

GEOMETRIC BRANCHING PATTERNS BASED ON P-FIBONACCI SEQUENCES: SELF-SIMILARITY ACROSS DIFFERENT DEGREES OF BRANCHING AND MULTIPLE DIMENSIONS

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ABSTRACT. Branching patterns occur throughout nature and are often described by the Fibonacci numbers. While the regularity of these branching patterns in biology can be described by the Fibonacci numbers, the branches (leaves, petals, offshoots, limbs, etc.) are often variegated (size, color, shape, etc.). To begin to understand how these patterns arise, we considered different branching patterns based on p -Fibonacci sequences. In our model, different branching patterns were created based on a specific number of decreasing-sized branches that arise from a main branch (termed the degree of branching). It was assumed that the ratio between the sizes of pairs of consecutive branches (ordered by size) equals the ratio of the largest branch size to the sum of the largest and smallest branch sizes. Generation of these branching structures illustrates that pattern self-similarities occur across different degrees of branching and multiple dimensions. Conclusion: studying geometric branching patterns based on p -Fibonacci sequences begins to show how the regularity in branching patterns might occur in biology.

1. INTRODUCTION

The lives of all multicellular organisms depend on the ability to maintain an organized branching structure of fully developed tissues. Animal tissue structure (e.g., lung, breast, intestine and vasculature) often consists of branched organized tubular systems that arise through the process of “branching morphogenesis.” This process involving the growth and branching of epithelial tubes is usually studied to understand mechanisms involved in tissue development during embryogenesis. But this branching process also functions in adult tissues to maintain highly organized branched structures that underlie tissue shape and size. The regularity of these branching patterns in biology can be described by Fibonacci sequences [1, 2, 3], but the mathematical laws and biological rules that control branching patterns have not been elucidated.

Previously, we modeled spatial and temporal asymmetries of cell division and discovered that simple mathematical laws and biological rules can explain how tissue organization is precisely maintained during tissue renewal [4]. Our model was based on biology of cell division that builds upon the concept posed in the Spears and Bicknell-Johnson model for asymmetric cell division [5]. However, these models do not explain how tissue branching and shape are dynamically maintained.

Another approach to model branching based on quantitative analyses of large-scale organ reconstructions and proliferation kinetics measurements was used in [6]. However this model is not based on Fibonacci sequences. Accordingly, we created a model for branching processes that provides a mechanism that controls branching structure of adult tissues and explains how tissue structure may be linked to the p -Fibonacci sequences. We emphasize that branching in our model occurs due to an inherited cell division mechanism, *cell maturation age*, and *degree*

of branching; it is not like crystal growth or “Lego models” where addition of “cells” occurs only at the edges of the structure. Our model describes an emergent branching structure based on maturation age, namely the number c of cycles before a new branch is formed [4, 5] and the degree of branching (d below, $d = c - 1$). Branching patterns emerge depending on a specific ratio between branch sizes as explained below, depending on the degree of branching, consonant with the fact that tissue branching is dynamically maintained in a highly ordered fashion.

In order to explain how our model design is connected to the p -Fibonacci sequences, consider positive real numbers a_0, a_1, \dots, a_d satisfying $a_0 > a_1 > \dots > a_d$ ($d \geq 1$ an integer). Let these be known as the *branch lengths*, or simply as *branches*. Assume that

$$\frac{a_1}{a_0} = \frac{a_2}{a_1} = \dots = \frac{a_d}{a_{d-1}} = \frac{a_0}{a_0 + a_d} = q, \tag{1.1}$$

where q is called the *ratio*, d is called the degree of branching (later related to the p -Fibonacci sequence, where $p = d$), and a_0 is the main branch length. Let $a_d = k a_0$, where $0 < k < 1$. Then, from (1.1) above,

$$q = \frac{1}{1 + k},$$

$0 < q < 1$, and also from (1.1), we conclude that

$$\frac{1}{q} - 1 = k = \frac{a_d}{a_0} = q^d,$$

so that

$$q^{d+1} + q - 1 = 0. \tag{1.2}$$

It is trivial to show that this polynomial has only one positive root, hence the ratio $q = q(d)$ is uniquely determined by the degree of branching d , and the d relations in (1.1). It is also easy to see that q is the unique positive root of (1.2) if and only if $\varphi = 1/q$ is the unique positive root of

$$\varphi^{d+1} - \varphi^d - 1 = 0.$$

For the case $d = 1$, φ is known as the *golden ratio*; for $d > 1$, $\varphi = \varphi(d)$ is known as the golden d -ratio (see Figure 1, where $\tau_p = \varphi(d)$).

