FIBONACCI-PRODUCING RATIONAL POLYNOMIALS

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Abstract. We study a family of interpolating rational polynomials that produce Fibonacci numbers on certain integer values. By construction, the degree m polynomial gives Fibonacci numbers for $m+1$ arguments; we show that several additional outputs are also closely related to Fibonacci numbers. Also, we use an alternative version of the polynomials to establish several identities.

1. Introduction

There are many polynomials associated with the Fibonacci numbers. Our interest here is in certain interpolating polynomials determined by points (i, F_j) for particular integers i, j where F_j refers to the Fibonacci sequence. There has been previous attention to such interpolating polynomials [1, 2]; both use the condition $i = j$. That is, they consider polynomials determined by the points $(0, F_0)$, $(1, F_1)$, etc. (actually, [1] allows a generalization of Fibonacci numbers).

We consider instead the degree n polynomial determined by points whose abscissas are Fibonacci numbers and whose index depends on n. In particular, we use the points $(0, F_{n+2})$, $(1, F_{n+3})$, $\ldots$, (n, F_{2n+2}), that is, (i, F_j) with $j = n + i + 2$. These particular choices allow us to relate many additional values of these polynomials to Fibonacci numbers. In addition, alternative forms of these polynomials (Theorem 2.1 and Proposition 2.3) allow us to establish various identities (Corollary 3.2).

We use the standard definition of the Fibonacci numbers extended to the negative indices:

$$F_0 = 0,\ F_1 = 1,\ F_n = F_{n-1} + F_{n-2} \text{ for } n \geq 2, \text{ and } F_n = (-1)^{n+1} F_{-n} \text{ for } n < 0.$$

We also use the falling factorial notation defined by $(x)_0 = 1$ and $(x)_n = x(x-1)\cdots(x-n+1)$ for $n \geq 1$. Therefore, $(n)_n = (n)_{n-1} = n!$. By extension, $\binom{x}{n} = (x)_n/n!$. This also allows us to use binomial coefficients with negative integers, for example,

$$\binom{-1}{n} = (-1)^n, \quad \binom{-2}{n} = (-1)^n(n+1), \quad \binom{-3}{n} = (-1)^n\binom{n+2}{2}.$$

Further, we will use the identity [4, Eq. 6] valid for all integers n and nonnegative integers k,

$$\binom{n}{k} = (-1)^k\binom{k-n-1}{k}. \tag{1.1}$$

2. Interpolating Polynomials

Our primary result gives alternative forms for a family of interpolating polynomials determined by points involving the Fibonacci numbers.

Theorem 2.1. *Given a nonnegative integer n, let $P_n(x)$ be the interpolating polynomial determined by the points $(0, F_{n+2})$, $(1, F_{n+3})$, $\ldots$, (n, F_{2n+2}). For $n = 0$, we let $P_0(x)$ be the* *constant polynomial $F_2 = 1$. For $n > 0$, using Lagrange's formula, we have*

$$P_n(x) = \sum_{i=0}^{n} F_{i+n+2} \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x-j}{i-j}.$$

The polynomial $P_n(x)$ has degree n with leading coefficient $1/n!$ and an alternative form

$$P_n(x) = \sum_{i=0}^{n} F_{i+n+2} \binom{x}{i} \binom{n-x}{n-i}. \tag{2.1}$$

Further, for each integer $k > n$,

$$P_n(k) = F_{k+n+2} - \sum_{i=1}^{k-n} F_i \binom{k-i}{n}. \tag{2.2}$$

Also, for each integer $k < 0$,

$$P_n(k) = F_{k+n+2} + \sum_{i=1}^{-k-1} F_{-i} \binom{k+i}{n}. \tag{2.3}$$

Table 1 shows the first few of these polynomials.

