PROOF OF THE TAGIURI HISTOGRAM CONJECTURE

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Abstract. Several recent papers and presentations have introduced and studied the index histograms of one-parameter Tagiuri Generated Families of Fibonacci identities. The purpose of this paper is to prove a Unification Theorem which: (i) provides a unified proof of several results, (ii) reduces proofs to routine computations, and (iii) allows formulation and proof of the Tagiuri Histogram Conjecture for a broad range of cases.

1. Introduction

One-parameter Tagiuri Generated Families (TGF) of Fibonacci identities were presented in [1], [2], [3], and [4]. Index histograms of these identities show interesting properties. The main purpose of this paper is to present a Unification Theorem, unifying the proofs of the main theorems in [2], [3], and [4]. Besides unifying the proofs of these main theorems, the Unification Theorem reduces these proofs to routine computations. The Unification Theorem allows us to formulate and prove the Tagiuri Histogram Conjecture for a wide range of special cases.

Each TGF is a one-parameter family of Fibonacci identities. Throughout this paper, we let $q \geq 1$ be the parameter indexing the members of this family. To generate the q-th member of a TGF, we must specify certain items. We do not assume familiarity with the prior papers on this subject. Rather, we provide a self-contained presentation in this paper. Each of Sections 2 through 6 introduces one item needed for production of the q-th identity of a TGF and its associated index graph. For illustrative purposes, throughout the paper, besides presenting the general case, we use the $q = 2$nd identity of the TGF presented in [4]. Throughout the paper, this identity is called the *illustrative example.* Historically, the idea of presenting the definitions and proofs by interleaving the illustrative example with general proofs was introduced in [2].

2. The Product P

To produce the q-th identity of a TGF we first need to provide a product $P_q = P$ of Fibonacci numbers (throughout the paper we drop the subscript q when it will cause no confusion). To do this, we must specify a positive integer m, the number of multiplicands in P, as well as an index set of integers,

$$I = \langle i_1, \dots, i_m \rangle. \tag{2.1}$$

Throughout the paper, angle brackets indicate ordered sets while braces and parentheses indicate unordered sets.

Using the specifications of I and m just given, we then define

$$P = \prod_{j=1}^{m} F_{n+i_j}, \tag{2.2}$$

where as usual, here, and throughout the paper, n is a parameter varying over the integers.

Note that there is some abuse of language since the actual indices are $\langle n+i_1,\dots,n+i_m\rangle$, not $\langle i_1,\dots,i_m\rangle$, but this will cause no confusion.

Example 2.1. *For the illustrative example, we have*

$$m=5, \qquad I=\langle i_1,\dots,i_5\rangle=\langle -2,-1,\dots,2\rangle, \qquad P=\prod_{j=-2}^{2} F_{n+j}. \tag{2.3}$$

3. The Start Identity

After defining P, we next must specify a positive integer s_p and a non-negative integer s_n with $s_p > s_n$. These integers respectively indicate the number of positive and negative copies of P in the start identity which is given by

$$(s_p - s_n)P = s_pP - s_nP. \tag{3.1}$$

Note that the start identity is trivially true.

We let

$$s = s_p + s_q, \tag{3.2}$$

the total number of copies of P on the right-hand side of the start identity.

Example 3.1. *For the illustrative example,*

$$s_p=3, s_q=2, \qquad s=s_p+s_q=5, \qquad P=3P-2P. \tag{3.3}$$

To explain the "s" in s_p, s_n, s note that if we expand the right-most equation in (3.3) we obtain

$$P=P+P+P-P-P \tag{3.4}$$

so that the right-hand-side of this equation has five summands corresponding to the five copies of P used. In the sequel we will refer to these five copies of P as the 1st copy of P, the second copy of P, etc., where we count the copies of P sequentially from left to right, so that the first three copies of P have a plus sign while the last two copies of P have a negative sign.

4. The Tagiuri Replacement Transformation

Tagiuri's identity [5, p. 114] states that for all integers n,x,y we have

$$F_{n+x}F_{n+y} = F_nF_{n+x+y} + (-1)^n F_xF_y. \tag{4.1}$$

The Tagiuri identity generalizes both the Cassini identity [5, p. 74] and the Catalan identity [5, p. 83].

The idea introduced in [1] is that the Tagiuri Identity can be used to transform products and the identities containing them. Using (2.1), two indices $i_{k_1}, i_{k_2} \in I, 1 \le k_1, k_2 \le m$ are selected. Using (2.2), we say that *we apply Tagiuri at the indices i_{k_1}, i_{k_2} to the product P* if we replace the product $F_{n+i_{k_1}}F_{n+i_{k_2}}$ in P with the right-hand side of (4.1) with $x=i_{k_1}, y=i_{k_2}$.

For future reference, by (4.1), (2.2), and (2.1), the result of applying Tagiuri at i_{k_1}, i_{k_2} to P is

$$F_nF_{n+i_{k_1}+i_{k_2}} \prod_{\substack{1\le j\le m_q \\ i_j\notin\{i_{k_1},i_{k_2}\}}} F_{n+i_j} + (-1)^n F_{i_{k_1}}F_{i_{k_2}} \prod_{\substack{1\le j\le m_q \\ i_j\notin\{k_1,k_2\}}} F_{n+i_j}. \tag{4.2}$$

We refer to (4.2) as the *transformed* P or the P transformed at indices i_{k_1}, i_{k_2}.

