REPDIGITS AS PRODUCTS OF CONSECUTIVE BALANCING OR LUCAS-BALANCING NUMBERS

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Abstract. Repdigits are natural numbers formed by the repetition of a single digit. In this paper, we explore the presence of repdigits in the product of consecutive-balancing or Lucas-balancing numbers.

1. Introduction

The balancing sequence $\{B_n : n \geq 0\}$ and the Lucas-balancing sequence $\{C_n : n \geq 0\}$ are solutions of the binary recurrence $x_{n+1} = 6x_n - x_{n-1}$ with initial terms $B_0 = 0$, $B_1 = 1$, and $C_0 = 1$, $C_1 = 3$, respectively. The balancing sequence is a variant of the sequence of natural numbers since natural numbers are solutions of the binary recurrence $x_{n+1} = 2x_n - x_{n-1}$ with initial terms $x_0 = 0$ and $x_1 = 1$. The balancing numbers have certain properties identical with those of natural numbers [9]. It is important to note that the balancing sequence is a strong divisibility sequence, that is, $B_m \mid B_n$ if and only if $m \mid n$ [6].

In 2004, Liptai [2] searched for Fibonacci numbers in the balancing sequence and observed that 1 is the only number of this type. In a recent paper [5], the second author proved that there is no perfect square in the balancing sequence other than 1. Subsequently, Panda and Davala [7] verified that 6 is the only balancing number that is also a perfect number.

For a given integer $g > 1$, a number of the form $N = a\left(\frac{g^m-1}{g-1}\right)$ for some $m \geq 1$, where $a \in \{1, 2, \ldots, g-1\}$ is called a repdigit with respect to base g or g-repdigit. For $g = 10$, N is called a repdigit and if, in addition, $a = 1$, then N is called a repunit. Luca [3] identified the repdigits in Fibonacci and Lucas sequences. Subsequently, Faye and Luca [1] explored all repdigits in Pell and Pell-Lucas sequences. Marques and Togbé [4] searched for the repdigits that are products of consecutive Fibonacci numbers. In this paper, we search for repdigits in the balancing and Lucas-balancing sequences. In addition, we explore repdigits that are products of consecutive-balancing or Lucas-balancing numbers.

2. Main Results

In this section, we prove some theorems assuring the absence of certain classes of repdigits in the balancing and Lucas-balancing sequences. As generalizations, we also show that the product of consecutive-balancing or Lucas-balancing numbers is never a repdigit with more than one digit.

In the balancing sequence, the first two balancing numbers $B_1 = 1$ and $B_2 = 6$ are repdigits. We have checked the next 200 balancing numbers, but none is a repdigit. The following theorem excludes the presence of some specific types of repdigits in the balancing sequence.

Theorem 2.1. *If m, n, and a are natural numbers, $m \geq 2$, $a \neq 6$, and $1 \leq a \leq 9$, then the Diophantine equation*

$$B_n = a\Big(\frac{10^m - 1}{9}\Big) \tag{2.1}$$

has no solution.

Proof. To prove this theorem, we need all the least residues of the balancing sequence modulo 3, 4, 5, 7, 8, 9, 11, and 20 (see [8]). We list them in the following table.

Row no.	m	B_n mod m	Period
1	3	$0, 1, 0, 2$	4
2	4	$0, 1, 2, 3$	4
3	5	$0, 1, 1, 0, 4, 4$	6
4	7	$0, 1, 6$	3
5	8	$0, 1, 6, 3, 4, 5, 2, 7$	8
6	9	$0, 1, 6, 8, 6, 1, 0, 8, 3, 1, 3, 8$	12
7	11	$0, 1, 6, 2, 6, 1, 0, 10, 5, 9, 5, 10$	12
8	20	$0, 1, 6, 15, 4, 9, 10, 11, 16, 5, 14, 19$	12

Table 1

Since $m \geq 2$, it follows that $n \geq 3$. We claim that m is odd. Observe that if m is even, then

$$11 \ \Big|\ \frac{10^m - 1}{9} \ \Big|\ B_n$$

and from the seventh row of Table 1, it follows that $6 \mid n$ and consequently $B_6 \mid B_n$. Since $10 \mid B_6$, it follows that $10 \mid B_n = a \cdot \frac{10^m-1}{9}$, which is a contradiction. Now, to complete the proof, we distinguish eight different cases corresponding to the values of a.