TABLE 1. $P_n(x)$ for small values of n.

n	$P_n(x)$
0	1
1	$x+2$
2	$(x^2+3x+6)/2$
3	$(x^3+3x^2+14x+30)/6$
4	$(x^4+2x^3+23x^2+94x+192)/24$
5	$(x^5+35x^3+180x^2+744x+1560)/120$
6	$(x^6-3x^5+55x^4+255x^3+1744x^2+7308x+15120)/720$

Before proving the theorem, we collect some facts about other polynomials related to the Fibonacci numbers. For each positive integer k, let

$$\mathcal{F}_k(x) = F_1 x^{k-1} + F_2 x^{k-2} + \cdots + F_{k-1} x + F_k.$$

Lemma 2.2. *The polynomial $\mathcal{F}_k(x)$ satisfies the following statements.*

(i) The nth derivative of $\mathcal{F}_k(x)$ with respect to x is

$$\mathcal{F}_k^{(n)}(x) = \begin{cases} \displaystyle\sum_{i=1}^{k-n} x^{k-n-i} F_i \cdot (k-i)_n & \text{if } n \leq k-1, \\ 0 & \text{if } n \geq k. \end{cases}$$

(ii) $(x^2 - x - 1)\mathcal{F}_k(x) = x^{k+1} - F_{k+1}x - F_k$.

(iii) The derivatives with respect to x of the product in (ii) are

$$(x^2-x-1)\mathcal{F}_k'(x)+(2x-1)\mathcal{F}_k(x)=(k+1)x^k-F_{k+1},$$
$$(x^2-x-1)\mathcal{F}_k^{(n)}(x)+n(2x-1)\mathcal{F}_k^{(n-1)}(x)+n(n-1)\mathcal{F}_k^{(n-2)}(x)$$
$$=\begin{cases} x^{k-n+1}(k+1)_n, & \text{if } 2\le n\le k+1,\\ 0, & \text{if } n\ge k+2.\end{cases}$$

Proof. Differentiation with respect to x yields (i) and (iii). The product in (ii) telescopes. □

Proof of Theorem 2.1. Algebraic manipulation gives (2.1). For $n>0$,

$$\begin{aligned} P_n(x) &= \sum_{i=0}^{n} F_{i+n+2}\prod_{\substack{j=0\\ j\neq i}}^{n}\frac{x-j}{i-j}\\ &= \sum_{i=0}^{n} F_{i+n+2}\prod_{j\neq i}\frac{1}{i-j}\prod_{j\neq i}(x-j)\\ &= \sum_{i=0}^{n} F_{i+n+2}\cdot(-1)^{n-i}\frac{1}{i!(n-i)!}\cdot i!\binom{x}{i}(n-i)!\binom{x-i-1}{n-i}\\ &= \sum_{i=0}^{n} F_{i+n+2}\binom{x}{i}\binom{n-x}{n-i} \end{aligned}$$

where the last step uses identity (1.1). This formula holds for $n=0$ as well, since by definition $(x)_0=1$.

We prove the remaining assertions by induction on n. For $n=0$, the polynomial $P_0(x)$ is defined to be the constant polynomial 1, which has degree 0 and leading coefficient $1=1/0!$ as claimed. We want to show that, for all positive integers k,

$$P_0(k)=1=F_{k+2}-\sum_{i=1}^{k}F_i$$

and, for all negative integers k,

$$P_0(k)=1=F_{k+2}+\sum_{i=1}^{-k}F_{-i}.$$

This is immediate from substituting $x=1$ and $x=-1$, respectively, into the formula of Lemma 2.2(ii).

For $n=1$, the polynomial $P_1(x)$ satisfies $P_1(0)=F_3=2$ and $P_1(1)=F_4=3$, so $P_1(x)=x+2$, which has degree 1 and has leading coefficient $1=1/1!$, as claimed. Now, we want to show that, for all positive integers $k>1$,

$$P_1(k)=k+2=F_{k+3}-\sum_{i=1}^{k-1}F_i\cdot(k-i) \tag{2.4}$$

and, for all negative integers k,

$$P_1(k)=k+2=F_{k+3}+\sum_{i=1}^{-k-1}F_i\cdot(k+i). \tag{2.5}$$

For (2.4), assume $k > 1$. Substituting $x = 1$ into the equations of Lemma 2.2 gives

$$\mathcal{F}_k'(1) = \sum_{i=1}^{k-1} F_i \cdot (k-i), \quad \mathcal{F}_k'(1) = \mathcal{F}_k(1) + F_{k+1} - (k+1), \quad \mathcal{F}_k(1) = F_{k+1} + F_k - 1 = F_{k+2} - 1.$$

