TRIANGULAR REPBLOCKS

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Abstract. In this paper, we prove a finiteness theorem concerning repblocks of two digits in base 10, which are represented by a fixed quadratic polynomial. We also show that the only repblocks of two digits that are triangular numbers are 10, 15, 21, 28, 36, 45, 55, 66, 78, 91, 5050, 5151.

1. Introduction

Positive integers of the form $n(n+1)/2$ are called triangular numbers. A positive integer is called a *repdigit* (sequence A010785 in the OEIS [11]), if it has only one distinct digit in its decimal expansion. In 1975, Ballew and de Weger [1] stated that E. B. Escott in 1905 proved that 1, 3, 6, 55, 66, and 666 are the only triangular repdigits of less than 30 digits. In their paper, they extended the result of Escott and proved that these are the only triangular repdigits.

In 1989, L. Ming [8] proved that the only triangular numbers in the Fibonacci sequence are 1, 3, 21, and 55. That settled a conjecture of V. Hoggatt [5]. Later in 1991, L. Ming [9], proved that the only triangular numbers in the companion Lucas sequence of the Fibonacci sequence are 1, 3, and 5778. W. McDaniel [7], proved that 1 is the only triangular number in the Pell sequence. For further results concerning triangular numbers in the Pell sequence, see [4]. Recently, J. H. Jaroma [6], proved that 1 is the only integer that is both triangular and *repunit*, that is a positive integer whose decimal expansion consists entirely of the digit 1. Now, we extend the notion of repdigits and consider the positive integer with repeated blocks of two digits, which we call *repblocks of two digits*. Such a number has the form

$$c\left(\frac{10^{2m}-1}{99}\right), \qquad \text{for some} \quad m \geq 1 \quad \text{and} \quad c \in \{10, 11, \ldots, 99\}.$$

In this paper, we prove the finiteness of the solutions of some equations that involves repblocks of two digits. In particular, our result is the following.

Theorem 1. *Let A, B, C be fixed rational numbers with $A \neq 0$. Then, the Diophantine equation*

$$c\left(\frac{10^{2m}-1}{99}\right) = An^2 + Bn + C, \tag{1}$$

has only finite number of solutions, in integers $m, n \geq 1$ and $c \in \{10, 11, \ldots, 99\}$ provided $99B^2 - 396AC - 4Ac \neq 0$.

In case $99B^2 - 396AC - 4Ac = 0$, the equation may have infinitely many solutions, or none. For example, if $(A, B, C, c) = (1/99, 0, -1/99, 1)$, then $99B^2 - 396AC - 4Ac = 0$ and in this case, $n = 10^m$ is a solution that satisfies (1) identically. On the other hand, if $99B^2 - 396AC - 4Ac = 0$ but $99/(4Ac)$ is not a square of a rational number (see (2)), then equation (1) does not have any solution (n, m). We give no further details.

Proof. We begin by multiplying both sides of equation (1) by $4A$ and rearranging the terms, which gives us

$$4Ac\left(\frac{10^{2m}-1}{99}\right)+B^2-4AC=y^2,$$

where $y=2An+B$. Further, letting $m=3m_1+r$ with $r\in\{0,1,2\}$ and performing some algebra, we obtain

$$4Ac\cdot 10^{6m_1+2r}+\big(99(B^2-4AC)-4Ac\big)=99y^2. \tag{2}$$

Multiplying both sides of equation (2) by $4^2 11^3 A^2 c^2 10^{4r}$, we get

$$V^2=U^3+L, \tag{3}$$

where $U=44Ac10^{2(m_1+r)}$, $V=1452Ac10^{2r}y$, and $L=21296A^2c^2 10^{4r}(99B^2-396AC-4Ac)$. The hypothesis implies that $L\neq 0$. Thus, we obtain an elliptic curve over $\mathbb{Q}$ given by (3). By a theorem of Siegel (see [10], p. 313), this curve has a finite number of integer points. Therefore, equation (1) has a finite number of positive integer solutions. □

The following result comes as a corollary of the Theorem 1.