Case 1: $a = 1$. Assume that B_n is of the form $\frac{10^m-1}{9}$ for some m. Since m is odd, $B_n \equiv 1 \pmod{11}$ and also $B_n \equiv 11 \pmod{20}$. From the last row of Table 1, it follows that if $B_n \equiv 11 \pmod{20}$, then $n \equiv 7 \pmod{12}$. But, from the seventh row of Table 1, it follows that whenever $n \equiv 7 \pmod{12}$, $B_n \equiv 10 \pmod{11}$, a contradiction to $B_n \equiv 1 \pmod{11}$. Hence, no B_n is of the form $\frac{10^m-1}{9}$.

Case 2: $a = 2$. If $B_n = 2 \cdot \frac{10^m-1}{9}$, then $B_n \equiv 2 \pmod 5$. But, in view of the third row of Table 1, it follows that for no value of n, $B_n \equiv 2 \pmod 5$. Hence, B_n cannot be of the form $2 \cdot \frac{10^m-1}{9}$.

Case 3: $a = 3$. If $B_n = 3 \cdot \frac{10^m-1}{9}$, then $B_n \equiv 0 \pmod 3$. But, in view of the first row of Table 1, $n \equiv 0, 2 \pmod 4$. So, $B_2 \mid B_n$, and consequently, $2 \mid \frac{10^m-1}{9}$, which is a contradiction. Hence, B_n cannot be of the form $3 \cdot \frac{10^m-1}{9}$.

Case 4: $a = 4$. If $B_n = 4 \cdot \frac{10^m-1}{9}$, then $B_n \equiv 0 \pmod 4$ and, in view of the second row of Table 1, $4 \mid n$, which implies $B_4 \mid B_n$. Since $17 \mid B_4$, it follows that $17 \mid (10^m - 1)$. But this is possible if $16 \mid m$, which is a contradiction, since m is odd. Hence, B_n cannot be of the form $B_n = 4 \cdot \frac{10^m-1}{9}$.

Case 5: $a = 5$. If $B_n = 5 \cdot \frac{10^m-1}{9}$, then $B_n \equiv 0 \pmod 5$ and in view of the third row of Table 1, this is possible only if $3 \mid n$. Hence, $B_3 \mid B_n$ and since $7 \mid B_3$, it follows that $7 \mid \frac{10^m-1}{9}$, which implies that $6 \mid m$, a contradiction since m is odd. Hence, B_n cannot be of the form $B_n = 5 \cdot \frac{10^m-1}{9}$.

Case 6: $a = 7$. If $B_n = 7 \cdot \frac{10^m-1}{9}$, then $B_n \equiv 0 \pmod 7$ and, in view of the fourth row of Table 1, this is possible only if $3 \mid n$. Hence, $B_3 \mid B_n$ and since $5 \mid B_3$, it follows that $5 \mid \frac{10^m-1}{9}$, which is a contradiction. Hence, B_n cannot be of the form $B_n = 7 \cdot \frac{10^m-1}{9}$.

Case 7: $a = 8$. If $B_n = 8 \cdot \frac{10^m-1}{9}$, then $B_n \equiv 0 \pmod 8$, and in view of the fifth row of Table 1, this is possible only if $8 \mid n$. Hence, $B_8 \mid B_n$ and since $17 \mid B_8$, it follows that $17 \mid (10^m - 1)$. But this is possible if $16 \mid m$, which is a contradiction, since m is odd. Hence, B_n cannot be of the form $B_n = 8 \cdot \frac{10^m-1}{9}$.

Case 8: $a = 9$. If $B_n = 9 \cdot \frac{10^m-1}{9}$, then $B_n \equiv 0 \pmod 9$ and, in view of the sixth row of Table 1, this is possible only if $6 \mid n$. Consequently, $B_6 \mid B_n$ and since $11 \mid B_6$, it follows that $11 \mid \frac{10^m-1}{9}$. But this is possible only if m is even, which is a contradiction since m is odd. Hence, B_n cannot be of the form $B_n = 9 \cdot \frac{10^m-1}{9}$.