Example 4.1. *For the illustrative example, by* (2.3)*, if say* $i_{k_1} = i_1 = -2, i_{k_2} = i_2 = -1$*, then the transformed* P *is*

$$F_n F_{n-3} F_n F_{n+1} F_{n+2} + (-1)^n F_{-2} F_{-1} F_n F_{n+1} F_{n+2}. \tag{4.3}$$

As we noted above in connection with (2.1), there is some abuse of language in saying that we apply Tagiuri to the indices i_{k_1}, i_{k_2} since the actual indices are $n + i_{k_1}, n + i_{k_2}$, but this will cause no confusion in the sequel.

5. The Transformation Set

To motivate the definition in this section, we first examine the illustrative example.

Example 5.1. *Equation* (4.3) *presents the result of applying the Tagiuri transformation to the first and second indices in the index set,* (2.3)*, of the illustrative example. By* (3.3) *there are five copies of* P *in the start identity of the illustrative example. To complete the construction of the illustrative example, we must specify for each of these* $s = 5$ *copies of* P *in the start identity the pairs of indices to which Tagiuri is applied. We indicate this collection of pairs of indices with* T*.* T *is an ordered set of* s *members, one for each copy of* P *in the start identity, with each member of* T *equaling an unordered pair of indices.*

$$T = \langle(-2, -1), (-1, 0), (0, 1), (1, 2), (2, -2)\rangle \tag{5.1}$$

.

We interpret T *in the manner just described. First, we apply Tagiuri to the indices* $i_1 = -2, i_2 = -1$ *in the first copy of* P *in the start identity; then we apply Tagiuri to the indices* $i_2 = -1, i_3 = 0$ *in the second copy of* P *in the start identity. We continue this application process to all five copies of* P *in the start identity. The result is the transformed start identity and is in fact the* $q = 2$*nd identity of the TGF studied in* [4]*,*

$$F_{n-2}F_{n-1}F_nF_{n+1}F_{n+2} = \tag{5.2}$$

$$F_nF_{n-3}\left(F_nF_{n+1}F_{n+2}\right) + (-1)^n F_{-2}F_{-1}\left(F_nF_{n+1}F_{n+2}\right) + \tag{5.3}$$

$$F_nF_{n-1}\left(F_{n-2}F_{n+1}F_{n+2}\right) + \tag{5.4}$$

$$F_nF_{n+1}\left(F_{n-2}F_{n-1}F_{n+2}\right) - \tag{5.5}$$

$$F_nF_{n+3}\left(F_{n-2}F_{n-1}F_n\right) - (-1)^n F_1F_2\left(F_{n-2}F_{n-1}F_n\right) - \tag{5.6}$$

$$F_nF_n\left(F_{n-1}F_nF_{n+1}\right) - (-1)^n F_2F_{-2}\left(F_{n-1}F_nF_{n+1}\right), \tag{5.7}$$

where the summands containing F_0 *have been removed (because* $F_0 = 0$*) but no other simplification was done. The reasons for this lack of simplification will be discussed below. For future reference, we note that* (5.2) – (5.7) *will be useful in illustrating various cases in the proof of the Unification Theorem.*

For the general case, we define an ordered set of m unordered pairs by

$$T = \langle(k_{1,1}, k_{1,2}), (k_{2,1}, k_{2,2}), \ldots, (k_{s,1}, k_{s,2})\rangle, \quad \text{with } k_{j,1}, k_{j,2} \in I, k_{j,1} \neq k_{j,2}, 1 \leq j \leq s. \tag{5.8}$$

The q-th identity of the general TGF is obtained by applying Tagiuri to the s copies of P in the start identity at the indices $i_{k_{j,1}}, i_{k_{j,2}}, 1 \leq j \leq s$. Since (3.1) is true, the result of applying the Tagiuri identity (4.1) to the start identity must yield a true result. This proves that all identities of a TGF are true.

In all previous papers and presentations, there was concern about the compatibility of the transformation sets, $T_q, q = 1, 2, \ldots$ for a given TGF. This necessitated some complicated notation including use of infinite words and complicated definitions using indices. However, these compatibility criteria are not used in the proof of the Unification Theorem. Consequently, we are dropping this requirement. This greatly simplifies the notation and presentation used in this paper.

6. Index Graphs

To motivate the concepts presented in this section, we first discuss the illustrative example and then present the general theory.

Example 6.1. *The illustrative example is given in* (5.2)–(5.7) *which describes a Fibonacci identity true for all integer values of* n. *However,* (5.2)–(5.7) *appears unintuitive, lacking elegance, and lacking punchiness. The approach advocated in* [1] *to identify elegant patterns is to count the indices of the form* $n + x$, x *an integer, occurring on the right-hand side of* (5.2)–(5.7).

