THE CONVERSE OF EXACT DIVISIBILITY BY POWERS OF THE FIBONACCI AND LUCAS NUMBERS

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ABSTRACT. In 2014, Pongsriiam obtained the results on exact divisibility by powers of the Fibonacci and Lucas numbers. For instance, he proved that if $F_n^k \parallel m$, $n \geq 3$, and $n \not\equiv 3 \pmod 6$, then $F_n^{k+1} \parallel F_{nm}$. In this article, we give the converse of those theorems.

1. Introduction

Let $(F_n)_{n\geq 1}$ be the Fibonacci sequence defined by $F_1 = F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$, and let $(L_n)_{n\geq 1}$ be the Lucas sequence given by $L_1 = 1$, $L_2 = 3$ with the same recursive pattern as the Fibonacci sequence. We also assume throughout that k, m, and n are positive integers and p is a prime. The exact divisibility $m^k \| n$ means that $m^k | n$ and $m^{k+1} \nmid n$. The order of appearance of n in the Fibonacci sequence, denoted by $z(n)$, is the smallest positive integer k such that $n \,| F_k$. The p-adic valuation (or p-adic order) of n, denoted by $v_p(n)$, is the exponent of p in the prime factorization of n. So, $v_p(n) = k$ if and only if $p^k \| n$.

The divisibility property of the Fibonacci numbers and the behavior of the order of appearance have been a popular area of research, see for example in [6, 7, 8, 9, 10, 15, 18] and references therein. In particular, the divisibility by powers of the Fibonacci numbers attracts some attention because it is used in Matiyasevich's solution to Hilbert's 10th problem [11]. For example, Hoggatt and Bicknell-Jonhson [5] show that

$$\text{if } F_n^k \mid m, \text{ then } F_n^{k+1} \mid F_{nm}, \tag{1.1}$$

which is proved again later by Benjamin and Rouse using a different method [1]. In addition, Tangboonduangjit and Wiboonton [29], Panraksa et al. [14], and Onphaeng and Pongsriiam [13] obtain the divisibility by powers of the Fibonacci numbers, in particular subsequences of $(F_n)_{n\geq 1}$. The most general result in this direction is given by Pongsriiam [17] as follows.

Theorem 1.1. [17, Theorem 2] *For $n \geq 3$, we have*

(i) *if $F_n^k \parallel m$ and $n \not\equiv 3 \pmod 6$, then $F_n^{k+1} \parallel F_{nm}$;*
(ii) *if $F_n^k \parallel m$, $n \equiv 3 \pmod 6$, and $\frac{F_n^{k+1}}{2} \nmid m$, then $F_n^{k+1} \parallel F_{nm}$;*
(iii) *if $F_n^k \parallel m$, $n \equiv 3 \pmod 6$, and $\frac{F_n^{k+1}}{2} \mid m$, then $F_n^{k+2} \parallel F_{nm}$.*

Theorem 1.2. [17, Theorem 3] *Let m be an odd integer. Then,*

(i) *if $L_n^k \mid m$, then $L_n^{k+1} \mid L_{nm}$;*
(ii) *if $n \geq 2$ and $L_n^k \parallel m$, then $L_n^{k+1} \parallel L_{nm}$.*

Theorem 1.3. [17, Theorem 4] *Let m be even and $n \geq 2$. Then, the following statements hold.*

(i) *If $L_n^k \mid m$, then $L_n^{k+1} \mid F_{nm}$.*
(ii) *If $L_n^k \parallel m$ and $n \not\equiv 0 \pmod 3$, then $L_n^{k+1} \parallel F_{nm}$.*
(iii) *If $L_n^k \parallel m$, $n \equiv 0 \pmod 6$, and $\frac{L_n^{k+1}}{2} \nmid m$, then $L_n^{k+1} \parallel F_{nm}$.*

(iv) *If* $L_n^k \parallel m$, $n \equiv 0 \pmod 6$ *and* $\frac{L_n^{k+1}}{2} \mid m$, *then* $L_n^{k+2} \mid F_{nm}$.
(v) *If* $L_n^k \parallel m$, $n \equiv 3 \pmod 6$, *and* $\frac{L_n^{k+1}}{4} \nmid m$, *then* $L_n^{k+1} \parallel F_{nm}$.
(vi) *If* $L_n^k \parallel m$, $n \equiv 3 \pmod 6$, *and* $\frac{L_n^{k+1}}{4} \mid m$, *then* $L_n^{k+2} \mid 4F_{nm}$.