Corollary 1. *The complete list of repblocks of two digits, which are also triangular numbers, is*

$$10,\ 15,\ 21,\ 28,\ 36,\ 45,\ 55,\ 66,\ 78,\ 91,\ 5050,\ 5151.$$

Proof. To search for the triangular repblocks of two digits, we let $A=B=\frac{1}{2}$ and $C=0$ in equation (1), which then gives us the following Diophantine equation

$$c\left(\frac{10^{2m}-1}{99}\right)=\frac{n(n+1)}{2}, \tag{4}$$

in integers $m,n\geq 1$ and $c\in\{10,11,\ldots,99\}$. Furthermore, equation (3) reduces to

$$v^2=v^3+\ell, \tag{5}$$

where $u=22c10^{2(m_1+r)}$, $v=363c10^{2r}y$, and $\ell=(99-8c)c^2 10^{4r}11^3$. Note that $\ell\neq 0$, since this would lead to $c=99/8$, which is impossible. By Theorem 1, equation (4) has only a finite number of solutions in $m,n\geq 1$ and $10\leq c\leq 99$. Since $c\in\{10,11\ldots,99\}$ and $r\in\{0,1,2\}$, we obtain 243 elliptic curves given by (3). This leads us to determine the integer points (u,v) on each of these elliptic curves. For this, we used MAGMA [2].

The following table displays all the integer points (u,v)[1], as described above that produce the corresponding integer solutions (m,n) of equation (4). There are only 12 such elliptic curves given by equation (2). The other 231 equations do not have any integer points (u,v), or do not produce relevant solutions (m,n) and thus, we omit those equations.

[1] (u,v)'s in **bold** correspond to the integer solutions in the third column.

c, r	(u, v)	(m, n)
$c = 10,\ r = 1$	$(-2900, \pm 30000), (-2200, \pm 121000),\ (-200, \pm 159000),$ $(2761, \pm 215259),\ (8000, \pm 733000),\ (\mathbf{22000}, \pm\mathbf{3267000})$	$(1, 4)$
$c = 15,\ r = 1$	$(4425, \pm 154125),\ (8800, \pm 786500),\ (\mathbf{33000}, \pm\mathbf{5989500}),$ $(36600, \pm 6997500)$	$(1, 5)$
$c = 21,\ r = 1$	$(\mathbf{46200}, \pm\mathbf{9909900})$	$(1, 6)$
$c = 28,\ r = 1$	$(\mathbf{61600}, \pm\mathbf{15246000})$	$(1, 7)$
$c = 36,\ r = 1$	$(25524, \pm 3656232), (\mathbf{79200}, \pm\mathbf{22215600}),$ $(127600, \pm 45544400),\ (1753200, \pm 2321384400)$	$(1, 8)$
$c = 45,\ r = 1$	$(\mathbf{99000}, \pm\mathbf{31036500})$	$(1, 9)$
$c = 50,\ r = 2$	$(2120000, \pm 3070500000),\ (\mathbf{11000000}, \pm\mathbf{36481500000})$	$(2, 100)$
$c = 51,\ r = 2$	$(1734000, \pm 2259810000), (3706000, \pm 7126910000),$ $(4686000, \pm 10138590000),\ (\mathbf{11220000}, \pm\mathbf{37581390000}),$ $(17217600, \pm 71442126000),\ (20476500, \pm 92657565000),$ $(166268400, \pm 2143949598000)$	$(2, 101)$
$c = 55,\ r = 1$	$(24200, \pm 665500),\ (27104, \pm 2486308),\ (48400, \pm 9982500),$ $(75625, \pm 20464125),\ (\mathbf{121000}, \pm\mathbf{41926500}),$ $(341825, \pm 199816375),\ (1694000, \pm 2204801500)$	$(1, 10)$
$c = 66,\ r = 1$	$(31944, \pm 2779128),\ (36300, \pm 4791600),$ $(121000, \pm 41793400),\ (\mathbf{145200}, \pm\mathbf{55103400}),$ $(2952400, \pm 5072973400),\ (15765816, \pm 62600049864)$	$(1, 11)$
$c = 78,\ r = 1$	$(46176, \pm 7479576), (\mathbf{171600}, \pm\mathbf{70785000})$	$(1, 12)$
$c = 91,\ r = 1$	$(\mathbf{200200}, \pm\mathbf{89189100})$	$(1, 13)$

Table 1: Integer solutions.

The list of ordered pairs (m, n) in the third column of Table 1, with the corresponding values of c in the first column provide the complete list of the solutions (m, n, c) in positive integers with $c \in \{10, 11, \ldots, 99\}$ of equation (4). From this, we conclude that the only triangular numbers that are also the repblocks of two digits are given by the statement of Corollary 1. This completes the proof of Corollary 1.

$\square$

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