In counting occurrences certain conventions have to be observed to avoid the effect of cancellations. The various TGF examined so far seem to have little if any cancellation for large values of q. *However, we have no way of proving this. Therefore, we adopt the following conventions: After transforming each* P *in the start identity we further assume that i) all parenthetical expressions are expanded, ii) all summands with coefficient zero eliminated, iii) all powers are counted with multiplicity, iv) all non-zero constant coefficients and* $(-1)^n$ *do not contribute to the count, v) no similar terms are coalesced, and vi) only indices that have an occurrence of* n *are counted.*

To illustrate these conventions: $F_n(F_{n+1} + F_{n+2})$ *has a count of two occurrences of* F_n *after parenthetical expansion.* F_n^2 *has a count of two occurrences of* F_n *while* $2F_{-1}F_{n-1}$ *has only one occurrence of* F_{n-1}. *Although we do not allow gathering similar terms, constant coefficients can arise from the initial multiplicands in the second summand of* (4.2).

We let y_x *indicate the number of occurrences of* F_{n+x} *in* (5.3)–(5.7). *We have*

$$y_{-3} = 1, y_{-2} = 4, y_{-1} = 6, y_0 = 12, y_1 = 6, y_2 = 4, y_3 = 1,\ y_x = 0 \textit{ for } |x| > 3. \tag{6.1}$$

The collection of all pairs $(x, y_x), x \in \mathbb{N}$, *can naturally be represented by a graph, with integer points on the graph connected with line segments. Figure 1 presents the graph for the illustrative example. Historically, actual histograms were used in* [1], [4], *and* [2]. *However, for the TGF presented in* [3] *a graph was more convenient than a histogram. Accordingly, in this paper we use graphs, vs. histograms. For the Histogram Conjecture we retain use of the term histogram since the conjecture was presented with that terminology in earlier papers.*

For the general case, we define for integer x

$$y_x = \text{ the number of occurrences of } F_{n+x} \text{ on the right hand side of the transformed start identity.} \tag{6.2}$$

The main theorems of [4], [2], and [3] present a closed formula for $y_x, x \in \mathbb{N}$.

7. Previous TGF

Three TGF have been studied and presented in [2], [3], and [4]. There are many items defining a TGF. To provide the most intuitive appeal, we provide in Table 1 below the source

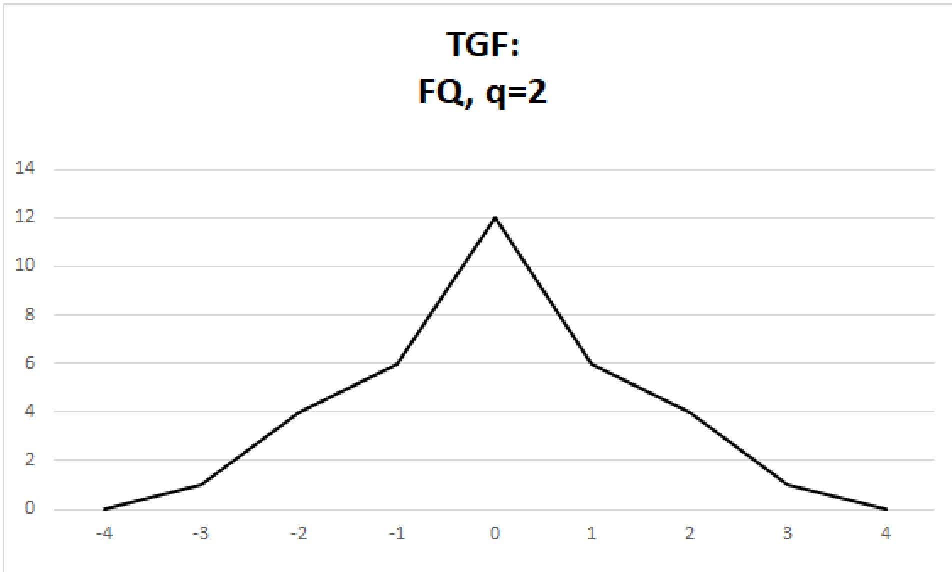

Figure 1. Index histogram for the Illustrative Example

of presentation, the start identity, the index set I_q and the transformation set T_q. Related functions such as m_q and s_q can be inferred from these. For example, for the TGF in source FQ, using (3.1), (3.2), and (2.1) respectively, we infer that $s_p = q+1, s_n = q, s = s_p + s_n = 2q+1$, and $m_q = 2q+1$, the number of integers between $-q$ and q including endpoints. P can then be constructed using (2.2).