Measuring the branches in units of the main branch (scaling selection without loss of generality), we can set $a_0 = 1$, so that all branches are given by

$$a_j = q^j, \quad j = 0, 1, 2, \dots, d. \tag{1.3}$$

The sum of the lengths of all branches is

$$\sum_{j=0}^d a_j = \frac{1 - q^{d+1}}{1 - q} a_0 = \frac{q a_0}{1 - q}.$$

The table below gives approximate values for $q = q(d)$ versus d (ratio vs. degree of branching), for $d = 1, \dots, 5$.

d	1	2	3	4	5
q	.6180339887	.6823278038	.7244919590	.7548776662	.7780895987

Consider the Fibonacci p -numbers defined by the recursive relation

$$F_p(n) = F_p(n - 1) + F_p(n - p - 1), \quad n = p + 1, p + 2, \dots,$$

with given seed values

$$F_p(0) = 0, F_p(1) = \dots = F_p(p) = 1.$$

It is well-known that

$$q(p) = \lim_{n \rightarrow \infty} \frac{F_{p-1}(n-1)}{F_{p-1}(n)}.$$

Hence, for the degree of the branching d case, the ratio $q = q(d)$ is the limit of ratios of consecutive Fibonacci p -numbers, where $p = d$.

2. MODEL DESIGN

Because branching patterns in biology can be described by Fibonacci sequences, to create a model for branching processes we first considered different geometric expressions that previously described the p -Fibonacci sequence given by the recursive relation $F_n = F_{n-1} + F_{n-p-1}$. Description of these geometric patterns dates back centuries and some of them can even be found in the Elements of Euclid. An excellent account of the history of geometric expression of the p -Fibonacci numbers is provided in the book by Stakhov and Olsen [7], pages 199–220 — their description of the generalized golden p -sections is shown in Figure 1. Generalized Binet formula, sums, combinatorial representations and generating function are presented in [8].

Based on the geometric expression of the p -Fibonacci numbers, we took a different approach to geometrically characterize the p -Fibonacci numbers by adding branches to the linear geometric structure. In our approach, different branch patterns were created based on a specific number of decreasing-sized branches that arise from a main branch (termed the degree of branching). It was assumed that the ratio between the sizes of pairs of consecutive branches (ordered by size) equals the ratio of the largest branch size to the sum of the largest and smallest branch sizes.

An example of creation of a geometric model for $p = 3$ is shown in Figure 2. It is seen that the original expression $CB/AC = (AB/CB)^3$ for the generalized golden p -sections is still retained by the expression $d/a = (a/(a+d))^3$ in our new three branched geometric structure, or $k = q^3$ with $d = 3$, of Section 1. We also created different branching structures by rearranging the location of the branches while keeping the sizes of the expanding branches the same (Figure 3). Our study builds on this geometric expression to identify patterns of self-similarities that occur across different degrees of branching and multiple dimensions (see Figure 4 and the identities that follow). This geometric structure also allowed us to create a corresponding agent-based model to further study different branching patterns and analyze the dynamics of these patterns as they emerge over time (Figure 5 and 6).

3. AGENT-BASED MODEL DESIGN FOR BRANCHING STRUCTURES

A corresponding agent-based model was created with the “agents” being the branches themselves, and each branch is considered as an agent that follows its own rules to grow (Figure 5), branches having the counting properties of age and generation. The model is simulated using Netlogo, a multi-agent programmable modeling environment. Netlogo allows for instructions to be given to hundreds or thousands of “agents” all operating independently, making it possible to explore the connection between the micro-level behavior of individual agents and the macro-level patterns that emerge from their interaction (the Netlogo code is available upon request).