In this article, we prove the converse of the above theorems. For some recent results concerning the Fibonacci and Lucas numbers, we refer the reader to [19, 20, 21, 22]. We also invite the reader to visit the second author's ResearchGate website [28], which contains freely downloadable articles on related topics of research [2, 12, 16, 23, 24, 25, 26, 27].

2. Preliminaries and Lemmas

In this section, we give some useful lemmas for the reader's convenience. First, Lengyel's result on the p-adic valuation of the Fibonacci and Lucas numbers is as follows.

Lemma 2.1. (Lengyel [9]) *For every* $n \geq 1$, *we have*

$$v_2(F_n) = \begin{cases} 0, & \text{if } n \equiv 1,2 \pmod 3; \\ 1, & \text{if } n \equiv 3 \pmod 6; \\ v_2(n)+2, & \text{if } n \equiv 0 \pmod 6, \end{cases}$$

$$v_2(L_n) = \begin{cases} 0, & \text{if } n \equiv 1,2 \pmod 3; \\ 2, & \text{if } n \equiv 3 \pmod 6; \\ 1, & \text{if } n \equiv 0 \pmod 6, \end{cases}$$

$v_5(F_n) = v_5(n)$, $v_5(L_n) = 0$, *and if* p *is a prime,* $p \neq 2$, *and* $p \neq 5$, *then*

$$v_p(F_n) = \begin{cases} v_p(n) + v_p(F_{z(p)}), & \text{if } n \equiv 0 \pmod{z(p)}; \\ 0, & \text{otherwise.} \end{cases}$$

$$v_p(L_n) = \begin{cases} v_p(n) + v_p(F_{z(p)}), & \text{if } z(p) \text{ is even and } n \equiv \frac{z(p)}{2} \pmod{z(p)}; \\ 0, & \text{otherwise.} \end{cases}$$

Next, we give the formulas for the order of appearance of F_n^k and L_n^k given by Marques [10] and Pongsriiam [15], respectively.

Lemma 2.2. (Marques[10]) *Let* n *be a positive integer.*
(i) *If* $n \equiv 3 \pmod 6$, *then* $z(F_n^2) = nF_n$ *and* $z(F_n^{k+1}) = n\frac{F_n^k}{2}$, *for* $k \geq 2$.
(ii) *Let* k *and* $n \geq 3$ *be integers, with* $k \geq 0$ *and* $n \not\equiv 3 \pmod 6$, *then* $z(F_n^{k+1}) = nF_n^k$.

Lemma 2.3. (Pongsriiam[15]) *Let* $n \geq 2$. *Then, the following statements hold.*
(i) $z(L_n) = 2n$.
(ii) *If* $k \geq 2$ *and* $n \equiv 1,2 \pmod 3$, *then* $z(L_n^k) = 2nL_n^{k-1}$.
(iii) *If* $k \geq 2$ *and* $n \equiv 3 \pmod 6$, *then* $z(L_n^k) = nL_n^{k-1}$.
(iv) *If* $k \geq 2$ *and* $n \equiv 0 \pmod 6$, *then*

$$z(L_n^k) = \begin{cases} \frac{nL_n^{k-1}}{2^{v_2(n)+1}}, & \text{if } k \geq v_2(n)+3; \\ \frac{nL_n^{k-1}}{2^{k-2}}, & \text{if } k < v_2(n)+3. \end{cases}$$

Lemma 2.4. *For each* n *and* m, *we have* $n | F_m$ *if and only if* $z(n) | m$.

Proof. This is a well-known result. For the proof, see for example Halton [4, Lemma 8, p. 222]. Note that Halton [4] used $\alpha(n)$ instead of $z(n)$ to denote the order of appearance of n and called it by the old name: the rank of apparition. □

The next result is easy but we use it often in the proof of the main theorems, so we state it as a lemma.

Lemma 2.5. *Let n and m be positive integers. Then, the following statements are equivalent.*
(i) $n \mid m$.
(ii) $v_p(n) \le v_p(m)$ *for every prime* p.
(iii) $v_p(n) \le v_p(m)$ *for every prime* p *dividing* n.

Proof. This result is also well-known and easy to prove. So, we leave the details to the reader. □

In the proof of our main theorems, we apply Lemma 2.5 without further reference.

3. Main Results

We begin with the converse of the divisibility relation (1.1).

