HOW INTEGER SEQUENCES FIND THEIR WAY INTO AREAS OUTSIDE PURE MATHEMATICS

BURGHAH HERRMANN

Abstract. The integer sequences A190249, A190250 and A190251 of the On-Line Ency-lopedia OEIS are defined by mathematical conditions related to the golden ratio Φ. The botanical study of phyllotaxis models plants such as pineapples or sunflowers and describes the angular position of the nth leaf, scale, or seed by the fractional part of nΦ. Front num-ers and back numbers are distinguished by turning the model such that the polar axis faces forward. This paper shows that A295085, the sequence of front numbers, is the intertwining of A190249 and A190251. Moreover, A190250 is the sequence of back numbers.

The main theorem states that any generalized Fibonacci sequence eventually is a sequence of front numbers.

1. Arithmetics of the integer and fractional part

For real numbers x, y the integer part ⌊x⌋ is the greatest integer not exceeding x, and ⟨x⟩ denotes the fractional part x − ⌊x⌋. Of course, x = ⌊x⌋ + ⟨x⟩. For an integer k we have

⌊k + x⌋ = k + ⌊x⌋,

i.e., the integer part is additive with respect to integers. However, it is not additive in general since

⌊x + y⌋ = ⌊x⌋ + ⌊y⌋ + ⌊⟨x⟩ + ⟨y⟩⌋. (1)

The fractional part is invariant with respect to integers,

⟨k + x⟩ = ⟨x⟩, (2)

and thus,

⟨kx⟩ = ⟨k⟨x⟩⟩. (3)

Furthermore,

⟨−x⟩ = 1 − ⟨x⟩. (4)

The following implications are useful in the proof of the upcoming Theorem 4.1:

⟨x⟩ + ⟨y⟩ < 1 implies ⟨x + y⟩ = ⟨x⟩ + ⟨y⟩ (5)

and

⟨x⟩ + ⟨y⟩ > 1 implies 1 − ⟨x + y⟩ = 1 − ⟨x⟩ + 1 − ⟨y⟩ (6)

since, by (2), ⟨x + y⟩ = ⟨⟨x⟩ + ⟨y⟩⟩ = ⟨x⟩ + ⟨y⟩ − ⌊⟨x⟩ + ⟨y⟩⌋.

2. The function A190248

The number Φ = (√5 + 1)/2 ≈ 1.618 is the golden ratio. Note that

Φ2 = 1 + Φ. (7)

Applying (7) twice yields Φ3 = Φ(1 + Φ) = Φ + (1 + Φ) and hence,

Φ3 = 1 + 2Φ. (8)
Let \( n \) be a positive integer and \( f(n) \) the value at the \( n^{th} \) position of the sequence A190248 in [4] defined as
\[
f(n) = \lfloor n\Phi + n\Phi^2 + n\Phi^3 \rfloor - \lfloor n\Phi \rfloor - \lfloor n\Phi^2 \rfloor - \lfloor n\Phi^3 \rfloor
\]
and shown in Table 1. From (1) generalized to three terms we obtain
\[
f(n) = \lfloor \langle n\Phi \rangle + \langle n\Phi^2 \rangle + \langle n\Phi^3 \rangle \rfloor.
\]
Thus, \( f(n) = \lfloor \langle n\Phi \rangle + \langle n(1 + \Phi) \rangle + \langle n(1 + 2\Phi) \rangle \rfloor = \lfloor \langle n\Phi \rangle + \langle n\Phi \rangle + \langle 2n\Phi \rangle \rfloor \), where for the first equation we have used (7) and (8) and for the second equation we have used (2). Hence, by (3),
\[
f(n) = 2\lfloor \langle n\Phi \rangle + \langle 2n\Phi \rangle \rfloor.
\]
(9)
As \( \langle x \rangle < 1/2 \) implies \( 2\langle x \rangle = 2\langle x \rangle \), and by (9), we have
\[
\langle n\Phi \rangle < 1/2 \Rightarrow f(n) = \lfloor 4\langle n\Phi \rangle \rfloor.
\]
(10)
As \( \langle x \rangle > 1/2 \) implies \( 2\langle x \rangle = 2\langle x \rangle - 1 \), and by (9), we have
\[
\langle n\Phi \rangle > 1/2 \Rightarrow f(n) = \lfloor 4\langle n\Phi \rangle \rfloor - 1.
\]
(11)

**Lemma 2.1.** Let \( f(n) \) be the value of the \( n^{th} \) position of OEIS sequence A190248. Then
\[
f(n) = \begin{cases} 
0 & \text{if } \langle n\Phi \rangle < 1/4, \\
1 & \text{if } 1/4 < \langle n\Phi \rangle < 3/4, \\
2 & \text{if } \langle n\Phi \rangle > 3/4.
\end{cases}
\]

**Proof.** By (10), the 1\(^{st}\) and 2\(^{nd}\) and by (11), the 3\(^{rd}\) and 4\(^{th}\) of the following implications hold
\[
\langle n\Phi \rangle < 1/4 \Rightarrow f(n) = 0,
1/4 < \langle n\Phi \rangle < 1/2 \Rightarrow f(n) = 1,
1/2 < \langle n\Phi \rangle < 3/4 \Rightarrow f(n) = 1,
\langle n\Phi \rangle > 3/4 \Rightarrow f(n) = 2.
\]
This covers all the cases and, thus, finishes the proof. \( \square \)

The next section interprets Lemma 2.1 in areas outside pure mathematics.