Source	Start Identity	Index Set, I
FQ [4]	$P = (q+1)P - qP$	$\langle -q, \dots, q \rangle$
WCNT [2]	$P = (q+1)P - qP$	$\langle -(q+1), \dots, -1, 1, \dots, q+1 \rangle$
MASON II [3]	$2qP = 4qP - 2qP$	$\langle -2q, \dots, -1, 1, \dots, 2q \rangle$

Source	Transformation set, T
FQ [4]	$\langle (-q, -(q-1)), (-(q-1), -(q-2)), \dots, (q-1, q), (q, -q) \rangle$
WCNT [2]	$\langle (-q-1), -q), \dots, (-q-1, -1), (-q-1, 1), \dots, (-q-1, q+1) \rangle$
MASON II [3]	See below in the caption

Table 1. Presentation medium, start identity, index set I, and transformation set T for three TGF. For MASON II, let $\langle x_1, x_2, \dots x_{4q} \rangle = I = \langle -2q, \dots, -1, 1, \dots 2q \rangle$. Then the first six members of T are $\langle (x_1, x_2), (x_1, x_3), (x_1, x_4), (x_2, x_3), (x_2, x_4), (x_3, x_4) \rangle$; the next six members of T are $\langle (x_5, x_6), (x_5, x_7), (x_5, x_8), (x_6, x_7), (x_6, x_8), (x_7, x_8) \rangle$. This process continues until all members of I are used, each four consecutive members of I generating six members of T as just shown.

8. Proof Overview

Our goal in the next four sections is to derive a closed formula for y_x, as defined by (6.2), for the qth identity of an arbitrary TGF. For the next four sections we fix q. We first review all necessary background.

Recall that the start identity, presented in (3.1), which has s (see (3.2)) copies of P (see (2.2)) on its right-hand side is transformed by applying Tagiuri to these s copies of P at the s index pairs in T (see (5.8)). Equation (4.2) provides the result for applying Tagiuri to P at one pair of indices. The right-hand side of the qth identity is the sum of the s copies of transformed P.

For the illustrative example, the index set I, the product P, the start identity, s, and T are given by (2.3), (3.3), and (5.1). The values of y_x are given by (6.1). The $q = 2$nd identity is given by (5.2)–(5.7).

It will be useful in the sequel to use the following terminology about (4.2). We refer to $\prod_{\substack{1 \le j \le m_q \\ i_j \notin \{i_{k_1}, i_{k_2}\}}} F_{n+i_j}$ as the *product in the first summand.* We refer to $\prod_{\substack{1 \le j \le m_q \\ i_j \notin \{k_1, k_2\}}} F_{n+i_j}$ as *the product in the second summand.* We refer to $(-1)^n F_{i_{k_1}} F_{i_{k_2}} \prod_{\substack{1 \le j \le m_q \\ i_j \notin \{k_1, k_2\}}} F_{n+i_j}$ as *the second summand.* Finally, we refer to $F_n F_{n+i_{k_1}+i_{k_2}}$ as the *the initial multiplicands.* (Note: We will never, in the proof, need to refer to the initial multiplicands in the first summand; therefore, the phrase *initial multiplicands* has a unique meaning.)

We can illustrate use of this terminology with the illustrative example. Technically, (5.4) has no second summand. Nevertheless, we will, for example, say that the second summand in the transformed copy of P in (5.4) vanishes. This should cause no confusion.

Each of the next four sections will prove a formula for y_x for one case. Each section will provide a general proof and may also use the illustrative example which is presented in (5.2)–(5.7). We note that the parenthetical expressions in (5.2)–(5.7) contain the products in the first and second summand of (4.2) so that the initial multiplicands are clearly indicated. This will be useful in the sequel

In general, the closed formula for y_x will be a sum. We will use the language of "contribution" to describe how each summand or the underlying items it counts "contributes" to y_x.

9. The case when $x \neq 0, x \notin I$

Since by assumption $x \notin I$, it follows from (4.2) that only the initial multiplicands could contribute to y_x. For each of the s transformed copies of P, the products in the first and second summands could not contribute to y_x since each multiplicand in them satisfies F_{n+i_j}, for some $i_j \in I$.

This naturally motivates defining for integer x,

$$C_x = \text{ the number or count of } (i_{k_1}, i_{k_2}) \in T \text{ such that } i_{k_1} + i_{k_2} = x. \tag{9.1}$$

.

We then have the result,

$$y_x = C_x, \text{ for } x \neq 0, x \notin I. \tag{9.2}$$

Example 9.1. *For the illustrative example, by* (2.3)*,* $\pm 3 \notin I$*. By* (9.1) *and* (5.1)*,* $C_{-3} = C_3 = 1$*. Therefore, by* (9.2)*,* $y_{-3} = y_3 = 1$*, which in fact is confirmed by* (6.1)*.*

10. The case when $x = 0, 0 \notin I$

In the illustrative example, $0 \in I$, so we cannot motivate the proof in this case with the illustrative example. However, the discussion in the previous section for the case $x \neq 0, x \notin I$, still applies. More specifically, we have the following:

C_0 contributes to y_0.

For each of the s transformed P, their initial multiplicands in (4.2) always have an occurrence of F_n. Therefore, we have an additional contribution of s to y_0.

As discussed in the last section, for each of the s transformed copies of P, their products in the first and second summand do not contribute to y_0 since we assume $0 \notin I$. We have therefore proved that

$$y_0 = C_0 + s, \qquad \text{if } 0 \notin I. \tag{10.1}$$

11. The case when $x = 0, 0 \in I$

The arguments made in Sections 9 and 10 hold for this case also. For each of the s copies of transformed P, their initial multiplicands in the first summand always have an occurrence of F_n. Thus, they contribute s to y_0.