Hence,

$$\sum_{i=1}^{k-1} F_i \cdot (k-i) = \mathcal{F}_k'(1) = (F_{k+2} - 1) + F_{k+1} - (k+1) = F_{k+3} - (k+2)$$

which, rearranged, establishes (2.4).

For (2.5), assume $k < 0$. Write $k = -m$, where $m > 0$. Recall that $F_k = (-1)^{m+1} F_m$. Substituting $x = -1$ into the equations of Lemma 2.2 gives

$$\mathcal{F}_m'(-1) = (-1)^{m-1} \sum_{i=1}^{m-1} (-1)^i F_i \cdot (m-i), \quad \mathcal{F}_m'(-1) = 3\mathcal{F}_m(-1) - F_{m+1} + (-1)^m (m+1),$$
$$\mathcal{F}_m(-1) = F_{m+1} - F_m + (-1)^{m+1} = F_{m-1} + (-1)^{m+1}.$$

Hence,

$$\begin{aligned}
\sum_{i=1}^{-k-1} F_{-i} \cdot (k+i) &= (-1)^{m+1} \sum_{i=1}^{m-1} (-1)^m F_{-i} \cdot (m-i) \\
&= (-1)^{m+1} \mathcal{F}_m'(-1) \\
&= (-1)^{m+1} \big(3(F_{m-1} + (-1)^{m+1}) - F_{m+1} + (-1)^m (m+1)\big) \\
&= (-1)^{m+1} (3F_{m-1} - F_{m+1}) - m + 2 = (-1)^{m+1} F_{m-3} - m + 2
\end{aligned}$$

which establishes (2.5), after putting the expression back in terms of k.

We have now established the base case of the induction. Let $n > 1$ and assume the theorem holds for all positive integers not greater than n. We want to prove that it also holds for $n+1$, i.e., $P_{n+1}(x)$ has degree $n+1$ with leading coefficient $1/(n+1)!$ and, for each integer $k > n+1$,

$$P_{n+1}(k) = F_{k+n+3} - \sum_{i=1}^{k-n-1} F_i \binom{k-i}{n+1} \tag{2.6}$$

and, for each integer $k < 0$,

$$P_{n+1}(k) = F_{k+n+3} + \sum_{i=1}^{-k-1} F_{-i} \binom{k+i}{n+1}. \tag{2.7}$$

We begin by proving (2.6). Let $k > n+1$. Then, $k+1 > n+1$ and, by Lemma 2.2(iii),

$$(x^2 - x - 1)\mathcal{F}_k^{(n+1)}(x) + (n+1)(2x-1)\mathcal{F}_k^{(n)}(x) + n(n+1)\mathcal{F}_k^{(n-1)}(x) = x^{k-n}(k+1)_{n+1}.$$

Substituting $x = 1$ yields

$$-\mathcal{F}_k^{(n+1)}(1) + (n+1)\mathcal{F}_k^{(n)}(1) + n(n+1)\mathcal{F}_k^{(n-1)}(1) = (k+1)_{n+1}. \tag{2.8}$$

By the induction hypothesis, polynomials $P_n(x)$ and $P_{n-1}(x)$ are of degrees n and $n-1$, and have leading coefficients $1/n!$ and $1/(n-1)!$, respectively, so that

$$P_n(k) = F_{k+n+2} - \frac{1}{n!}\mathcal{F}_k^{(n)}(1), \quad P_{n-1}(k) = F_{k+n+1} - \frac{1}{(n-1)!}\mathcal{F}_k^{(n-1)}(1).$$