Thus, (2.1) has no solution if $m \geq 2$ and $a \neq 6$. This completes the proof. □

Next, we study the presence of repdigits in the products of consecutive balancing numbers. The product $B_1B_2 = 6$ is a repdigit. So, a natural question is: "Is there any other repdigit that is a consecutive product of balancing numbers?" In the following theorem, we answer this question in negative.

Theorem 2.2. *If m, n, k, and a are natural numbers such that $m > 1$ and $1 \leq a \leq 9$, then the Diophantine equation*

$$B_nB_{n+1}\cdots B_{n+k} = a\Big(\frac{10^m-1}{9}\Big) \tag{2.2}$$

has no solution.

Proof. First, we show that (2.2) has no solution for $k \geq 2$. Assume, to the contrary that (2.2) has a solution in positive integers n, m, and a for $k \geq 2$. Then, $2 \mid (n+i)$ and $3 \mid (n+j)$ for some $i, j \in \{0, 1, \ldots, k\}$. Since $2 \mid B_2$ and $5 \mid B_3$, it follows that $2 \mid B_{n+i}$ and $5 \mid B_{n+j}$. Hence, $10 \mid B_nB_{n+1}\cdots B_{n+k} = a\Big(\frac{10^m-1}{9}\Big)$, which is a contradiction. Hence, (2.2) has no solution for $k \geq 2$.

Next, we show that (2.2) has no solution if $k = 1$. If $k = 1$, (2.2) reduces to

$$B_nB_{n+1} = a\Big(\frac{10^m-1}{9}\Big).$$

One of n and $n + 1$ is even and consequently, either B_n or B_{n+1} is also even. Hence, $a \in \{2, 4, 6, 8\}$. Since $m > 1$, $B_nB_{n+1} \geq 11$ and hence, n must be greater than 1.

In the following table, we list all the least residues of B_nB_{n+1} modulo 5 and 100, which will be useful in the proof.

If $a = 2$ or $a = 4$, then

$$B_nB_{n+1} = a \cdot \frac{10^m-1}{9} \equiv a \pmod 5.$$

m	B_nB_{n+1} mod m	Period
5	$0, 1, 0$	3
100	$0, 6, 10, 40, 56, 70, 30, 56, 80, 70, 6, 40, 60, 6, 50, 0, 56, 10, 90, 56, 20, 30, 6, 80, 20, 6, 90, 60, 56, 50, 50, 56, 60, 90, 6, 20, 80, 6, 30, 20, 56, 90, 10, 56, 0, 50, 6, 60, 40, 6, 70, 80, 56, 30, 70, 56, 40, 10, 6, 0$	60

TABLE 2

If $a = 8$, then

$$B_nB_{n+1} = 8 \cdot \frac{10^m - 1}{9} \equiv 3 \pmod{5}.$$

Similarly, if $a = 6$, then

$$B_nB_{n+1} = 6 \cdot \frac{10^m - 1}{9} \equiv 66 \pmod{100}.$$

Since the least residues of the last three congruences do not appear in the appropriate row of Table 2, it follows that B_nB_{n+1} is not a repdigit if $n > 1$. This completes the proof. □

In Theorem 2.1, we proved the absence of certain types of repdigits in the sequence of balancing numbers. However, in the case of Lucas-balancing numbers, $C_1 = 3$ and $C_3 = 99$ are two known repdigits. Thus, a natural question is: "Does this sequence contain any other larger repdigit?" In the following theorem, we answer this question in negative.

Theorem 2.3. *If m, n, and a are natural numbers and $1 \leq a \leq 9$, then the Diophantine equation*

$$C_n = a\Big(\frac{10^m - 1}{9}\Big) \tag{2.3}$$

has the only solutions $(m, n, a) = (1, 1, 3),\ (2, 3, 9)$.