Theorem 3.1. *Let k, m, and n be positive integers and $n \ge 3$. Then, the following statements hold.*

(i) *If* $F_n^{k+1} \mid F_{nm}$ *and* $n \not\equiv 3 \pmod 6$, *then* $F_n^k \mid m$.
(ii) *If* $F_n^{k+1} \mid F_{nm}$ *and* $n \equiv 3 \pmod 6$, *then* $F_n^k \mid 2m$ *and* $F_n^{k-1} \mid m$.
(iii) *If* $F_n^{k+1} \mid F_{nm}$, $n \equiv 3 \pmod 6$, *and* $2^k|m$, *then* $F_n^k \mid m$.

Proof. We use Lemma 2.2 and Lemma 2.4 to prove this theorem. Assume that $F_n^{k+1} \mid F_{nm}$. If $n \not\equiv 3 \pmod 6$, then $nF_n^k = z(F_n^{k+1})$ and $z(F_n^{k+1})|nm$, which implies $F_n^k|m$. This proves (i). So assume that $n \equiv 3 \pmod 6$. If $k = 1$, then the calculation is the same as the case $n \not\equiv 3 \pmod 6$ and we obtain $F_n|m$, which implies the desired result in both (ii) and (iii). So suppose $k \ge 2$. Then, we obtain $n\frac{F_n^k}{2} = z(F_n^{k+1})$ and $z(F_n^{k+1})|nm$, which implies $F_n^k|2m$. By Lemma 2.1, we know that $2|F_n$ and therefore,

$$F_n^{k-1}|F_n^{k-1}(F_n/2) = \frac{1}{2}F_n^k \text{ and } \frac{1}{2}F_n^k|m. \tag{3.1}$$

This proves (ii). Next suppose that $2^k|m$. Since $F_n^k|2m$, we have $\frac{F_n^k}{2^k}|\frac{m}{2^{k-1}}$. Since $k \ge 2$, we obtain $F_n^k|\frac{2F_n^k}{2^k}|\frac{2m}{2^{k-1}}|m$, as required. □

Theorem 3.2. *Let k, m, and n be positive integers and $n \ge 3$. Then, the following statements hold.*

(i) *If* $F_n^{k+1} \parallel F_{nm}$ *and* $n \not\equiv 3 \pmod 6$, *then* $F_n^k \parallel m$.
(ii) *If* $F_n^{k+1} \parallel F_{nm}$, $n \equiv 3 \pmod 6$, *and* $2^k| m$, *then* $F_n^k \parallel m$.
(iii) *If* $F_n^{k+1} \parallel F_{nm}$, $n \equiv 3 \pmod 6$, *and* $2^k \nmid m$, *then* $F_n^{k-1} \parallel m$.

Proof. Assume that $F_n^{k+1} \parallel F_{nm}$. We divide the proof into three cases as follows.

Case 1. $n \not\equiv 3 \pmod 6$. Then by Theorem 3.1, we obtain $F_n^k \mid m$. If $F_n^{k+1} \mid m$, then $F_n^\ell \| m$ for some $\ell \ge k+1$ and we obtain by Theorem 1.1 that $F_n^{k+2}|F_n^{\ell+1} \mid F_{nm}$, which contradicts the assumption that $F_n^{k+1} \parallel F_{nm}$. Therefore, $F_n^k \parallel m$ and (i) is proved.

Case 2. $n \equiv 3 \pmod 6$ and $2^k \mid m$. By Theorem 3.1(iii), $F_n^k \mid m$. If $F_n^{k+1} \mid m$, then we obtain by Theorem 1.1 that $F_n^{k+2} \mid F_{nm}$, which contradicts the assumption that $F_n^{k+1} \parallel F_{nm}$. So $F_n^k \| m$.

Case 3. $n \equiv 3 \pmod 6$ and $2^k \nmid m$. By Theorem 3.1(ii), $F_n^{k-1} \mid m$. If $F_n^k \mid m$, then $v_2(m) \geq v_2(F_n^k) = k$, which contradicts $2^k \nmid m$. So, $F_n^{k-1} \parallel m$, as required. □

Theorem 3.3. *Let k, m, and n be positive integers and $n \geq 2$. Then, the following statements hold.*

(i) *If $L_n^{k+1} \mid L_{nm}$, then $n \not\equiv 0 \pmod 3$, m is odd, and $L_n^k \mid m$.*

