$. In this note, we completely solve the Diophantine equation

$$F_n = 2^a \pm 2^b + 1$$

in positive integers (n, a, b) with $a > b \geqslant 1$.

1. Introduction

Let $(F_n)_{n\geqslant 0}$ be *Fibonacci sequence* given by $F_0 = 0$, $F_1 = 1$, and the recurrence formula

$$F_{n+2} = F_{n+1} + F_n \quad \text{for all} \quad n \geqslant 0.$$

Its few first terms are

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, \ldots$$

Let p be a prime number. In [8], Luca and Szalay study the Diophantine equation

$$F_n = p^a \pm p^b + 1 \tag{1}$$

in positive integers (n, p, a, b) with $n > 2$ and $\max\{a, b\} \geqslant 2$. They prove that equation (1) has only finitely many solutions and all of them are effectively computable. This result was generalized in [7]. In this note, we study the particular case $p = 2$. More precisely, we solve the Diophantine equation

$$F_n = 2^a \pm 2^b + 1 \tag{2}$$

in positive integers (n, a, b) with $a > b \geqslant 1$. Our result is the following theorem.

Theorem 1. *All solutions of equation (2) in positive integers (n, a, b) with $a > b \geqslant 1$ are*

$$F_7 = 2^3 + 2^2 + 1, \quad F_8 = 2^4 + 2^2 + 1,$$

and

$$F_4 = 2^2 - 2^1 + 1, \quad F_5 = 2^3 - 2^2 + 1, \quad F_7 = 2^4 - 2^2 + 1.$$

2. Tools

The method of proof of Theorem 1 is the classic one with lower bounds in logarithms and the reduction method of Baker-Davenport, used in [2, 3] for example. We collect these tools in this section. Let α be an algebraic number of degree d, let $a > 0$ be the leading coefficient of its minimal polynomial over $\mathbb{Z}$, and let $\alpha = \alpha^{(1)}, \ldots, \alpha^{(d)}$ denote its conjugates. The *logarithmic height* of α is defined as

$$h(\alpha) = \frac{1}{d}\left(\log a + \sum_{i=1}^{d} \log\max\left\{|\alpha^{(i)}|, 1\right\}\right).$$

This height has the following basic properties. For algebraic numbers α and β and $m \in \mathbb{Z}$, we have

- $h(\alpha+\beta) \leqslant h(\alpha) + h(\beta) + \log 2$.
- $h(\alpha\beta) \leqslant h(\alpha) + h(\beta)$.
- $h(\alpha^m) = |m| h(\alpha)$.

Now, let $\mathbb{L}$ be a real number field of degree $d_{\mathbb{L}}$, $\alpha_1, \ldots, \alpha_\ell \in \mathbb{L}$, and $b_1, \ldots, b_\ell \in \mathbb{Z} \setminus \{0\}$. Let $B \geqslant \max\{|b_1|, \ldots, |b_\ell|\}$ and

$$\Lambda = \alpha_1^{b_1} \cdots \alpha_\ell^{b_\ell} - 1.$$

Let $A_1, \ldots, A_\ell$ be real numbers with

$$A_i \geqslant \max\{d_{\mathbb{L}}\, h(\alpha_i), |\log \alpha_i|, 0.16\}, \qquad i = 1, 2, \ldots, \ell.$$

The first tool we need is the following result due to Matveev in [9] (see also Theorem 9.4 in [4]).

Theorem 2. *Assume that $\Lambda \neq 0$. Then,*

$$\log |\Lambda| > -1.4 \cdot 30^{\ell+3} \cdot \ell^{4.5} \cdot d_{\mathbb{L}}^2 \cdot (1 + \log d_{\mathbb{L}}) \cdot (1 + \log B) A_1 \cdots A_\ell.$$

In this note, we always use $\ell = 3$. Furthermore, $\mathbb{L} = \mathbb{Q}(\sqrt{5})$ has degree $d_{\mathbb{L}} = 2$. Throughout the paper, we fix the constant

$$C = 9.69742 \times 10^{11} > 1.4 \cdot 30^{3+3} \cdot 3^{4.5} \cdot 2^2 \cdot (1 + \log 2).$$

Our second tool is a version of the reduction method of Baker-Davenport, based on the Lemma in [1]. We shall use the one given by Bravo, Gómez, and Luca in [2]. For a real number x, we write $\|x\|$ for the distance from x to the nearest integer.