Proof. To prove this theorem, we need all the least residues of the Lucas-balancing sequence modulo 5, 7, and 8. We list them in the following table.

Row no.	m	C_n mod m	Period
1	5	$1, 3, 2, 4, 2, 3$	6
2	7	$1, 3, 3$	3
3	8	$1, 3$	2

TABLE 3

Among the first three Lucas-balancing numbers, $C_1 = 3$ and $C_3 = 99$ are repdigits and (2.3) is satisfied for $(m, n, a) = (1, 1, 3),\ (2, 3, 9)$. Now, let $n \geq 4$ and hence, $m \geq 3$. Since C_n is always odd, $a \in \{1, 3, 5, 7, 9\}$. Since no zero appears in the first two rows of Table 3, it follows that C_n is not divisible by 5 or 7 and hence, the possible values of a are limited to 1, 3, and 9.

If $a \in \{1, 9\}$, then

$$C_n = a \cdot \frac{10^m - 1}{9} \equiv 10^m - 1 \equiv 7 \pmod{8}.$$

Similarly, if $a = 3$, then

$$C_n = 3 \cdot \frac{10^m - 1}{9} \equiv 5 \pmod 8.$$

Since, the least residues 5 and 7 do not appear in the last row of Table 3, it follows that (2.3) has no solution for $n > 3$. This completes the proof. □

In Theorem 2.2, we noticed that no product of consecutive balancing numbers is a repdidit with more than one digit, although the only product $B_1B_2 = 6$ is a single digit repdigit. The following theorem negates the possibility of any repdigit as the product of consecutive Lucas-balancing numbers.

Theorem 2.4. *If m, n, k, and a are natural numbers and $1 \le a \le 9$, then the Diophantine equation*

$$C_nC_{n+1}\cdots C_{n+k} = a\left(\frac{10^m - 1}{9}\right) \tag{2.4}$$

has no solution.

Proof. All the Lucas-balancing numbers are odd and, in view of (2.4), $a \in \{1, 3, 5, 7, 9\}$. It is easy to see that (2.4) has no solution if $m = 1, 2$. In the following table, we list all the nonnegative residues of Lucas-balancing numbers and their consecutive product, modulo 5, 7, and 8, which will play an important role in proving this theorem.

m	C_n mod m	$C_nC_{n+1}\cdots C_{n+k}$ mod m
5	$1, 3, 2, 4, 2, 3$	$\in \{1, 2, 3, 4\}$
7	$1, 3, 3$	$\in \{1, 2, 3, 4, 5, 6\}$
8	$1, 3$	$\in \{1, 3\}$

TABLE 4

For $m \ge 3$, $C_nC_{n+1}\cdots C_{n+k} = a\left(\frac{10^m-1}{9}\right) \equiv 7a \pmod 8$. But from the last row of Table 4, it follows that $7a \equiv 1, 3 \pmod 8$ and hence, $a = 5$ or $a = 7$. Now, reducing (2.4) modulo a, we get $C_nC_{n+1}\cdots C_{n+k} \equiv 0 \pmod a$. Since, 0 does not appear as a residue of $C_nC_{n+1}\cdots C_{n+k}$ modulo 5 or 7, it follows that (2.4) has no solution for $m \ge 3$. This completes the proof. □

3. Conclusion

In the last section, we noticed that the Lucas-balancing sequence contains only two repdigits, namely $C_1 = 3$ and $C_3 = 99$, whereas we could not explore all repdigits in the balancing sequence. In Theorem 2.1, we proved that B_n is not a repdigit ($B_n \ne a\left(\frac{10^m-1}{9}\right)$), with more than one digit, if $a \ne 6$. Thus, repdigits in the balancing sequence having all digits 6 is yet unexplored. In this connection, one can verify that if $n \not\equiv 14 \pmod{96}$, then B_n is not a repdigit. Further, if $m \not\equiv 1 \pmod 6$, then also B_n is not a repdigit. We believe that, $B_1 = 1$ and $B_2 = 6$ are the only repdigits in the balancing sequence. It is still an open problem to prove the existence or nonexistence of repdigits that are 6 times of some repunit other than $B_2 = 6$.

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