(ii) *If $L_n^{k+1} \parallel L_{nm}$, then $L_n^k \parallel m$.*

Proof. Assume that $L_n^{k+1} \mid L_{nm}$. We first show that $n \not\equiv 0 \pmod 3$. If $n \equiv 3 \pmod 6$, then we obtain by Lemma 2.1 that $2(k+1) = v_2(L_n^{k+1}) \leq v_2(L_{nm}) \leq 2$, which is a contradiction. Similarly, if $n \equiv 0 \pmod 6$, then $k+1 = v_2(L_n^{k+1}) \leq v_2(L_{nm}) = 1$, which is not the case. Hence, $n \not\equiv 0, 3 \pmod 6$ and so $n \not\equiv 0 \pmod 3$. Next, we show that m is odd. So, suppose for a contradiction that m is even. By the primitive divisor theorem of Carmichael [3], for each $n \notin \{1, 3\}$, there exists a prime $p \notin \{2, 5\}$ such that $p | L_n$. By Lemma 2.1, $z(p)$ is even and $n \equiv \frac{z(p)}{2} \pmod{z(p)}$. Then $nm \equiv 0 \pmod{z(p)}$ so by Lemma 2.1, $v_p(L_{nm}) = 0$, which contradicts the fact that $p | L_n$ and $L_n | L_{nm}$. Hence, m is odd. Recall the well-known identity that $F_{2j} = F_j L_j$ for every $j \geq 1$. Then $L_{nm} | F_{2nm}$ and so $L_n^{k+1} | F_{2nm}$. By Lemmas 2.4 and 2.3, we obtain $2nL_n^k = z\left(L_n^{k+1}\right)$ and $z\left(L_n^{k+1}\right) | 2nm$, which implies $L_n^k | m$, as required.

Next, we prove (ii). Assume that $L_n^{k+1} \parallel L_{nm}$. Then by (i), m is odd and $L_n^k | m$. If $L_n^{k+1} \mid m$, then by Theorem 1.2, $L_n^{k+2} \mid L_{nm}$, a contradiction. So, $L_n^k \parallel m$. □

It may be possible to prove the next theorem by applying Lemma 2.3, but it seems simpler to calculate straightforwardly using Lemma 2.1.

Theorem 3.4. *Let k, m, and n be positive integers and $n \geq 2$. If $L_n^{k+1} | F_{nm}$, then m is even. Moreover, the following statements hold.*

(i) *If $L_n^{k+1} \mid F_{nm}$ and $n \not\equiv 0 \pmod 6$, then $L_n^k \mid m$.*

(ii) *If $L_n^{k+1} \parallel F_{nm}$ and $n \not\equiv 0 \pmod 6$, then $L_n^k \parallel m$.*

(iii) *If $L_n^{k+1} \mid F_{nm}$ and $n \equiv 0 \pmod 6$, then $L_n^{min\{v_2(m), k\}} \mid m$.*

(iv) *If $L_n^{k+1} \parallel F_{nm}$ and $n \equiv 0 \pmod 6$, then $L_n^{min\{v_2(m), k\}} \parallel m$.*

Proof. Assume that $L_n^{k+1} \mid F_{nm}$. Then, for each prime p dividing L_n, $v_p(L_n^{k+1}) \leq v_p(F_{nm})$. By Lemma 2.1, we obtain the following inequalities:

$$v_5(L_n^k) = 0 \leq v_p(m)$$

and for every prime $p \notin \{2, 5\}$ and $p \mid L_n$, we have

$$\begin{aligned} v_p(n) + v_p(F_{z(p)}) + v_p(L_n^k) = v_p(L_n^{k+1}) &\leq v_p(F_{nm}) \\ &\leq v_p(nm) + v_p(F_{z(p)}) \\ &= v_p(n) + v_p(m) + v_p(F_{z(p)}). \end{aligned}$$

The above inequalities imply that

$$v_p(L_n^k) \leq v_p(m) \text{ for each prime } p \neq 2. \tag{3.2}$$

Note that (3.2) holds whenever $L_n^{k+1} \mid F_{nm}$. Suppose $n \neq 3$. Then, by the primitive divisor theorem of Carmichael [3], there exists a prime $p \neq 2$ such that $p | L_n$. Then, by Lemma 2.1, $n \equiv \frac{z(p)}{2} \pmod{z(p)}$. Since $p | L_n$, we have $p | F_{nm}$. So, $nm \equiv 0 \pmod{z(p)}$. From this and the congruence $n \equiv \frac{z(p)}{2} \pmod{z(p)}$, we see that m is even. If $n = 3$ and m is odd, then $1 \geq v_2(F_{3m}) = v_2(F_{nm}) \geq v_2(L_n^{k+1}) = v_2(4^{k+1}) = 2k+2$, which is not the case. So in any case, m is even. This proves the first part of this theorem.