Based on an asymmetric division process, branches are distinguished as immature or mature. The age of a branch is defined by the number of time steps since its initial production. A branch divides if and only if it is a mature branch — it has reached *maturation age*. The

generation of a branch is the number of divisions removed from the original meristematic (clonogenic) branch (zeroth generation branch). In the code, the position of all branches is defined by their place within an implicit tree structure that is rooted by the meristem branch's position. The c value of the meristem branch defines the pattern of the entire structure [4, 5]. The output was plotted as branching trees to illustrate the numerical evolution of the model. The tree tracks all divisions that occur at each time step. Beginning with time zero and a single meristem branch, each successive growth in the tree indicates a new time step. If a branch divides, then it is connected to both of its resulting offshoot branches. Output was also plotted as very large lineage trees at low resolution.

Both 2D and 3D models were created to explore branching rules based on p -Fibonacci sequences. The simulation starts with an immature meristem branch at the bottom of the structure. When the branch becomes mature, it will produce immature branches upwards. When these become mature, they will produce their own offspring branches in alternative directions. The same rules apply to all the mature branches in the model. By manipulating parameters, many different growth patterns can be generated.

4. BRANCHING STRUCTURES GENERATED BY AGENT-BASED MODELING

NetLogo was used to compute the resulting branches (Figure 5). In the 2D model, there was typically significant overlap of branches, a problem that did not appear in the 3D version. The 3D model followed a similar pattern to the 2D model, but allowed the structure to grow in more directions because customized angle options are available: the rotational angle and the offset angle. By manipulating parameters (such as maturation age and angles), different growth patterns were generated (Figure 6). In particular, we considered different rotational angles, including the Golden angle [9], pages 101–107, and [10]. Our model generated complex dynamic patterns in which organization of branches remained the same despite continuous growth within the branching structure. No branch overlaps was observed in our 3D models. Our findings reveal that basic branching structures can be generated in 3D models that simulate branching patterns in plants found in nature.

5. DISCUSSION

Varied kinds of trees and flowers with distinct shapes exist in nature. The design of trees, leaves and flower petals follows specific patterns [9]. The sequence that appears in nature is frequently a Fibonacci sequence. Since Fibonacci sequences appear so frequently in nature, it clearly indicates that this does not happen randomly or by accident. Indeed, it is believed that the Fibonacci sequences prompt possible rules for plant branching [7]. So the question becomes: what regulates the plant branching pattern? This work focused on creating mathematical and agent-based models to explore plant branching rules based on Fibonacci p -sequences.

Indeed, our model output on the number of branches generated over time fits specific p -Fibonacci sequences depending on the maturation time. Taking this approach, it was possible to geometrically characterize any p -Fibonacci sequence as a branching structure. Branching characteristics can be described mathematically based on branch ratio (q) that is a function of the degree of branching ($p = d$). The relations assumed in equation (1.1) and the conclusion in (1.3) served as the basis for the design of the agent-based model. Although organized patterns were conceptualized in 2D, the 3D agent-based model incorporated two additional design parameters based on the fact that plant and other tissue structure sizes are regulated by maturation age and 3D rotation of branching direction.

While understanding how branching structures normally appear in nature is an important question in its own right, another reason for our modeling branching in normal tissues is to understand how branching disorganization might occur in cancer and other diseases [11].

6. CONCLUSIONS

- (1) Branching structures are vital to lives of multicellular organisms.
- (2) Geometric branching patterns may be described by p -Fibonacci sequences.
- (3) Geometric characterization of p -Fibonacci numbers can help us understand mathematical laws and biological rules that underlie branching patterns in nature.

7. ACKNOWLEDGEMENTS

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Figure Legends

- Figure 1:** Geometric expression of the p -Fibonacci numbers.
- Figure 2:** A geometric branching pattern was created for $p = 3$ based on a specific number of decreasing-sized branches that arise from a main branch. In the model, the ratio between the sizes of pairs of consecutive branches (ordered by size) equals the ratio of the largest branch size to the sum of the largest and smallest branch sizes.
- Figure 3:** Different branching structures can also be created by rearranging the location of the branches while keeping the sizes of the expanding branches the same.
- Figure 4:** The geometric expression for $p = 3$ illustrates how to identify patterns of self-similarities that occur across different degrees of branching and multiple dimensions.
- Figure 5:** An agent-based code was created for our model with the “agents” being the branches themselves and each branch is considered as an agent that follows its own rules to grow.
- Figure 6:** Examples of branching structures generated by selecting different rotational angles (angle for short in the captions), including the Golden Angle (Figure 6(i)), and maturation age (age for short in the captions) = 6. The top views show more clearly the effect of different angles on the branching pattern.