By Lemma 2.2(i) (with $x = 1$) and (2.8), we have

$$-\sum_{i=1}^{k-n-1} F_i \cdot (k-i)_{n+1} + (n+1)[n!F_{k+n+2} - n!P_n(k)] + n(n+1)[(n-1)!F_{k+n+1} - (n-1)!P_{n-1}(k)] = (k+1)_{n+1}.$$

Rearranging terms gives

$$\begin{aligned}(k+1)_{n+1} &+ (n+1)!P_n(k) + (n+1)!P_{n-1}(k)\\ &= (n+1)!F_{k+n+2} + (n+1)!F_{k+n+1} - \sum_{i=1}^{k-n-1} F_i \cdot (k-i)_{n+1}\\ &= (n+1)!F_{k+n+3} - \sum_{i=1}^{k-n-1} F_i \cdot (k-i)_{n+1},\end{aligned}$$

or equivalently,

$$\binom{k+1}{n+1} + P_n(k) + P_{n-1}(k) = F_{k+n+3} - \sum_{i=1}^{k-n-1} F_i \binom{k-i}{n+1}.$$

Call the left side of this equation

$$Q(x) = \binom{x+1}{n+1} + P_n(x) + P_{n-1}(x). \tag{2.9}$$

Showing $Q(x) = P_{n+1}(x)$ will establish (2.6). Notice that the polynomial $Q(x)$ has degree $n+1$ and leading coefficient $1/(n+1)!$. It suffices then to verify that $Q(k) = F_{k+n+3}$ for $k = 0, \ldots, n+1$. Consider three cases.

Case 1. For $0 \le k \le n-1$, we have $P_n(k) = F_{k+n+2}$ and $P_{n-1}(k) = F_{k+n+1}$ by the definitions of the polynomials. Also, $\binom{k+1}{n+1} = 0$ in these cases. Hence,

$$Q(k) = \binom{k+1}{n+1} + F_{k+n+2} + F_{k+n+1} = F_{k+n+3}.$$

Case 2. For $k = n$, we have $P_n(n) = F_{2n+2}$ by definition and

$$P_{n-1}(n) = F_{2n+1} - F_1\binom{n-1}{n-1} = F_{2n+1} - 1$$

by the induction hypothesis. Also, $\binom{k+1}{n+1} = \binom{n+1}{n+1} = 1$. Hence,

$$Q(k) = Q(n) = 1 + F_{2n+2} + F_{2n+1} - 1 = F_{2n+3}.$$

Case 3. For $k = n+1$, by the induction hypothesis,

$$P_n(n+1) = F_{2n+3} - F_1\binom{n}{n} = F_{2n+3} - 1,$$

$$P_{n-1}(n+1) = F_{2n+2} - F_1\binom{n}{n-1} - F_2\binom{n-1}{n-1} = F_{2n+2} - n - 1.$$

Also, $\binom{k+1}{n+1} = \binom{n+2}{n+1} = n+2$. Hence,

$$Q(k) = Q(n+1) = n + 2 + F_{2n+3} - 1 + F_{2n+2} - n - 1 = F_{2n+4}.$$

From these three cases, we conclude $Q(x) = P_{n+1}(x)$.

It remains to establish (2.7). Let $k < 0$ and write $k = -m$, where $m > 0$. Also, let $m' = m + n$. Then, again using the identity (1.1), we can write the sum in (2.7) as

$$\begin{aligned}
\sum_{i=1}^{-k-1} F_{-i}\binom{k+i}{n+1} &= \sum_{i=1}^{m-1} F_{-i}\binom{-m+i}{n+1} \\
&= (-1)^{n+1}\sum_{i=1}^{m-1} F_{-i}\binom{m+n-i}{n+1} \\
&= \frac{(-1)^{n+1}\cdot(-1)^{m'-n}}{(n+1)!}\sum_{i=1}^{m'-(n+1)} (-1)^{m'-n}F_{-i}\cdot(m'-i)_{n+1} \\
&= \frac{(-1)^{m+n+1}}{(n+1)!}\mathcal{F}_{m+n}^{(n+1)}(-1).
\end{aligned}$$