From this point on, we assume that m is even and we also use Lemma 2.1 without further reference. To prove (i), assume that $n \not\equiv 0 \pmod 6$. If $n \equiv 1, 2, 4, 5 \pmod 6$, then $v_2(L_n^k) = 0 \le v_2(m)$. If $n \equiv 3 \pmod 6$, then we obtain

$$v_2(L_n^k) + 2 = v_2(L_n^{k+1}) \le v_2(F_{nm}) = v_2(nm) + 2 = v_2(m) + 2,$$

which implies $v_2(L_n^k) \le v_2(m)$. So in any case, $v_2(L_n^k) \le v_2(m)$. From this and (3.2), we conclude that $v_p(L_n^k) \le v_p(m)$ for each prime p dividing L_n. Therefore, $L_n^k \mid m$.

Next, we prove (ii). Assume that $L_n^{k+1} \parallel F_{nm}$ and $n \not\equiv 0 \pmod 6$. By (i), $L_n^k \mid m$. If $L_n^{k+1} \mid m$, then by Theorem 1.3(i), $L_n^{k+2} \mid F_{nm}$, a contradiction. So $L_n^k \| m$.

Next, we prove (iii). Assume that $L_n^{k+1} \mid F_{nm}$ and $n \equiv 0 \pmod 6$. If $v_2(m) \ge k$, then

$$v_2(L_n^{min\{v_2(m),k\}}) = v_2(L_n^k) = k \le v_2(m).$$

If $v_2(m) < k$, then

$$v_2(L_n^{min\{v_2(m),k\}}) = v_2(L_n^{v_2(m)}) = v_2(m).$$

In any case, $v_2(L_n^{min\{v_2(m),k\}}) \le v_2(m)$. By (3.2), we also obtain $v_p(L_n^{min\{v_2(m),k\}}) \le v_p(L_n^k) \le v_p(m)$ for every $p \ne 2$. Therefore, $L_n^{min\{v_2(m),k\}} \mid m$.

Next we prove (iv). Assume that $L_n^{k+1} \parallel F_{nm}$ and $n \equiv 0 \pmod 6$. By (iii), $L_n^{min\{v_2(m),k\}} \mid m$. Suppose for a contradiction that $L_n^{min\{v_2(m),k\}+1} \mid m$. If $v_2(m) \ge k$, then $L_n^{k+1} \mid m$, and we obtain by Theorem 1.3 that $L_n^{k+2} \mid F_{nm}$, which is not the case. If $v_2(m) < k$, then $L_n^{v_2(m)+1} \mid m$, and so $v_2(m) \ge v_2(L_n^{v_2(m)+1}) = v_2(m) + 1$, a contradiction. This completes the proof. $\square$

Theorems 3.1 to 3.4 can be stated in a different form. For example, suppose $F_n^{k+1} | F_\ell$. Then, $F_n | F_\ell$ and so $n | \ell$. Therefore, we can write $\ell = nm$ for some $m \in \mathbb{N}$, and Theorem 3.1 can be changed to the following.

Corollary 3.5. *Let k, ℓ, and n be positive integers and $n \ge 3$. Then, the following statements hold.*

(i) *If $F_n^{k+1} | F_\ell$ and $n \not\equiv 3 \pmod 6$, then $F_n^k \mid \frac{\ell}{n}$,*
(ii) *If $F_n^{k+1} | F_\ell$ and $n \equiv 3 \pmod 6$, then $F_n^k \mid 2\frac{\ell}{n}$ and $F_n^{k-1} \mid \frac{\ell}{n}$,*
(iii) *If $F_n^{k+1} | F_\ell$, $n \equiv 3 \pmod 6$, and $2^k | \frac{\ell}{n}$, then $F_n^k \mid \frac{\ell}{n}$.*

Similarly, another version of Theorem 3.2 is as follows.

Corollary 3.6. *Let k, m, and n be positive integers and $n \ge 3$. Then, the following statements hold.*

(i) *If $F_n^{k+1} \parallel F_m$ and $n \not\equiv 3 \pmod 6$, then $F_n^k \parallel \frac{m}{n}$.*
(ii) *If $F_n^{k+1} \parallel F_m$, $n \equiv 3 \pmod 6$, and $2^k | \frac{m}{n}$, then $F_n^k \parallel \frac{m}{n}$.*
(iii) *If $F_n^{k+1} \parallel F_m$, $n \equiv 3 \pmod 6$, and $2^k \nmid \frac{m}{n}$, then $F_n^{k-1} \parallel \frac{m}{n}$.*

Theorems 3.3 and 3.4 can also be given in another form. We leave the details to the reader.

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