### 3. The phyllotactic front

Chapter 11.5 in [1] is about phyllotaxis and discusses the role of \( \langle n\Phi \rangle \) in the model of pineapples. It is considered as an angle of rotation. Thus, the intervals in Lemma 2.1 distinguish whether the angle in degree is smaller than 90°, between 90° and 270°, or larger than 270°.

**Definition 3.1.** A positive integer \( n \) such that \( 1/4 < \langle n\Phi \rangle < 3/4 \) is called a back number, otherwise it is called a front number.

The sequence of front numbers, 2, 3, 5, 8, 10, 11, 13, 16, 18, 21, ..., is registered as A295085 in [4]. The front numbers provide masks that are useful for labeling plants for which the model of golden spiral phyllotaxis fits; see Figure 3 in [2].

The sequences A190249, A190250, A190251, are respectively defined as positions of 0, 1, 2 in A190248 as shown in Table 1.

Lemma 2.1 yields

**Corollary 3.2.** The sequence of front numbers is the intertwining of A190249 and A190251. The sequence of back numbers coincides with A190250.
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| \(n\) | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|-------|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| \(f(n)\) | 1  | 0  | 2  | 1  | 0  | 1  | 2  | 1  | 0  | 2  | 1  | 0  | 1  | 1  | 2  |
| A190249 | 2  | 5  | 10 | 13 | 18 | 23 | 26 | 31 | 34 | 39 | 44 | 47 | 52 | 57 | 60 |
| A190250 | 1  | 4  | 6  | 7  | 9  | 12 | 14 | 15 | 17 | 19 | 20 | 22 | 25 | 27 | 28 | 30 |
| A190251 | 3  | 8  | 11 | 16 | 21 | 24 | 29 | 32 | 37 | 42 | 45 | 50 | 53 | 55 | 58 | 63 |

Table 1. Initial values of \(f\) and the position sequences of value 0 (A190249), value 1 (A190250), and value 2 (A190251).

4. Application to generalized Fibonacci sequences

A non-trivial sequence \((G_n)_n\geq 0\) of non-negative integers such that \(G_n = G_{n-1} + G_{n-2}\) for all \(n \geq 2\) is called a generalized Fibonacci sequence. Non-trivial means that the constant sequence of 0’s is excluded. The Fibonacci sequence is determined from \(F_0 = 0\) and \(F_1 = 1\). It is useful to prefix \(F_{-1} = 1\).

According to Exercise 31 in [3], p. 85, for \(n \geq 0\) we have

\[
\langle F_{n}\Phi\rangle = \begin{cases} 
\psi^n & \text{if } n \text{ is odd}, \\
1 - \psi^n & \text{if } n \text{ is even}, 
\end{cases}
\]

where \(\psi = 1/\Phi \approx 0.618\). Note that \(\psi^3 < 1/4\). Thus, Fibonacci numbers \(F_n\) are front numbers for \(n \geq 3\); see Figure 1.

The Lucas sequence is determined from \(L_0 = 2\) and \(L_1 = 1\). Lucas numbers \(L_n \geq 11\) are front numbers; see Figure 1. We show that any generalized Fibonacci sequence is eventually a sequence of front numbers.

**Theorem 4.1.** For a generalized Fibonacci sequence \((G_n)_n\geq 0\) there exists \(n_0\) such that \(G_n\) is a front number for all \(n \geq n_0\).
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Proof. It suffices to show

\[ \lim_{n \to \infty} \delta(G_n, \Phi) = 0 \]  

(13)

where \( \delta(x) = \min(\langle x \rangle, 1 - \langle x \rangle) \). We first give a decomposition in (15) and then proceed with various cases.

Equation (8) from the List of Formulae in the appendix of [5] states \( G_n = G_0 F_{n-1} + G_1 F_n \) and, hence,

\[ G_n = (G_0 - G_1) F_{n-1} + G_1 F_{n+1}. \]  

(14)

Without loss of generality, we assume \( G_0 > G_1 \) (otherwise, \( G_1 - G_0 \) could be prefixed). Put \( a = G_0 - G_1 \) and \( b = G_1 \) and note that \( a > 0 \). By (14),

\[ G_n = a F_{n-1} + b F_{n+1}. \]  

(15)

Case 1. Assume \( b = 0 \) and let \( n \) be large such that \( \psi^{n-1} < 1/a \).