By Lemma 2.2(iii) (with $x = -1$), since $n+1 \le m+n+1$, the last expression can be written

$$\frac{(-1)^{m+n+1}}{(n+1)!}\left(3(n+1)\mathcal{F}_{m+n}^{(n)}(-1) - n(n+1)\mathcal{F}_{m+n}^{(n-1)}(-1) + (-1)^m(m+n+1)_{n+1}\right). \tag{2.10}$$

Now, we determine $\mathcal{F}_{m+n}^{(n)}(-1)$ and $\mathcal{F}_{m+n}^{(n-1)}(-1)$. By Lemma 2.2(i), identity (1.1), and the induction hypothesis, we have

$$\begin{aligned}
\mathcal{F}_{m+n}^{(n)}(-1) &= \sum_{i=1}^{m}(-1)^{m+1}F_{-i}\cdot(m+n-i)_n \\
&= (-1)^{m+1}n!\sum_{i=1}^{m}(-1)^n F_{-i}\binom{-m-1+i}{n} \\
&= (-1)^{m+n+1}n!\sum_{i=1}^{(m+1)-1} F_{-i}\binom{-(m+1)+i}{n} \\
&= (-1)^{m+n+1}n!\left(P_n(-m-1) - F_{n-m+1}\right).
\end{aligned}$$

Similarly, $\mathcal{F}_{m+n}^{(n-1)}(-1) = (n-1)!(-1)^{m+n+1}\left(P_{n-1}(-m-2) - F_{n-m-1}\right)$. Substituting these expressions into (2.10) gives

$$\begin{aligned}
&\frac{(-1)^{m+n+1}}{(n+1)!}\Big(3(n+1)(-1)^{m+n+1}n!\left(P_n(-m-1) - F_{n-m+1}\right) \\
&\qquad - n(n+1)(-1)^{m+n+1}(n-1)!\left(P_{n-1}(-m-2) - F_{n-m-1}\right) + (-1)^m(m+n+1)_{n+1}\Big) \\
&= -3F_{n-m+1} + F_{n-m-1} + 3P_n(-m-1) - P_{n-1}(-m-2) + (-1)^{n+1}\binom{m+n+1}{n+1} \\
&= -F_{n-m+3} + 3P_n(-m-1) - P_{n-1}(-m-2) + \binom{-m-1}{n+1}.
\end{aligned}$$

Define $R(x)$ as

$$R(x) = \binom{x-1}{n+1} + 3P_n(x-1) - P_{n-1}(x-2).$$

Showing that $R(x) = P_{n+1}(x)$ will establish (2.7). As with $Q(x)$ above, it suffices to establish $R(k) = F_{k+n+3}$ for $k = 0, \ldots, n+1$, which again takes three cases.

Case 1. For $k = 0$, we have

$$R(0) = \binom{-1}{n+1} + 3P_n(-1) - P_{n-1}(-2) = \binom{-1}{n+1} + 3F_{n+1} - F_{n-1} - F_{-1}\binom{-1}{n-1} = F_{n+3}.$$

Case 2. For $k = 1$, we have

$$R(1) = \binom{0}{n+1} + 3P_n(0) - P_{n-1}(-1) = 3F_{n+2} - F_n = F_{n+4}.$$

Case 3. For $2 \le k \le n+1$, we have $1 \le k-1 \le n$ and $0 \le k-2 \le n-1$, so that $P_n(k-1) = F_{k-1+n+2} = F_{k+n+1}$ and $P_{n-1}(k-2) = F_{k-2+n-1+2} = F_{k+n-1}$. Also, $\binom{k-1}{n+1} = 0$ in these cases. Hence, $R(k) = 3F_{k+n+1} - F_{k+n-1} = F_{k+n+3}$.

We conclude $R(x) = P_{n+1}(x)$, which completes the proof. □

In the proof, we showed that the polynomials $P_n(x)$ satisfy the recurrence relations

$$P_{n+1}(x) - P_n(x) - P_{n-1}(x) = \binom{x+1}{n+1}, \tag{2.11}$$

$$P_{n+1}(x) - 3P_n(x-1) + P_{n-1}(x-2) = \binom{x-1}{n+1}. \tag{2.12}$$

Now, we prove that the converses are also true in the following sense.