Figures

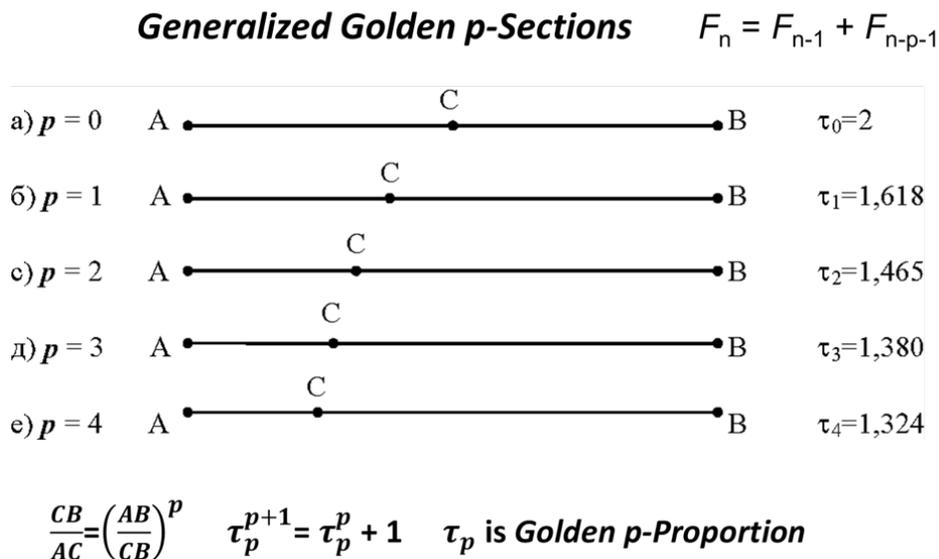
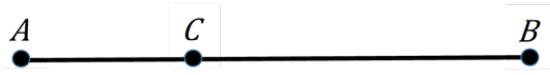
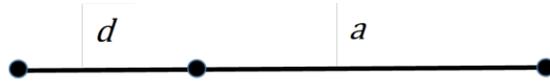


FIGURE 1. Geometric expression of the p -Fibonacci numbers [7], pages 199–220.

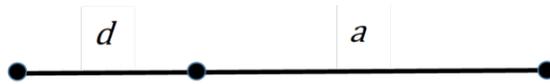
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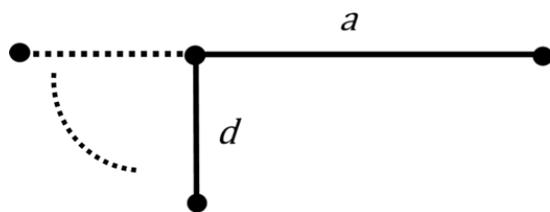
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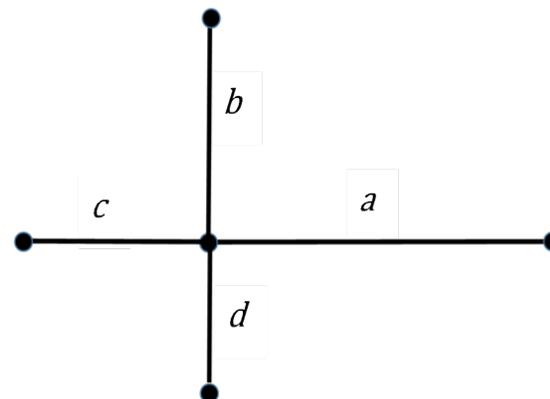
$$\frac{a}{d} = \left(\frac{a+d}{a}\right)^3$$