Similarly, by (9.1), there is a contribution of C_x to y_x from the initial multiplicands in the first summands of the s transformed copies of P that arise from the C_x pairs $(i_{k_1}, i_{k_2}) \in T$ such that $i_{k_1} + i_{k_2} = 0$.

By assumption, $0 \in I$. This implies that for each of the s copies of transformed P, the products in their first summands will contain F_n unless the pair of indices of T to which Tagiuri is applied to P contains a 0. One can see this by the bounds for the product in the first summand of (4.2) which require $i_j \in I, 1 \leq j \leq m, i_j \notin (i_{k_1}, i_{k_2})$. This motivates generally defining

$$c_x = \text{number of members of } T \text{ whose components contain an occurrence of } x. \tag{11.1}$$

It follows that the total contribution to y_0 from the products in the first summands of the s transformed copies of P is $s - c_0$.

We use lower case "c" in (11.1) and capital "C" in (9.1); this should cause no confusion.

A similar argument shows that there is a total contribution of $s - c_0$ arising from the second summands in the s transformed copies of P. (Note, the second summand will only vanish if 0 is a component of the member pair of T to which Tagiuri is applied to P; but these c_0 pairs are already subtracted from s.)

We summarize as follows:

$$y_0 = 2(s - c_0) + C_0 + s, \ \text{if } 0 \in I. \tag{11.2}$$

Example 11.1. *To illustrate these contributions, we may inspect the illustrative example* (5.2)–(5.7)*. As can be seen: (1) there are $s = 5$ copies of F_n in the initial multiplicands of* (5.3)–(5.7)*. (2) There is an additional occurrence of F_n in the initial multiplicands in the first summand of* (5.7) *corresponding to the index pair* $(2, -2)$ *of T whose components sum to 0. (3) Finally, by* (5.1)*, 0 occurs in two member pairs of T,* $(-1, 0)$ *and* $(0, 1)$*, implying $c_0 = 2$. We then confirm that there is a contribution of $s - c_0 = 5 - 2 = 3$ to y_0 from the products in the first summands as seen in* (5.3), (5.6), (5.7)*; there is a similar contribution of* 3 *to y_0 from the second summands. Contrastively, the products in* (5.4) *and* (5.5) *do not contribute occurrences of F_n to y_0.*

In total, $y_0 = 5 + 1 + 3 + 3 = 12$ which can be confirmed by either inspecting (5.3)–(5.7) *or by* (6.1).

12. The case $x \neq 0, x \in I$.

This is the most complex case. We therefore slowly go through the illustrative example, step by step.

Example 12.1. *By* (2.3), $x = -2, -1, 1, 2$ *each satisfy* $x \neq 0, x \in I$. *For purposes of illustration we fix* $x = 1$. *By* (6.1), $y_1 = 6$.

First, as in other cases of the proof, by (5.1) *and* (9.1) *we have a contribution of* $C_1 = 1$, *to* y_1 *arising from the pair* $(0, 1) \in T$.

Second, we have a total contribution of $s - c_1$ *to* y_x *arising from the products in the first summands of the* s *transformed copies of* P. *By* (5.1) *and* (11.1), $c_1 = 2$ *since* $(0, 1)$ *and* $(1, 2)$ *are both members of* T. *We can confirm that* $s - c_1 = 5 - 2 = 3$ *transformed copies of* P *in the illustrative identity,* $(5.3), (5, 4)$, *and* (5.7), *each have an occurrence of* F_{n+1} *in the product of their first summands.*

A problem however arises with the second summands. There are $s = 5$ *transformed copies of* P. *Two of these transformed coipes of* P, (5.5) *and* (5.6), *arise from a Tagiuri transformation involving* 1; *therefore* F_{n+1} *does not occur in their second summands and consequently, these two transformed copies of* P *must be subtracted from* $s = 5$. *Two transformed* $P, (5.4), (5.5)$, *have no second summand because when applying Tagiuri to* P *at indices* $(-1, 0)$ *or* $(0, 1)$, *the second summand vanishes because it is multiplied by* $F_0 = 0$.

If we proceeded naively, we would find a total contribution of $s - c_1 - c_0 = 5 - 2 - 2 = 1$ *from the second summands in the 5 transformed copies of* P. *Thus* y_1 *would have contributions of* 1 *from* C_1, $s - c_1 = 5 - 2 = 3$ *from the products in the first summands of the 5 transformed copies of* P *and* $s - c_1 - c_0 = 5 - 2 - 2 = 1$ *from the second summands in the 5 transformed copies of* P. *But then* $y_1 = 1 + 3 + 1 = 5$ *when in reality by inspection of* (5.3)–(5.7) *or by* (6.1), $y_6 = 1$. *What went wrong?*

The problem arises because the two transformed P *with a Tagiuri transformation involving a* 1 *are* (5.5) *and* (5.6) *and the two transformed* P *arising from a Tagiuri transformation involving a* 0 *are* (5.4) *and* (5.5). *Thus, we should subtract* 3 *corresponding to these three transformed copies, but instead have subtracted* 4. *The reason for this discrepancy is that the second summand in* (5.5) *simultaneously contains a* 1 *and* 0 *corresponding to the member* $(0, 1) \in T$. *Thus we subtracted it twice. We therefore have to add one of the subtracted copies back in.*