Proposition 2.3. *The polynomials $P_n(x)$ are uniquely determined by the initial conditions and (2.11). That is, let $(Q_n(x))$ be a sequence of polynomials defined by $Q_0(x) = 1$, $Q_1(x) = x+2$, and, for $n \ge 1$,*

$$Q_{n+1}(x) = Q_n(x) + Q_{n-1}(x) + \binom{x+1}{n+1}.$$

Then, $Q_n(x) = P_n(x)$ for all $n \ge 0$.

Proof. We proceed by induction. For $n = 0$ and 1, $Q_0(x) = 1 = P_0(x)$ and $Q_1(x) = x+2 = P_1(x)$, respectively. For $n = 2$,

$$Q_2(x) = Q_1(x) + Q_0(x) + \binom{x+1}{2} = (x+2) + 1 + \frac{(x+1)x}{2} = \frac{x^2+3x+6}{2} = P_2(x).$$

Assume $Q_k(x) = P_k(x)$ for all $k \le n$; we want to show that $Q_{n+1}(x) = P_{n+1}(x)$. By the definition of $(Q_n(x))$ and the induction hypothesis, we see that $Q_{n+1}(x)$ has degree $n+1$. Hence, it suffices to verify that $Q_{n+1}(k) = F_{k+n+2}$ for $k = 0, \ldots, n+1$. This justification follows the same details as the step of the proof of Theorem 2.1, where we verified that $Q(x)$, defined by (2.9), satisfies $Q(k) = F_{k+n+2}$ for $k = 0, \ldots, n+1$. We conclude that $Q_n(x) = P_n(x)$ for all $n \ge 0$. □

The analogous result for the second recurrence relation has a similar proof, which we omit.

Proposition 2.4. *The polynomials $P_n(x)$ are uniquely determined by the initial conditions and (2.12). That is, let $(R_n(x))$ be a sequence of polynomials defined by $R_0(x) = 1$, $R_1(x) = x+2$, and, for $n \ge 1$,*

$$R_{n+1}(x) = 3R_n(x-1) - R_{n-1}(x-2) + \binom{x-1}{n+1}.$$

Then, $R_n(x) = P_n(x)$ for all $n \ge 0$.

We will use the following result on the derivative of $P_n(x)$ in the next section.

Lemma 2.5. *Let n be a nonnegative integer. The derivative of the polynomial $P_n(x)$, defined in Theorem 2.1 with respect to x, is given by*

$$P_n'(x) = P_n(x)\sum_{i=0}^{n}\frac{1}{x-i} - \sum_{i=0}^{n}\frac{1}{x-i}F_{i+n+2}\binom{x}{i}\binom{n-x}{n-i}.$$

In particular,

$$P_n'(n) = H_nF_{2n+2} + \sum_{i=0}^{n-1}\frac{(-1)^{n-i}}{n-i}F_{i+n+2}\binom{n}{i},$$

where H_n is the harmonic number defined by $H_0 = 0$ and $H_n = \sum_{i=1}^{n}\frac{1}{i}$ for $n \geq 1$.

Proof. Differentiating with respect to x, using $\frac{d}{dx}\binom{x}{n} = \binom{x}{n}\sum_{i=0}^{n-1}\frac{1}{x-i}$, gives

$$\begin{aligned}
\frac{d}{dx}\binom{x}{i}\binom{n-x}{n-i} &= \binom{x}{i}\binom{n-x}{n-i}\sum_{j=0}^{n-1-i}\frac{(-1)}{n-x-j} + \binom{n-x}{n-i}\binom{x}{i}\sum_{j=0}^{i-1}\frac{1}{x-j}\\
&= \binom{x}{i}\binom{n-x}{n-i}\left(\sum_{j=0}^{n-1-i}\frac{1}{x-n+j} + \sum_{j=0}^{i-1}\frac{1}{x-j}\right)\\
&= \binom{x}{i}\binom{n-x}{n-i}\left(\sum_{j=0}^{n}\frac{1}{x-j} - \frac{1}{x-i}\right)
\end{aligned}$$

