12 TWO-PARAMETER FAMILIES OF RECIPROCAL SUMS OF PRODUCTS OF THE SINE AND COSINE FUNCTIONS

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Abstract. In this paper, we give closed forms for 12 two-parameter families of finite sums. In each of the aforementioned 12 families of finite sums, the denominator of the summand consists of a product of the sine or cosine functions, and the length of this product can be made as large as we please.

1. Introduction

In this paper, we present closed forms for 12 families of finite reciprocal sums of products that involve the sine or cosine functions. Each of the 12 sums that we consider is parametrized by an integer $j \geq 0$, and a rational number $k \neq 0$. The number of factors in the denominator of each summand increases with j, and is therefore arbitrarily large. In this regard, we refer the interested reader to [1], which is the only reference of which we are aware that presents closed forms for families of finite reciprocal sums of arbitrarily long products of the sine or cosine functions. In [1], each of the finite sums that we consider has a so-called weight term.

Throughout this paper, we take the running variable i to be the dummy variable, so that, for instance, $[\cos(ki+1)]_0^n$ is taken to mean $\cos(kn+1) - \cos 1$.

The following is an example of the kinds of sums that arise from the main results in the present paper. For $n \geq 1$, we have

$$\sum_{i=1}^{n} \frac{\sin(i+j+1)}{\cos i \cos(i+1) \cdots \cos(i+2j+2)} = \frac{1}{2\sin(j+1)} \left[\frac{1}{\cos(i+1) \cdots \cos(i+2j+2)} \right]_0^n, \quad (1.1)$$

in which $j \geq 0$ is an integer. In (1.1), which is a special case of (3.1), the product in the denominator of the summand has $2j+3$ factors, and so can be arbitrarily long. For $j = 2$, (1.1) becomes

$$\sum_{i=1}^{n} \frac{\sin(i+3)}{\cos i \cos(i+1)\cos(i+2) \cdots \cos(i+6)} = \frac{1}{2\sin 3} \left[\frac{1}{\cos(i+1) \cdots \cos(i+6)} \right]_0^n.$$

In Section 2, we define the 12 (families of finite reciprocal) sums that we consider in this paper. In Section 3, we give the closed form for each of the 12 sums in question. In Section 4, we demonstrate a sample proof, and in Section 5 we give special cases of a selection of our main results.

2. The Finite Sums

Throughout this paper, the upper limit of summation is an integer $n \geq 1$. Furthermore, in each of the sums that we define, the parameter $j \geq 0$ is taken to be an integer, and $k \neq 0$ is a rational number. We now define the 12 sums whose closed forms we give in the next section.

The first four sums S_1, S_2, S_2, and S_4 are defined as

$$S_1(n,j,k)=\sum_{i=1}^{n}\frac{\sin(k(i+j+1))}{\cos(ki)\cos(k(i+1))\cdots\cos(k(i+2j+2))},$$
$$S_2(n,j,k)=\sum_{i=1}^{n}\frac{\cos(k(i+j+1))}{\sin(ki)\sin(k(i+1))\cdots\sin(k(i+2j+2))},$$
$$S_3(n,j,k)=\sum_{i=1}^{n}\frac{\sin(k(2i+2j+1))}{\cos(2ki)\cos(2k(i+1))\cdots\cos(2k(i+2j+1))},$$
$$S_4(n,j,k)=\sum_{i=1}^{n}\frac{\cos(k(2i+2j+1))}{\sin(2ki)\sin(2k(i+1))\cdots\sin(2k(i+2j+1))}.$$

The sums S_5, S_6, S_7, and S_8 are alternating, and are defined as

$$S_5(n,j,k)=\sum_{i=1}^{n}\frac{(-1)^i\sin(k(i+j+1))}{\sin(ki)\sin(k(i+1))\cdots\sin(k(i+2j+2))},$$
$$S_6(n,j,k)=\sum_{i=1}^{n}\frac{(-1)^i\cos(k(i+j+1))}{\cos(ki)\cos(k(i+1))\cdots\cos(k(i+2j+2))},$$
$$S_7(n,j,k)=\sum_{i=1}^{n}\frac{(-1)^i\sin(k(2i+2j+1))}{\sin(2ki)\sin(2k(i+1))\cdots\sin(2k(i+2j+1))},$$
$$S_8(n,j,k)=\sum_{i=1}^{n}\frac{(-1)^i\cos(k(2i+2j+1))}{\cos(2ki)\cos(2k(i+1))\cdots\cos(2k(i+2j+1))}.$$

Finally for this section, we define the sums S_9, S_{10}, S_{11}, and S_{12}, each of which contains a run of squared terms in the denominator of the summand.

