A SIMPLE PROOF OF AN IDENTITY GENERALIZING FIBONACCI-LUCAS IDENTITIES

ANDREAS N. PHILIPPOU AND SPIROS D. DAFNIS

Abstract. Let $F_n^{(k)} = 0$ for $-k+1 \le n \le 0$, $F_1^{(k)} = 1$, and $F_n^{(k)} = \sum_{j=1}^{k} F_{n-j}^{(k)}$ for $n \ge 2$. Also let $L_0^{(k)} = k$, $L_1^{(k)} = 1$, $L_n^{(k)} = n + \sum_{j=1}^{n-1} L_{n-j}^{(k)}$ for $2 \le n \le k$, and $L_n^{(k)} = \sum_{j=1}^{k} L_{n-j}^{(k)}$ for $n \ge k+1$. The identity $\sum_{i=0}^{n} m^i \Big(\big(L_i^{(k)} + (m-2)F_{i+1}^{(k)} - \sum_{j=3}^{k} (j-2) F_{i-j+1}^{(k)} \big) \Big) = m^{n+1} F_{n+1}^{(k)} + k - 2$ ($m \ge 2, k \ge 2$), derived recently by means of colored tiling [4], is presently proved using only the definitions of $F_n^{(k)}$ and $L_n^{(k)}$, and the identity $L_n^{(k)} = \sum_{j=1}^{k} j F_{n-j+1}^{(k)}$ ($n \ge 1$).

1. Introduction and Summary

Let $m \ge 2$ be a fixed positive integer, and let n be a nonnegative integer, unless otherwise specified. Denote by F_n and L_n the Fibonacci and Lucas numbers, respectively, i.e., $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$ ($n \ge 2$) and $L_0 = 2$, $L_1 = 1$, $L_n = L_{n-1} + L_{n-2}$ ($n \ge 2$). The first, the second, and the third of the following well-known Fibonacci-Lucas identities

$$\sum_{i=0}^{n} 2^i L_i = 2^{n+1} F_{n+1}, \ \sum_{i=0}^{n} 3^i (L_i + F_{i+1}) = 3^{n+1} F_{n+1},$$
$$\sum_{i=0}^{n} m^i (L_i + (m-2) F_{i+1}) = m^{n+1} F_{n+1}, \tag{1.1}$$

are due to Benjamin and Quinn [1, 2], Marques [8] and Edgar [5], respectively. See also Sury [12] and Kwong [7] for the first and Martinjak [9] for the second.

Let $k \ge 2$ be a fixed positive integer. Dafnis, Philippou, and Livieris [4] generalized the above identities to the Fibonacci and Lucas numbers of order k, deriving the following theorem by means of color tiling.

Theorem 1. *Let $(F_n^{(k)})_{n \ge 0}$ be the sequence of Fibonacci numbers of order k [9], and set $F_{-1}^{(k)} = \cdots F_{-k+1}^{(k)} = 0$, i.e., $F_n^{(k)} = 0$ for $-k+1 \le n \le 0$, $F_1^{(k)} = 1$, and $F_n^{(k)} = \sum_{j=1}^{k} F_{n-j}^{(k)}$ for $n \ge 2$. Also let $(L_n^{(k)})_{n \ge 0}$, be the sequence of Lucas numbers of order k [3], i.e., $L_0^{(k)} = k$, $L_1^{(k)} = 1$, $L_n^{(k)} = n + \sum_{j=1}^{n-1} L_{n-j}^{(k)}$ for $2 \le n \le k$, and $L_n^{(k)} = \sum_{j=1}^{k} L_{n-j}^{(k)}$ for $n \ge k+1$. Then,*

$$\sum_{i=0}^{n} m^i \Big(\big(L_i^{(k)} + (m-2) F_{i+1}^{(k)} - \sum_{j=3}^{k} (j-2) F_{i-j+1}^{(k)} \big) \Big) = m^{n+1} F_{n+1}^{(k)} + k - 2,$$

where $\sum_{j=a}^{b} g(j) = 0$ if $a > b$.