for each $i = 0, \ldots, n$. Combining this with the definition of $P_n(x)$ gives the desired formula. To evaluate $P_n'(x)$ when $x = n$, rewrite the expression of $P_n'(x)$ as

$$P_n'(x) = P_n(x)\sum_{i=0}^{n-1}\frac{1}{x-i} - \sum_{i=0}^{n-1}\frac{1}{x-i}F_{i+n+2}\binom{x}{i}\binom{n-x}{n-i} + \frac{1}{x-n}P_n(x) - \frac{1}{x-n}F_{2n+2}\binom{x}{n}.$$

By the definition of $P_n(x)$, the last two terms in the sum above can be expressed as

$$\begin{aligned}
\frac{1}{x-n}\sum_{i=0}^{n-1}F_{i+n+2}\binom{x}{i}\binom{n-x}{n-i} &= \frac{1}{x-n}\sum_{i=0}^{n-1}F_{i+n+2}\binom{x}{i}\frac{n-x}{n-i}\binom{n-x-1}{n-i-1}\\
&= -\sum_{i=0}^{n-1}\frac{1}{n-i}F_{i+n+2}\binom{x}{i}\binom{n-x-1}{n-i-1}.
\end{aligned}$$

Substituting $x = n$ into this alternative form of $P_n'(x)$ gives the desired expression. □

3. Summation Identities Involving the Fibonacci Numbers

We can now motivate our definition of $P_n(x)$ by detailing how it is closely related to Fibonacci numbers beyond the defining values $x = 0, \ldots, n$. For instance, applying (2.3) to $k = -3, -2, -1$ and (2.2) to $k = n+1, n+2$ gives

$$\begin{aligned}
P_n(-3) = F_{n-1} + (-1)^n n, P_n(-2) = F_n + (-1)^n, P_n(-1) = F_{n+1};\\
P_n(n+1) = F_{2n+3} - 1, P_n(n+2) = F_{2n+4} - (n+2),
\end{aligned}\tag{3.1}$$

respectively. Let us highlight that these results depend on our particular choice of the points (i, F_{n+i+2}) for $i = 0, \ldots, n$ to determine the interpolating polynomials; most choices of points (i, F_j) do not lead to comparable results. In future work, we will consider some other choices of points that do lead to interesting polynomials.

Our last result makes use of the following lemma, an easy extension of Theorem 1, and Corollary 2 in [3] to arbitrary polynomials.

Lemma 3.1. *Let n be a nonnegative integer and $p(x)$ a degree k polynomial with leading coefficient a_n.*

If $k<n$, then $\sum_{i=0}^{n}(-1)^i p(i)\binom{n}{i}=0$. If $k=n$, then $\sum_{i=0}^{n}(-1)^i p(n-i)\binom{n}{i}=n!a_n$.

We conclude with various identities that come from the alternative formulations of $P_n(k)$ developed in Theorem 2.1.

Corollary 3.2. *Given positive integers m,n, and nonnegative integer k,*

$$\sum_{i=0}^{n}(-1)^i F_{i+n+2}\binom{n+1}{i+1}=F_{n+1}, \tag{3.2}$$

$$\sum_{i=0}^{n}(-1)^i F_{i+n+k}\binom{n}{i}=(-1)^n F_k, \tag{3.3}$$

$$\sum_{i=0}^{n}(-1)^i F_{2n+2-i}\binom{n}{i}=1, \tag{3.4}$$

$$\sum_{i=1}^{m}(-1)^{i+1}F_i\left(\binom{m+n-i+1}{n+1}+\binom{m+n-i-1}{n}\right)=\binom{m+n}{n+1}, \tag{3.5}$$

$$\begin{aligned}\sum_{i=1}^{n-1}\frac{(-1)^{n-i}}{n-i}&\left(F_{i+n}\binom{n-1}{i-1}-3F_{i+n+2}\binom{n}{i}+F_{i+n+4}\binom{n+1}{i+1}\right)\\ &=\frac{1}{n+1}+\frac{(-1)^n}{n(n+1)}\left(F_n-nF_{n+1}-n^2F_{n+4}\right)-H_{n-1}F_{2n}+3H_nF_{2n+2}-H_{n+1}F_{2n+4}.\end{aligned} \tag{3.6}$$

Proof. Identity (3.2) is essentially writing out $P_n(-1)$ using (3.1).