$$\frac{d}{a} = \left(\frac{a}{a+d}\right)^3$$



$$\frac{d}{a} = \left(\frac{a}{a+d}\right)^3$$



$$\frac{d}{a} = \left(\frac{a}{a+d}\right)^3$$

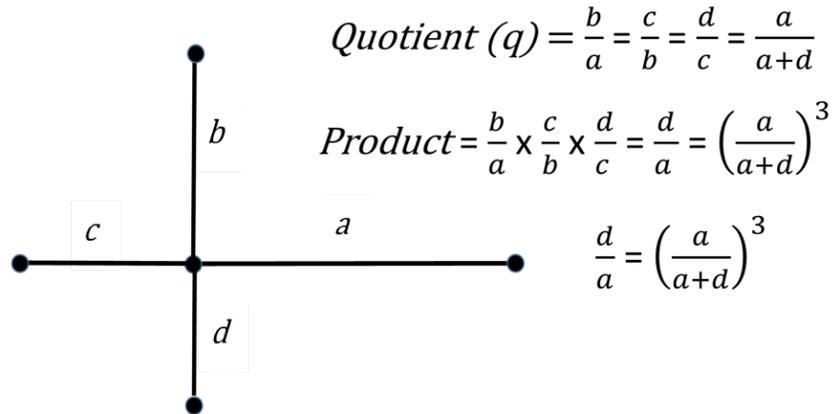


FIGURE 2. A geometric branching pattern was created for $p = 3$ based on a specific number of decreasing-sized branches that arise from a main branch. In the model, the ratio between the sizes of pairs of consecutive branches (ordered by size) equals the ratio of the largest branch size to the sum of the largest and smallest branch sizes.

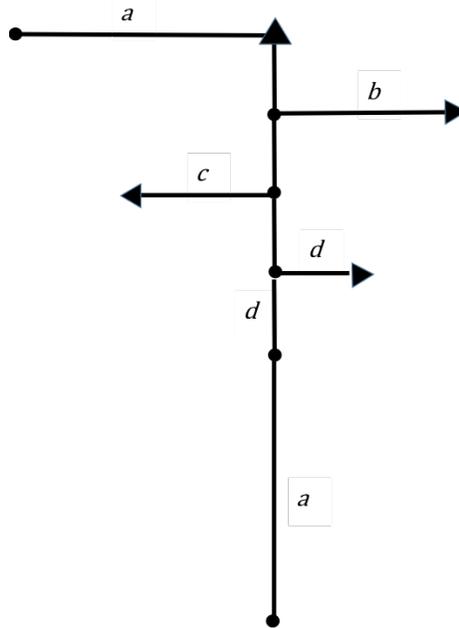


FIGURE 3. Different branching structures can also be created by rearranging the location of the branches while keeping the sizes of the expanding branches the same. This branching structure provided the basis for our agent-based model design (see Fig. 5).

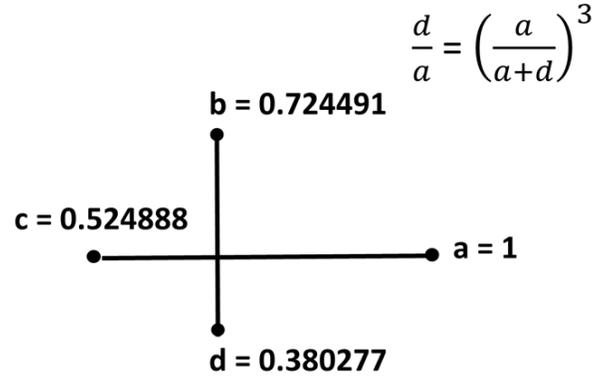


FIGURE 4. The geometric expression for $p = 3$ illustrates how to identify patterns of self-similarities that occur across different degrees of branching and multiple dimensions.