This motivates generally defining

$$z_x = \text{ number of members of } T \text{ equaling } (0, x). \tag{12.1}$$

By (5.1) *and* (12.1), $z_1 = 1$. *Therefore, the total contribution of all* s *transformed* P *from their second summands is* $s - c_0 - c_1 + z_1 = 5 - 2 - 2 + 1 = 2$. *We then have* $y_1 = C_1 + (s - c_1) + (s - c_1 - c_0 + z_1) = 1 + 3 + 2 = 6$, *as required.*

For the general case, (12.1) is still used to define z_x. We have 1) contributions of C_x to y_x from members of T whose components add to x, 2) a total contribution from all s transformed copies of P of $s - c_x$ from the products of their first summands, and 3) a total contribution for all s transformed copies of P of $s - c_0 - c_x + z_x$ from their second summands. We have proven

$$y_x = C_x + (s - c_x) + (s - c_x - c_0 + z_x), \text{ if } x \neq 0 \text{ and } x \in I. \tag{12.2}$$

13. The Unification Theorem and the Tagiuri Histogram Conjecture

We summarize the previous 4 sections. The Unification Theorem unifies previous proofs as well as reduces these proofs to routine computations.

Theorem 13.1. ***The Unification Theorem.*** *Consider a TGF whose identities are indexed by $q \geq 1$. For $q \geq 1$, I_q, P_q, s_q, the start identity, and T_q are given by* (2.1), (2.2), (3.2), (3.1), *and* (5.8), *respectively. Then, for integer x, the count of occurrences of F_{n+x} in the right-hand side of the q-th identity of the TGF, is given by* (9.2), (10.1), (11.2) *and* (12.2).

14. Previous Results

The Unification Theorem both unifies the proofs of previously proven results as well as reduces them to computations. The specifications of the three TGF used in previous papers and presentations are presented in Table 1. Tables 2 and 3 present y_x, s, c_x, C_x, z_x for the FQ and WCNT TGF. The MASON II TGF will not be dealt with via a table.

F_{n+x}	Restrictions on x	y_x	s	c_x	C_x	z_x
F_n		$6q$	$2q+1$	2	1	0
$F_{n\pm1}$		$4q-2$	$2q+1$	2	1	1
$F_{n\pm e}$	$2 \leq \lvert e \rvert \leq q$, e even	$4q-4$	$2q+1$	2	0	0
$F_{n\pm o}$	$3 \leq \lvert o \rvert \leq q$, o odd	$4q-3$	$2q+1$	2	1	0
$F_{n\pm x}$	$q+1 \leq \lvert x \rvert \leq 2q$, x odd	1	$2q+1$	0	1	0
$F_{n\pm x}$	$\lvert x \rvert > 2q$	0	$2q+1$	0	0	0

Table 2. Values of y_x, s, c_x, C_x, z_x for the FQ TGF.

In this section we illustrate how routine computations yield a proof of the formula for y_x for the FQ TGF. The proof for the WCNT TGF is similar and omitted. The MASON II TGF is treated in a separate section.

Table 1 presents T for the FQ example. Using the T presented in Table 1 for the FQ TGF we see that by (11.1), $c_x = 2$ for $x \in I$, and $c_x = 0$ otherwise. By (12.1), we see that $z_x = 1$ for $x \in \{-1, 1\}$, and $z_x = 0$ otherwise. Equation (9.1) motivates adding the components of each member of T yielding $\{-(2q-1), -(2q-3), \dots, 2q-1, 0\}$. This proves that $C_x = 1$ if $x = 0$ or $x \in \{-(2q-1), -(2q-3), \dots, 2q-1\}$ and $C_x = 0$ otherwise.

Using these values of c_x, C_x, z_x and the value of s inferred by applying (3.1) and (3.2) to the start identity in Table 1, we can easily plug in the formulas (11.2), (12.2), (9.2), and (10.1) and verify the values of y_x. For example, when $x = 1$, (in the row of Table 2 beginning $F_{n\pm1}$) we are in the case $x \neq 0, x \in I$. By (12.2), $y_x = C_x + s - c_x + s - c_x - c_0 + z_x = 1 + 2q + 1 - 2 + 2q + 1 - 2 - 2 + 1 = 4q - 2$, as required. If $|x| > 2q$ then $x \neq 0, x \notin I$, so that by (9.2), $y_x = 0$ (as seen in the last row of Table 2.)

15. Tagiuri Histogram Conjecture

The Tagiuri Histogram conjecture was only partially formulated in [4]. It asserted that for a given TGF, there is a constant independent of q such that the number of distinct y_x (as q varies) is uniformly bounded by this constant. However, the conditions which would enable this to be true were never formulated. We are now able to formulate and prove the conjecture using the Unification Theorem. An index graph for the FQ TGF was presented in Figure 1. Sample index graphs for the TGFs of WCNT and MASON II are presented in Figures 2 and 3. A glance at these three graphs and similar graphs for other values of q shows that the cardinality of distinct (integer) y_x is very small.