2. New Proof of Theorem 1

We presently give a new proof of Theorem 1, using only the definitions of $(F_n^{(k)})_{n\geq -k+1}$ and $(L_n^{(k)})_{n\geq 0}$, and the relation $L_n^{(k)} = \sum_{j=1}^{k} jF_{n-j+1}^{(k)}$, $n\geq 1$, which readily follows from (2.18) of Charalambides [3].

Proof. Using $L_0^{(k)} = k$, $F_1^{(k)} = 1$, and adding and subtracting $F_i^{(k)}$ in the parenthesis, we have

$$\sum_{i=0}^{n} m^i \Big(L_i^{(k)} + (m-2)F_{i+1}^{(k)} - \sum_{j=3}^{k} (j-2)F_{i-j+1}^{(k)} \Big) \tag{2.1}$$

$$= k + m - 2 + \sum_{i=1}^{n} m^i \Big(L_i^{(k)} + (m-2)F_{i+1}^{(k)} - F_i^{(k)} - \sum_{j=1}^{k} (j-2)F_{i-j+1}^{(k)} \Big).$$

Next, using $F_{i+1}^{(k)} = \sum_{j=1}^{k} F_{i-j+1}^{(k)}$ for $i\geq 1$, which hold true by definition, and $L_i^{(k)} = \sum_{j=1}^{k} jF_{i-j+1}^{(k)}$ for $i\geq 1$ [3], we get

$$\sum_{j=1}^{k} (j-2)F_{i-j+1}^{(k)} = \sum_{j=1}^{k} jF_{i-j+1}^{(k)} - 2\sum_{j=1}^{k} F_{i-j+1}^{(k)} = L_i^{(k)} - 2F_{i+1}^{(k)},$$

which implies

$$k + m - 2 + \sum_{i=1}^{n} m^i \Big(L_i^{(k)} + (m-2)F_{i+1}^{(k)} - F_i^{(k)} - \sum_{j=1}^{k} (j-2)F_{i-j+1}^{(k)} \Big)$$

$$= k + m - 2 + \sum_{i=1}^{n} m^i (mF_{i+1}^{(k)} - F_i^{(k)}) \tag{2.2}$$

$$= k + m - 2 + m^{n+1}F_{n+1}^{(k)} - mF_1^{(k)} = m^{n+1}F_{n+1}^{(k)} + k - 2.$$

Relations (2.1) and (2.2) establish the theorem. □

The following obvious corollary to the theorem is the analogue of (1.1) for the Lucas numbers of order 3 (or 3-step Lucas numbers) and the Tribonacci numbers.

Corollary 2. *Let $(T_n)_{n\geq 0}$ be the sequence of Tribonacci numbers* [6, 9] *i.e., $T_0 = 0$, $T_1 = 1$, and $T_n = T_{n-1} + T_{n-2} + T_{n-3}$ for $n\geq 3$. Also let $(V_n)_{n\geq 0}$ be the sequence of Lucas numbers of order* 3 [3] *(or 3-step Lucas numbers* [11], *A001644), i.e., $V_0 = 3$, $V_1 = 1$, $V_2 = 3$, and $V_n = V_{n-1} + V_{n-2} + V_{n-3}$ for $n\geq 3$. Set $T_{-2} = T_{-1} = 0$. Then,*

$$\sum_{i=0}^{n} 2^i (V_i - T_{i-2}) = 2^{n+1}T_{n+1} + 1, \ \sum_{i=0}^{n} 3^i (V_i + T_{i+1} - T_{i-2}) = 3^{n+1}T_{n+1} + 1,$$

$$\sum_{i=0}^{n} m^i (V_i + (m-2)T_{i+1} - T_{i-2}) = m^{n+1}T_{n+1} + 1.$$

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Department of Mathematics, University of Patras, 265-00 Patras, Greece
E-mail address: `anphilip@math.upatras.gr`

Department of Mathematics, University of Patras, 265-00 Patras, Greece
E-mail address: `dafnisspyros@gmail.com`