We prove (3.3) by induction on k. For $k=0$, Lemma 3.1 applied to the polynomial $P_{n-2}(x)$ and the specific formulas (3.1) yield

$$\begin{aligned}0&=\sum_{i=0}^{n}(-1)^i P_{n-2}(i)\binom{n}{i}\\ &=\sum_{i=0}^{n-2}(-1)^i P_{n-2}(i)\binom{n}{i}+(-1)^{n-1}P_{n-2}(n-1)\binom{n}{n-1}+(-1)^n P_{n-2}(n)\binom{n}{n}\\ &=\sum_{i=0}^{n-2}(-1)^i F_{i+n}\binom{n}{i}+(-1)^{n-1}(F_{2n-1}-1)n+(-1)^n(F_{2n}-n)\\ &=\sum_{i=0}^{n}(-1)^i F_{i+n}\binom{n}{i}.\end{aligned}$$

Similarly, for $k=1$, applying Lemma 3.1 to $P_{n-1}(x)$ gives

$$\sum_{i=0}^{n}(-1)^i F_{i+n+1}\binom{n}{i}=(-1)^n=(-1)^n F_1.$$

The inductive step then follows from the recurrence relation of the Fibonacci numbers.

Identity (3.4) follows immediately from applying Lemma 3.1 to $P_n(x)$.

Identity (3.5) follows from evaluating the recurrence relation (2.11) at negative integers whose values are supplied by (2.3).

To prove (3.6), differentiate the recurrence relation (2.12) with respect to x and then evaluate at $x = n+1$ to obtain

$$P'_{n+1}(n+1) - 3P'_n(n) + P'_{n-1}(n-1) = \frac{1}{n+1}.$$

Replacing each derivative with the formula from Lemma 2.5 gives the identity. □

The identity (3.3) for $k = 2$ can also be derived by equating $1/n!$, the leading coefficient of the polynomial $P_n(x)$, to the leading coefficient of the polynomial appearing on the right side of (2.1). Moreover, (3.2)–(3.4) can also be proved directly using Binet's formula for the nth Fibonacci number. For example, to prove (3.2),

$$\begin{aligned}
\sum_{i=0}^{n}(-1)^i F_{i+n+2}\binom{n+1}{i+1} &= \sum_{i=0}^{n}(-1)^i\Big(\frac{\alpha^{i+n+2}-\beta^{i+n+2}}{\sqrt{5}}\Big)\binom{n+1}{i+1}\\
&= \frac{\alpha^{n+2}}{\sqrt{5}}\sum_{i=0}^{n}(-\alpha)^i\binom{n+1}{i+1} - \frac{\beta^{n+2}}{\sqrt{5}}\sum_{i=0}^{n}(-\beta)^i\binom{n+1}{i+1}\\
&= \frac{\alpha^{n+2}}{\sqrt{5}}(-\alpha)^{-1}((1-\alpha)^{n+1}-1) - \frac{\beta^{n+2}}{\sqrt{5}}(-\beta)^{-1}((1-\beta)^{n+1}-1)\\
&= -\frac{\alpha^{n+1}}{\sqrt{5}}(\beta^{n+1}-1) + \frac{\beta^{n+1}}{\sqrt{5}}(\alpha^{n+1}-1)\\
&= \frac{\alpha^{n+1}-\beta^{n+1}}{\sqrt{5}}\\
&= F_{n+1}.
\end{aligned}$$

In contrast, identities (3.5) and (3.6) do not seem to have a more direct proof of this type.

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MSC2010: 11B39, 33C05

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