The proof of the Unification Theorem provides necessary insight to this observation. For example if $x \neq 0, x \notin I$ then (9.2) states that $y_x = C_x$. Equation (9.1) states that C_x is the

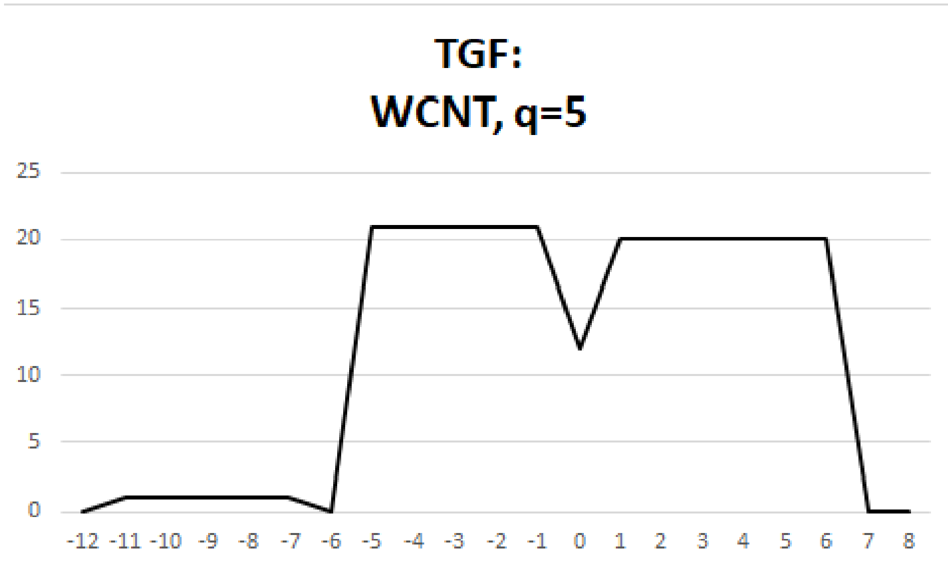

FIGURE 2. Index graph for the $q = 5$th identity of the TGF presented in WCNT.

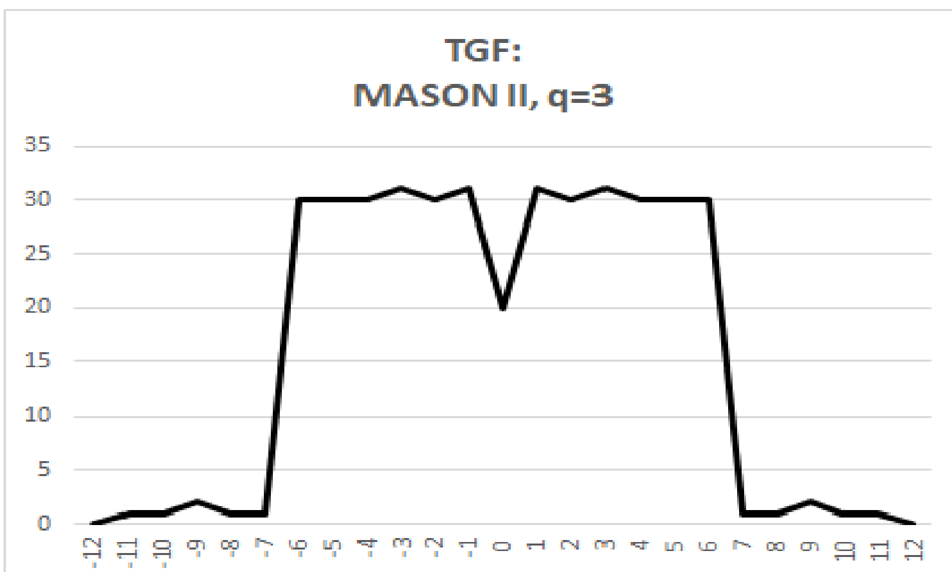

FIGURE 3. Index histogram for the TGF presented in MASON II with $q = 3$.

number of solutions to $i_{k_1} + i_{k_2} = x, (i_{k_1}, i_{k_2}) \in T$. In the three TGF studied, the number of solutions to this equation is finite and hence the number of possible y_x is also finite.

Theorem 15.1. ***Tagiuri Histogram Conjecture.*** *Fix a TGF. Suppose there is a constant independent of q such that as q varies, c_x, C_x, and z_x are uniformly bounded in absolute value by this constant. Then there is a constant independent of q such that for each q the number of distinct values of y_x is bounded by that constant.*

Proof. If q is fixed, then s is fixed. By the Unification Theorem the only way y_x can vary is according to the values of c_x, C_x, and z_x. This completes the proof. □

Note, that although the number of y_x for each q is uniformly bounded, the values of y_x will differ for each q, and in fact the maximum of these values is unbounded as q goes to infinity.

In practice, in the three TGF studied so far, the bound on the number of distinct y_x is under 10. Nevertheless, the graphs show variety and richness.

We have already pointed out that we are retaining the term "histogram" for historical reasons. However, the illustrative figures presented use graphs.