The following identities follow from Figure 4 and the corresponding results in Section 1.

$$\begin{aligned}
 q &= \frac{b}{a} = \frac{c}{b} = \frac{d}{c} = \frac{a}{a+d} = \frac{b+c}{a+b} = \frac{b+d}{a+c} = \frac{c+d}{b+c} = \frac{a+c}{a+b+d} = \frac{a+d}{a+c+d} \\
 &= \frac{b+c+d}{a+b+c} = \frac{a+c+d}{a+b+c+d} \\
 q^2 &= \frac{c}{a} = \frac{d}{b} = \frac{b}{a+d} = \frac{c+d}{a+b} = \frac{a}{a+c+d} = \frac{b+d}{a+b+d} = \frac{a+d}{a+b+c+d} \\
 q^3 &= \frac{d}{a} = \frac{c}{a+d} = \frac{b}{a+c+d} = \frac{a}{a+b+c+d} \\
 q^4 &= \frac{d}{a+d} = \frac{c}{a+c+d} = \frac{b}{a+b+c+d} \\
 q^5 &= \frac{d}{a+c+d} = \frac{c}{a+b+c+d} \\
 q^6 &= \frac{d}{a+b+c+d}
 \end{aligned}$$

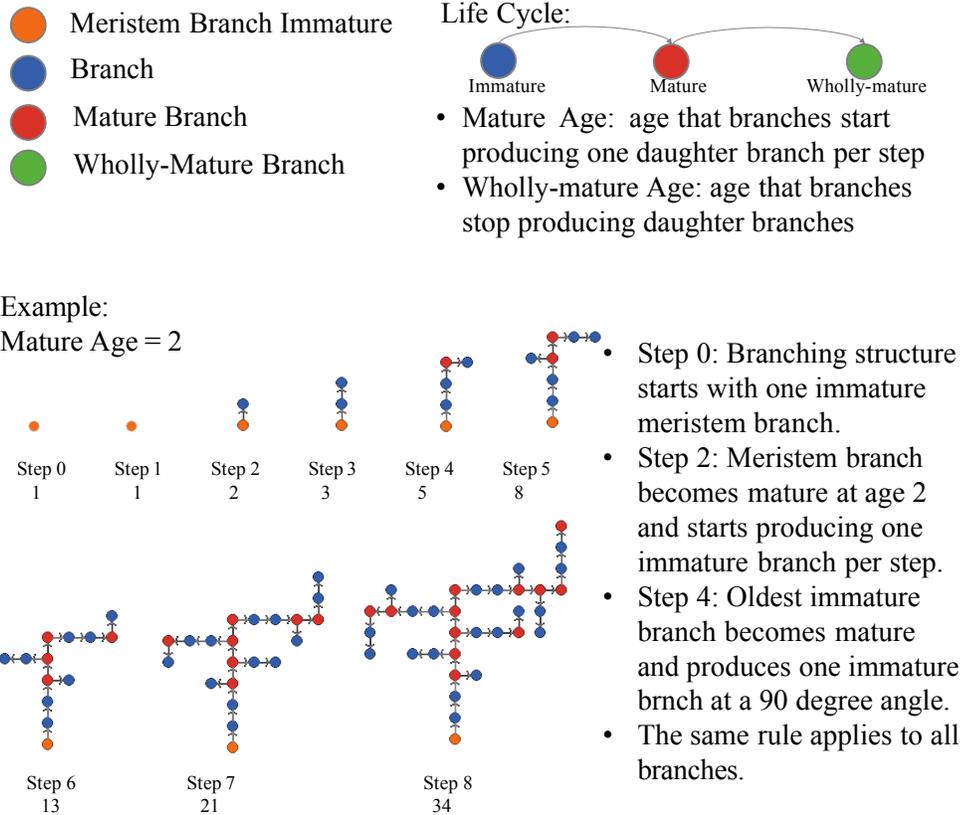
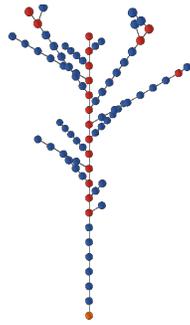
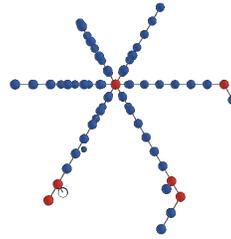


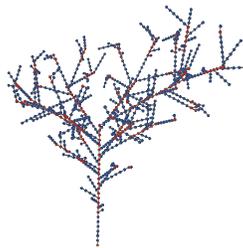
FIGURE 5. An agent-based code was created for our model with the “agents” being the branches themselves and each branch is considered as an agent that follows its own rules to grow. The sums of the sub-populations of each branch type can be calculated based on the generating function of [4].