Case 1.1. Say \( n \) is even. We show

\[ \langle G_n, \Phi \rangle = a \psi^{n-1}. \]

By (15), \( \langle G_n, \Phi \rangle = \langle a F_{n-1}, \Phi \rangle \) and it remains to show \( \langle a F_{n-1}, \Phi \rangle = a \psi^{n-1} \). By (3), \( \langle a F_{n-1}, \Phi \rangle = \langle a \langle F_{n-1}, \Phi \rangle \rangle \) and, by (12), \( \langle a \langle F_{n-1}, \Phi \rangle \rangle = \langle a \psi^{n-1} \rangle \) and, finally, \( \langle a \psi^{n-1} \rangle = a \psi^{n-1} \), since \( n \) is large such that \( a \psi^{n-1} < 1 \).

Case 1.2. Say \( n \) is odd. We show

\[ 1 - \langle G_n, \Phi \rangle = a \psi^{n-1}. \]

By (15), (3), (12) and (2), \( \langle G_n, \Phi \rangle = \langle a F_{n-1}, \Phi \rangle = \langle a \langle F_{n-1}, \Phi \rangle \rangle = \langle a(1 - \psi^{n-1}) \rangle = \langle -a \psi^{n-1} \rangle \) and it remains to show \( 1 - \langle -a \psi^{n-1} \rangle = a \psi^{n-1} \). By (4), \( 1 - \langle -a \psi^{n-1} \rangle = \langle a \psi^{n-1} \rangle \) and, finally, \( \langle a \psi^{n-1} \rangle = a \psi^{n-1} \), since \( n \) is large such that \( a \psi^{n-1} < 1 \).

Thus, in both subsets we have \( \delta(G_n, \Phi) \leq a \psi^{n-1} \) and the value converges to zero as \( n \) goes to infinity. This takes care of (13) for Case 1.

Case 2. Assume \( b > 0 \) and let \( n \) be large such that \( \psi^{n-1} < 1/(a + b \psi^2) \).

We have \( \psi^{n-1} < 1/a \) and \( \psi^{n+1} < 1/b \).

Case 2.1. Say \( n \) is even. We show

\[ \langle G_n, \Phi \rangle = a \psi^{n-1} + b \psi^{n+1}. \]

As above, \( \langle a F_{n-1}, \Phi \rangle = a \psi^{n-1} \) and, similarly, \( \langle b F_{n+1}, \Phi \rangle = b \psi^{n+1} \). Thus, \( \langle a F_{n-1}, \Phi \rangle + \langle b F_{n+1}, \Phi \rangle = a \psi^{n-1} + b \psi^{n+1} = \psi^{n-1}(a + b \psi^2) < 1 \). By (15) and (5), \( \langle G_n, \Phi \rangle = \langle a F_{n-1}, \Phi \rangle + \langle b F_{n+1}, \Phi \rangle = a \psi^{n-1} + b \psi^{n+1} \).

Case 2.2. Say \( n \) is odd. We show

\[ 1 - \langle G_n, \Phi \rangle = a \psi^{n-1} + b \psi^{n+1}. \]

As above, \( 1 - \langle a F_{n-1}, \Phi \rangle = a \psi^{n-1} \) and, similarly, \( 1 - \langle b F_{n+1}, \Phi \rangle = b \psi^{n+1} \). Thus, \( \langle a F_{n-1}, \Phi \rangle + \langle b F_{n+1}, \Phi \rangle = 1 - a \psi^{n-1} + 1 - b \psi^{n+1} = 2 - \psi^{n-1}(a + b \psi^2) > 1 \). By (15) and (6), \( 1 - \langle G_n, \Phi \rangle = 1 - \langle a F_{n-1}, \Phi \rangle + 1 - \langle b F_{n+1}, \Phi \rangle = a \psi^{n-1} + b \psi^{n+1} \).

Thus, in both subsets we have \( \delta(G_n, \Phi) \leq a \psi^{n-1} + b \psi^{n+1} \) and the value converges to zero. This takes care of (13) for Case 2.

In any case \( n_0 \) can be taken as the smallest \( n \) such that \( a \psi^{n-1} + b \psi^{n+1} = \psi^{n-1}(a + b \psi^2) < 1/4 \) or, equivalently, \( \psi^{n-1} < 1/(4(a + b \psi^2)). \)
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Burghard Herrmann, Venloer Str. 253, 50823 Köln, Germany
E-mail address: bfgb@netcologne.de, burghard.herrmann@fu-berlin.de