Lemma 3. *Let M be a positive integer. Let τ, μ, $A > 0$, and $B > 1$ be given real numbers. Assume that p/q is a convergent of τ such that $q > 6M$ and $\varepsilon = \|\mu\, q\| - M\|\tau\, q\| > 0$. Then, there is no solution to the inequality*

$$0 < |n\,\tau - m + \mu| < \frac{A}{B^w}$$

in positive integers n, m, and w satisfying

$$n \leqslant M \quad \text{and} \quad w \geqslant \frac{\log(Aq/\varepsilon)}{\log(B)}.$$

Lemma 3 is a slight variation of the one given by Dujella and Pethő in [5]. Finally, the following result will be useful. This is Lemma 7 in [6].

Lemma 4. *If $m \geqslant 1$, $T > (4m^2)^m$, and $T > x/(\log x)^m$. Then,*

$$x < 2^m T (\log T)^m.$$

3. Proof of Theorem 1

To start with, let us to recall some basic properties of the Fibonacci sequence. Put

$$\alpha = \frac{1+\sqrt{5}}{2}, \qquad \beta = \frac{1-\sqrt{5}}{2}.$$

The well known Binet formula states that

$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}} \quad \text{for all} \quad n \geqslant 0. \tag{3}$$

Furthermore, the inequality

$$\alpha^{n-2} \leqslant F_n \leqslant \alpha^{n-1} \tag{4}$$

also holds for all $n \geqslant 1$.

Now, we start with the study of (2) in positive integer solutions (n, a, b) with $a > b \geqslant 1$. From (4), we obtain

$$\alpha^{n-2} \leqslant F_n = 2^a + 2^b + 1 < 2^{a+2}, \qquad \alpha^{n-1} \geqslant F_n = 2^a + 2^b + 1 > 2^a,$$

and

$$\alpha^{n-2} \leqslant F_n = 2^a - 2^b + 1 < 2^{a+1}, \qquad \alpha^{n-1} \geqslant F_n = 2^a - 2^b + 1 > 2^{a-1}.$$

Then, in both cases, we have

$$(n-2)\,\frac{\log\alpha}{\log 2} < a + 2 \quad \text{and} \quad (n-1)\,\frac{\log\alpha}{\log 2} > a - 1. \tag{5}$$

Since $\log\alpha/\log 2 = 0.69424\ldots$, we have that if $n \leqslant 200$, then $a \leqslant 139$. We ran a *Mathematica* program in the range $1 \leqslant n \leqslant 200$, $1 \leqslant b < a \leqslant 139$, and we obtained all the solutions listed in Theorem 1. We will prove that these are all of them.

From now on, we assume $n > 200$. Furthermore from (5), we obtain that $a > 135$ and $n > a$. From the Binet formula (3), we rewrite (2) as

$$\left|\frac{\alpha^n}{\sqrt{5}} - 2^a\right| \leqslant \frac{|\beta|^n}{\sqrt{5}} + 2^b + 1 < 2^{b+1}.$$

Dividing through by 2^a, we obtain

$$\left|\frac{1}{\sqrt{5}}\alpha^n 2^{-a} - 1\right| < \frac{1}{2^{a-b-1}}. \tag{6}$$