$$S_9(n,j,k)=\sum_{i=1}^{n}\frac{\sin(2k(i+j+1))}{\sin^2(ki)\sin^2(k(i+1))\cdots\sin^2(k(i+2j+2))},$$
$$S_{10}(n,j,k)=\sum_{i=1}^{n}\frac{\sin(2k(i+j+1))}{\cos^2(ki)\cos^2(k(i+1))\cdots\cos^2(k(i+2j+2))},$$
$$S_{11}(n,j,k)=\sum_{i=1}^{n}\frac{\sin(2k(2i+2j+1))}{\sin^2(2ki)\sin^2(2k(i+1))\cdots\sin^2(2k(i+2j+1))},$$
$$S_{12}(n,j,k)=\sum_{i=1}^{n}\frac{\sin(2k(2i+2j+1))}{\cos^2(2ki)\cos^2(2k(i+1))\cdots\cos^2(2k(i+2j+1))}.$$

3. The Closed Forms

In this section, we give the closed form for each of the 12 sums defined in Section 2. We present these closed forms in three theorems. Our first theorem gives the closed forms for S_1, S_2, S_3, and S_4.

Theorem 3.1. *The closed forms for S_1, S_2, S_3, and S_4 are*

$$S_1(n,j,k) = \frac{1}{2\sin(k(j+1))}\left[\frac{1}{\cos(k(i+1))\cdots\cos(k(i+2j+2))}\right]_0^n, \tag{3.1}$$

$$S_2(n,j,k) = \frac{-1}{2\sin(k(j+1))}\left[\frac{1}{\sin(k(i+1))\cdots\sin(k(i+2j+2))}\right]_0^n, \tag{3.2}$$

$$S_3(n,j,k) = \frac{1}{2\sin(k(2j+1))}\left[\frac{1}{\cos(2k(i+1))\cdots\cos(2k(i+2j+1))}\right]_0^n, \tag{3.3}$$

$$S_4(n,j,k) = \frac{-1}{2\sin(k(2j+1))}\left[\frac{1}{\sin(2k(i+1))\cdots\sin(2k(i+2j+1))}\right]_0^n. \tag{3.4}$$

Our second theorem gives the closed forms for S_5, S_6, S_7, and S_8.

Theorem 3.2. *The closed forms for S_5, S_6, S_7, and S_8 are*

$$S_5(n,j,k) = \frac{1}{2\cos(k(j+1))}\left[\frac{(-1)^i}{\sin(k(i+1))\cdots\sin(k(i+2j+2))}\right]_0^n, \tag{3.5}$$

$$S_6(n,j,k) = \frac{1}{2\cos(k(j+1))}\left[\frac{(-1)^i}{\cos(k(i+1))\cdots\cos(k(i+2j+2))}\right]_0^n, \tag{3.6}$$

$$S_7(n,j,k) = \frac{1}{2\cos(k(2j+1))}\left[\frac{(-1)^i}{\sin(2k(i+1))\cdots\sin(2k(i+2j+1))}\right]_0^n, \tag{3.7}$$

$$S_8(n,j,k) = \frac{1}{2\cos(k(2j+1))}\left[\frac{(-1)^i}{\cos(2k(i+1))\cdots\cos(2k(i+2j+1))}\right]_0^n. \tag{3.8}$$

Our final theorem gives the closed forms for S_9, S_{10}, S_{11}, and S_{12}.

Theorem 3.3. *The closed forms for S_9, S_{10}, S_{11}, and S_{12} are*

$$S_9(n,j,k) = \frac{-1}{\sin(2k(j+1))}\left[\frac{1}{\sin^2(k(i+1))\cdots\sin^2(k(i+2j+2))}\right]_0^n, \tag{3.9}$$

$$S_{10}(n,j,k) = \frac{1}{\sin(2k(j+1))}\left[\frac{1}{\cos^2(k(i+1))\cdots\cos^2(k(i+2j+2))}\right]_0^n, \tag{3.10}$$

$$S_{11}(n,j,k) = \frac{-1}{\sin(2k(2j+1))}\left[\frac{1}{\sin^2(2k(i+1))\cdots\sin^2(2k(i+2j+1))}\right]_0^n, \tag{3.11}$$

$$S_{12}(n,j,k) = \frac{1}{\sin(2k(2j+1))}\left[\frac{1}{\cos^2(2k(i+1))\cdots\cos^2(2k(i+2j+1))}\right]_0^n. \tag{3.12}$$

To conclude this section, we note that due to the parameter k, (3.9) and (3.10) generalize two results that appear in the first paragraph in Section 5 of [1].

4. A Sample Proof

We require some familiar identities from elementary trigonometry. First, we have

$$\begin{aligned}\sin\alpha - \sin\beta &= 2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right),\\ \sin\alpha + \sin\beta &= 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right),\end{aligned} \tag{4.1}$$

in which α and β are arbitrary real numbers. From the two identities in (4.1), it follows immediately that

$$\sin^2\alpha - \sin^2\beta = \sin(\alpha+\beta)\sin(\alpha-\beta). \tag{4.2}$$

Each of the 12 results stated in Theorem 3.1, Theorem 3.2, and Theorem 3.3 can be proved in the same manner. To illustrate the method, we now give a proof of (3.9).