F_{n+x}	Restrictions on x	y_x	s	c_x	C_x	z_x
F_n		$2q+2$	$2q+1$	0	1	0
F_{n+x}	$1 \leq x \leq q+1$	$4q$	$2q+1$	1	0	0
F_{n-x}	$1 \leq x \leq q,$	$4q+1$	$2q+1$	1	1	0
$F_{n-(q+1)}$		0	$2q+1$	2q+1	0	0
F_{n-x}	$q+2 \leq x \leq 2q+1,$	1	$2q+1$	0	1	0
$F_{n\pm x}$	$\lvert x\rvert > 2q+1$	0	$2q+1$	0	0	0

TABLE 3. Values of y_x, c_x, C_x, z_x, and s for the WCNT TGF.

16. MASON II TGF

The TGF presented in MASON II cannot be compactly presented in a table. However, the description of the TGF can be elegantly and compactly described using 7 very simply stated (and simply proven) formulae. To develop and prove the seven formulae we i) refer to Table 1 which presents the start identity, I, and T for the MASON II case, and ii) note, that the definitions of y_x, C_x, c_x, and z_x are given in (6.2), (9.1), (11.1), and (12.1) respectively.

Formula 1: $s = 6q$.

Proof. This follows immediately from (3.2), (3.1), and the start identity presented in Table 1, $2qP = 4qP - 2qP$. □

Formula 2: $z_x = 0$.

Proof. By Table 1, $0 \notin I$, and therefore 0 is not a component of any pair of T. □

Formula 3: $c_x = 3$ for $x \in I$ and $c_x = 0$ otherwise.

Proof. By the description of T in the caption of Table 1, each four consecutive members of I generate 6 members of T; for example, $x_1, x_2, x_3, x_4 \in I$ gives rise to the 6 members of T, $(x_1, x_2), (x_1, x_3), (x_1, x_4), (x_2, x_3), (x_2, x_4), (x_3, x_4)$. It should be immediately clear that each $x_i \in I$ occurs exactly three times. □

Formula 4: $C_x = 0$ if $|x| \geq 4q$.

Proof. By (9.1), C_x is the sum of the components of a member of T. Each of the members of T have two distinct components in I. By Table 1, the biggest two members of I are $2q$ and $2q-1$. Similarly, the smallest two members of I are $-(2q), -(2q-1)$. The result follows immediately. □

Formula 5: $c_x = c_{-x}; C_x = C_{-x}$.

Proof. Recall by Formula 3, $c_x = 3$ for $x \in I$. By Table 1, I is symmetric about 0. Similarly, using (9.1), if $i_{j_1} + i_{j_2} = x$, for $(i_{j_1}, i_{j_2}) \in T$, then since I and T are symmetric about 0, $(-i_{j_1}, -i_{j_2}) \in I$ and therefore $-i_{j_1} - i_{j_2} = -x$. Thus the number of members of T whose components sum to x equals the number of members of T whose components sum to $-x$ showing $C_x = C_{-x}$. □

Formula 6: We assume $0 \leq x < 4q$, since by Formulas 4 and 5, for $x < 0$, we have $C_{-x} = C_x$. There are two cases to consider according to the parity of q.

q **even.**

$$C_x = \begin{cases} 0 & \text{if } x \equiv 0, 1, 2 \pmod 8 \\ 1 & \text{if } x \equiv 3, 4, 6, 7 \pmod 8 \\ 2 & \text{if } x \equiv 5 \pmod 8. \end{cases}$$

q **odd.** First, for all q, $C_0 = 2, C_1 = C_3 = 1, C_2 = 0$. For $4 \leq x < 4q$,

$$C_x = \begin{cases} 0 & \text{if } x \equiv 4, 5, 6 \pmod 8 \\ 1 & \text{if } x \equiv 0, 2, 3, 7 \pmod 8 \\ 2 & \text{if } x \equiv 1 \pmod 8. \end{cases}$$

Formula 7: $y_x = y_{-x}$.

Proof. By the Unification Theorem, y_x is a function of s, c_x, C_x, z_x and these four functions are symmetric about 0. □

These 7 formulae are sufficient to calculate the values of s, c_x, C_x, and z_x. The Unification Theorem then allows computation of y_x. The 7 formulae can also be used to write a simple program to generate values. Figure 3 was generated by such a program.

17. Conclusion

In this paper, we have unified the treatment of several TGF. The unification allows a unified treatment of proofs, reduction of proofs to routine computations, as well as an elegant proof of the Tagiuri Histogram conjecture.

There are several future avenues of research. First, in this paper, we have assumed one application of Tagiuri per summand in the start identity. In Caen [4], an example was presented where several applications per summand were allowed. This invites generalization of the results of this paper.

Glancing at Figures 1–3, we see a wide variety of histogram shapes. The classification of histogram shapes (as well as the definition of histogram shape) is another interesting avenue of research.

For large enough q, say for $q \geq 3$, the identities of the TGF for FQ, WCNT, and MASON II do not simplify. To avoid the problem of simplification, the Unification Theorem has been developed under assumption that coalescing of like terms is not done. It would be worthwhile to identify conditions on T assuring lack of possible simplification for large q.

Finally, there is the open conjecture of recognizing that a complex identity is in fact a member of a TGF. This would allow short proofs of certain complex identities.

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