Let Λ be the expression inside the absolute value on the left side of (6). We note that $\Lambda \neq 0$. Actually, from (2) with the $+$ sign, we have that $\Lambda > 0$, whereas with the $-$ sign we have that $\Lambda < 0$. Indeed from (2), we have that

$$\frac{\alpha^n}{\sqrt{5}} - 2^a = \pm 2^b + 1 + \frac{\beta^n}{\sqrt{5}}$$

and we note that its right side is positive or negative according to the choice of the sign of 2^b. In particular, in both cases, we have that $\Lambda \neq 0$ and we apply Matveev's inequality to it. To do this, we take

$$\alpha_1 = \frac{1}{\sqrt{5}},\ \alpha_2 = \alpha,\ \alpha_3 = 2, \quad b_1 = 1,\ b_2 = n,\ b_3 = -a.$$

Thus, $B = n$. Furthermore, we have $h(\alpha_1) = \log\sqrt{5}$, $h(\alpha_2) = \log\alpha/2$, and $h(\alpha_3) = \log 2$. Thus, we take $A_1 = \log 5$, $A_2 = 0.5$, and $A_3 = 1.4$. Then,

$$\log|\Lambda| > -C \cdot (1 + \log n) \cdot \log 5 \cdot 0.5 \cdot 1.4.$$

Comparing this with (6), we obtain

$$(a-b)\log 2 < 1.09253 \times 10^{12}(1 + \log n). \tag{7}$$

Again from the Binet formula (3), we rewrite (2) as

$$\left|\frac{\alpha^n}{\sqrt{5}} - (2^{a-b} \pm 1)2^b\right| < 2.$$

Dividing through by $2^a \pm 2^b$, we obtain

$$\left|\frac{1}{\sqrt{5}(2^{a-b} \pm 1)}\alpha^n 2^{-b} - 1\right| < \frac{2}{2^a \pm 2^b} \leqslant \frac{4}{2^a} < \frac{1}{\alpha^{n-8}}, \tag{8}$$

where we use $\alpha^{n-2} < 2^{a+2}$ and $16 < \alpha^6$. Let Λ_1 be the expression inside of the absolute value on the left side of (8). We note that $\Lambda_1 > 0$. Indeed from (2), we obtain

$$\frac{\alpha^n}{\sqrt{5}} - (2^a \pm 2^b) = 1 + \frac{\beta^n}{\sqrt{5}} > 0.$$

Now, we apply Matveev's inequality to Λ_1. To do this, we consider

$$\alpha_1 = \frac{1}{\sqrt{5}(2^{a-b} \pm 1)},\ \alpha_2 = \alpha,\ \alpha_3 = 2, \quad b_1 = 1,\ b_2 = n,\ b_3 = -b.$$

Thus, $B = n$. The heights of α_2 and α_3 are already calculated. For α_1, we use the properties of the height and (7) to conclude that

$$h(\alpha_1) \leqslant h(\sqrt{5}) + h(2^{a-b} \pm 1) < 1.09254 \times 10^{12}(1 + \log n).$$

So we take $A_1 = 2.18508 \times 10^{12}(1 + \log n)$ and A_2 and A_3 as above. Hence, from Matveev's inequality we obtain

$$\log \Lambda_1 > -C(1 + \log n) \cdot \big(2.18508 \times 10^{12}(1 + \log n)\big) \cdot 0.5 \cdot 1.4,$$

which compared with (8) yields

$$n \log \alpha < 1.48328 \times 10^{24}(1 + \log n)^2.$$

Thus, $n < 1.23295 \times 10^{25}(\log n)^2$, and from Lemma 4 we conclude that

$$n < 1.64616 \times 10^{29}. \tag{9}$$

Now, we will reduce this bound on n. To do this, we consider

$$\Gamma = n \log \alpha - a \log 2 + \log\left(\frac{1}{\sqrt{5}}\right),$$

and we consider (6). Assume that $a - b \geqslant 10$. Note that $e^\Gamma - 1 = \Lambda \neq 0$. Thus, $\Gamma \neq 0$. If $\Gamma > 0$, we have that

$$0 < \Gamma < e^\Gamma - 1 = |\Lambda| < \frac{1}{2^{a-b-1}}.$$

If on the other hand, $\Gamma < 0$, we then have that $1 - e^\Gamma = |\Lambda| < 1/2$. Thus, $e^{|\Gamma|} < 2$. Hence,

$$0 < |\Gamma| < e^{|\Gamma|} - 1 = e^{|\Gamma|}|\Lambda| < \frac{2}{2^{a-b-1}}.$$