Proof. Denote the right side of (3.9) by $r(n,j,k)$. Then, the difference $r(n+1,j,k)-r(n,j,k)$ is

$$\begin{aligned}
&\frac{1}{\sin(2k(j+1))} \times \frac{\sin^2(k(n+2j+3)) - \sin^2(k(n+1))}{\sin^2(k(n+1))\cdots\sin^2(k(n+2j+3))}\\
&= \frac{1}{\sin(2k(j+1))} \times \frac{\sin(2k(n+j+2))\sin(2k(j+1))}{\sin^2(k(n+1))\cdots\sin^2(k(n+2j+3))}, \text{ by (4.2)}\\
&= \frac{\sin(2k(n+j+2))}{\sin^2(k(n+1))\cdots\sin^2(k(n+2j+3))}\\
&= S_9(n+1,j,k) - S_9(n,j,k).
\end{aligned} \tag{4.3}$$

In a similar manner, we see that

$$\begin{aligned}
r(1,j,k) &= \frac{1}{\sin(2k(j+1))} \times \frac{\sin^2(k(2j+3)) - \sin^2 k}{\sin^2 k \sin^2(2k)\cdots\sin^2(k(2j+3))}\\
&= \frac{1}{\sin(2k(j+1))} \times \frac{\sin(2k(j+2))\sin(2k(j+1))}{\sin^2 k \sin^2(2k)\cdots\sin^2(k(2j+3))}\\
&= \frac{\sin(2k(j+2))}{\sin^2 k \sin^2(2k)\cdots\sin^2(k(2j+3))}\\
&= S_9(1,j,k).
\end{aligned} \tag{4.4}$$

Together, (4.3) and (4.4) prove (3.9). □

5. Special Cases of a Selection of Our Main Results

In this section, we give some simple cases of a selection of our main results. To begin, with $k=1$, (3.3) becomes

$$\begin{aligned}
&\sum_{i=1}^{n} \frac{\sin(2i+2j+1)}{\cos(2i)\cos(2(i+1))\cdots\cos(2(i+2j+1))}\\
&= \frac{1}{2\sin(2j+1)}\left[\frac{1}{\cos(2(i+1))\cdots\cos(2(i+2j+1))}\right]_0^n.
\end{aligned} \tag{5.1}$$

With $j=0$ and $j=1$, (5.1) becomes, respectively

$$\sum_{i=1}^{n} \frac{\sin(2i+1)}{\cos(2i)\cos(2(i+1))} = \frac{1}{2\sin 1}\left(\frac{1}{\cos(2(n+1))} - \frac{1}{\cos 2}\right),$$

$$\sum_{i=1}^{n} \frac{\sin(2i+3)}{\cos(2i)\cos(2(i+1))\cdots\cos(2(i+3))} = \frac{1}{2\sin 3}\left[\frac{1}{\cos(2(i+1))\cdots\cos(2(i+3))}\right]_0^n.$$

Next, with $k = 1$, (3.5) becomes

$$\begin{aligned}\sum_{i=1}^{n} &\frac{(-1)^i \sin(i+j+1)}{\sin i \sin(i+1)\cdots\sin(i+2j+2)} \\ &= \frac{1}{2\cos(j+1)}\left[\frac{(-1)^i}{\sin(i+1)\cdots\sin(i+2j+2)}\right]_0^n.\end{aligned} \tag{5.2}$$

With $j = 0$ and $j = 1$, (5.2) becomes, respectively

$$\begin{aligned}\sum_{i=1}^{n} \frac{(-1)^i}{\sin i \sin(i+2)} &= \frac{1}{2\cos 1}\left(\frac{(-1)^n}{\sin(n+1)\sin(n+2)} - \frac{1}{\sin 1 \sin 2}\right), \\ \sum_{i=1}^{n} \frac{(-1)^i}{\sin i \sin(i+1)\sin(i+3)\sin(i+4)} &= \frac{1}{2\cos 2}\left[\frac{(-1)^i}{\sin(i+1)\cdots\sin(i+4)}\right]_0^n.\end{aligned} \tag{5.3}$$

To conclude, we remark that to present this paper succinctly, we have chosen to present all our results in an abbreviated manner. We now indicate how our results can be expressed in their most general form. Let θ be any real number that is not a rational multiple of π. This condition on θ eliminates the possibility of vanishing denominators. Then, this *entire* paper can be generalized in the following manner: take *every* occurrence of sin and cos, and multiply the argument by θ. For instance, the generalized form of the first sum in the array (5.3) is

$$\sum_{i=1}^{n} \frac{(-1)^i}{\sin(i\theta)\sin((i+2)\theta)} = \frac{1}{2\cos\theta}\left(\frac{(-1)^n}{\sin((n+1)\theta)\sin((n+2)\theta)} - \frac{1}{\sin\theta\sin(2\theta)}\right).$$

References

[1] R. S. Melham, *Sums of reciprocals of weighted products of the sine and cosine functions*, The Fibonacci Quarterly, **56.2** (2018), 99–105.

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