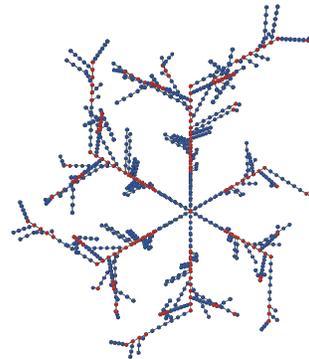
(a) Angle = 60° , age = 6



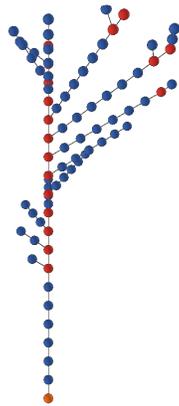
(b) Same as Fig. 6(a), top view



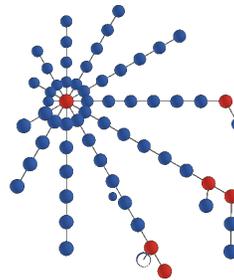
(c) Same as Fig. 6(a), more cells



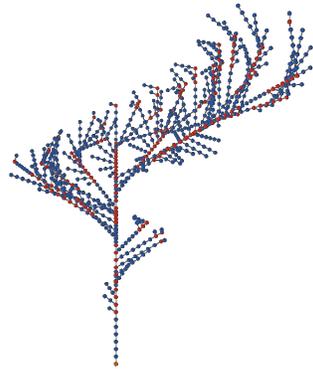
(d) Same as Fig. 6(c), top view



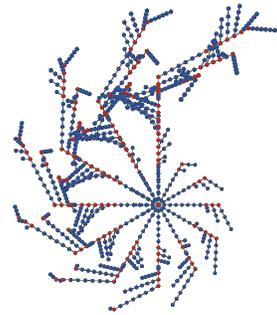
(e) Angle = 30° , age = 6



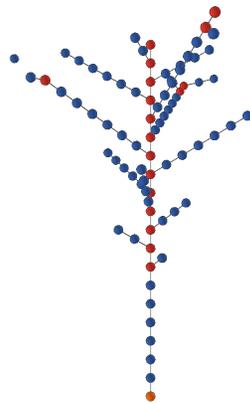
(f) Same as Fig. 6(e), top view



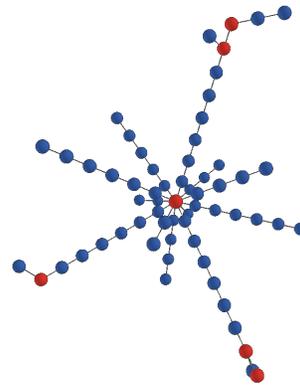
(g) Same as Fig. 6(e), more cells



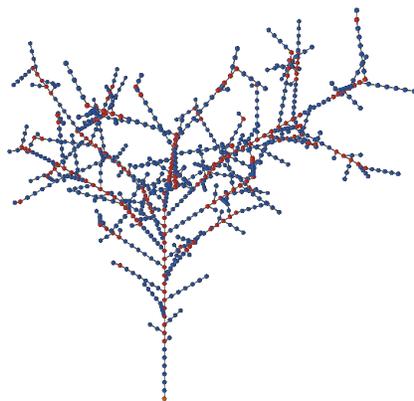
(h) Same as Fig. 6(g), top view



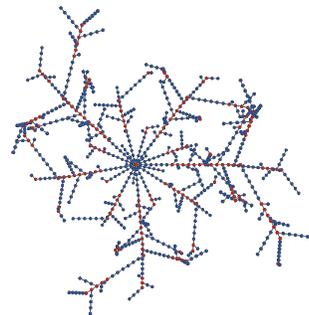
(i) Same as 6(a), except angle = Golden Angle = $137.507\dots^\circ$



(j) Same as Fig. 6(i), top view



(k) Same as Fig. 6(i), more cells



(l) Same as Fig. 6(k), top view

FIGURE 6. Examples of generation of branching structures in 3D models that simulate branching patterns in plants found in nature.

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