Thus in both cases, we have that

$$0 < |\Gamma| < \frac{2}{2^{a-b-1}}.$$

Dividing through by $\log 2$, we obtain

$$0 < |n\,\tau - a + \mu| < \frac{6}{2^{a-b}},$$

where

$$\tau = \frac{\log \alpha}{\log 2} \qquad \text{and} \qquad \mu = \frac{\log(1/\sqrt{5})}{\log 2}.$$

Now, we apply Lemma 3. To do this, we take $M = 1.64616 \times 10^{29}$, which from (9) is the upper bound on n. With the help of *Mathematica*, we found that the 70th convergent

$$\frac{p_{70}}{q_{70}} = \frac{143857379293355987619511933268732}{207215059288249261970895631754277}$$

of τ is such that $q_{70} > 6M$ and $\varepsilon = \|q_{70}\,\mu\| - M\|q_{70}\,\tau\| = 0.452806 > 0$. Thus from Lemma 3, with $A = 6$ and $B = 2$, we obtain that

$$a - b < \frac{\log\left(q_{70}\,6/\varepsilon\right)}{\log 2} < 108.$$

Now, we consider

$$\Gamma_1 = n\log\alpha - b\log 2 + \log\left(\frac{1}{\sqrt{5}\,(2^{a-b} \pm 1)}\right)$$

and we consider (8). Note that $e^{\Gamma_1} - 1 = \Lambda_1 > 0$. Thus, $\Gamma_1 > 0$ and we have

$$0 < \Gamma_1 < e^{\Gamma_1} - 1 = \Lambda_1 < \frac{1}{\alpha^{n-8}}.$$

Dividing through by $\log 2$, we obtain

$$0 < n\,\tau - b + \mu < \frac{68}{\alpha^n},$$

where τ is as above and

$$\mu = \frac{\log\left(1/\sqrt{5}\,(2^{a-b} \pm 1)\right)}{\log 2}.$$

Again, we apply Lemma 3. Consider

$$\mu_k = \frac{\log\left(1/\sqrt{5}\,(2^{k} \pm 1)\right)}{\log 2}, \qquad k = 1, 2 \ldots 107.$$

Again, with the help of *Mathematica*, we find that the 70th convergent of τ above also works well. That is, $q_{70} > 6M$ and $\varepsilon_k \geqslant 0.000905562$ for all $k = 1, \ldots, 107$. We calculated $\log(q_{70}\,68/\varepsilon_k)/\log\alpha$ for all $k = 1, \ldots, 107$ and found that the maximum of these values is at most 173. Thus $n \leqslant 173$, which contradicts the assumption on n. This finishes the proof of Theorem 1.

4. Remarks

a) In the same way, it can be proved that in the cases $p = 3$, 5, the only solutions of (1) in positive integers (n, a, b) with $a > b \geqslant 1$ are

$$F_7 = 3^2 + 3^1 + 1 \qquad \text{and} \qquad F_{10} = 3^4 - 3^3 + 1,$$

and

$$F_8 = 5^2 - 5^1 + 1,$$

respectively, whereas in the cases $p = 7$, 11, 13 it does not have any solution. Thus, one is tempted to conjecture that for all primes $p \geqslant 7$ (1) does not have any solution.

b) The reviewer pointed out that the conditions $a > b \geqslant 1$ in Theorem 1 can be relaxed to $a \geqslant b \geqslant 0$. Their argument says that, to do this, what we need is to additionally study the cases $F_n = 2^a$, $F_n = 2^a + 2$, for example. In the last case, we find that n is a multiple of 3, but not 6, and rewrite $F_n - F_3 = F_{n\pm 3/2}L_{n\mp 3/2}$. Then, apply the Primitive Divisor Theorem to all cases.

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