# LINEAR COMPLEMENTARY EQUATIONS AND SYSTEMS

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ABSTRACT. After a brief history of complementary equations, a definition is given for linear complementary equations, with particular attention to examples typified by $a_n = a_{n-1} + a_{n-2} + b_n$, where $(b_n)$ is the complement of $(a_n)$ in the set $\mathbb{N}$ of positive integers, and $a_n/a_{n-1} \to (1+\sqrt{5})/2$. Also introduced are systems of equations whose solutions are sequences that partition $\mathbb{N}$. An example is the system defined recursively by $a_n =$ least new $k$, $b_n =$ least new $k$, and $c_n = a_n + b_n$, where "least new $k$", also known as "mex", is the least integer in $\mathbb{N}$ not yet placed. The sequence $(c_n)$ in this example is the anti-Fibonacci sequence, A075326 in the Online Encyclopedia of Integer Sequences.

## 1. Introduction

Two sequences $(a_n)$ and $(b_n)$ are called complementary if they partition the set $\mathbb{N}$ of positive integers. Perhaps the history of such sequences dates back to the discovery of odd numbers and even numbers, but the present account jumps to the twentieth century when W. A. Wythoff, in his introduction to what is now called the Wythoff game, defined sequences that are now called the lower and upper Wythoff sequences. The first ten terms of these sequences appear in the final two rows of Table 1:

Table 1. lower and upper Wythoff sequences

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $a_n$ | 1 | 3 | 4 | 6 | 8 | 9 | 11 | 12 | 14 | 16 |
| $b_n$ | 2 | 5 | 7 | 10 | 13 | 15 | 18 | 20 | 23 | 26 |

Here are the rules: $a_n$ is always the least positive integer not yet in the final two rows, and

$$b_n = a_n + n. \tag{1.1}$$

This way of generating sequences illustrates two basic ideas for introducing the subject of complementary equations; first, "least positive integer not yet used", known as "mex", for minimal excludant, provides a natural way to account for—or construct—complementary sequences. (Aviezri Fraenkel [3] notes that the term "mex" may have originated with John H. Conway.) Second, (1.1) exemplifies a complementary equation; that is, an equation whose solution consists of sequences that partition $\mathbb{N}$. Except where otherwise stated (e.g., [1]) it is required that the solution sequences of a complementary equation be increasing. Under these conditions, certain historical examples having notable unique solutions were known before the term "complementary equation" gained currency sometime after 2001. Table 2 shows five of these, with solutions given by A-numbers as in OEIS [8].

| Table 2. Historic complementary equations and solutions | | | |
|---|---|---|---|
| $b_n = a_n + 1$ | $(a_n) =$ A005408 | $(b_n) =$ A005843 | odds and evens |
| $b_n = a_n + n$ | $(a_n) =$ A000201 | $(b_n) =$ A001950 | Wythoff sequences |
| $b_n = a_{a_n}$ | $(a_n) =$ A000201 | $(b_n) =$ A001950 | Wythoff sequences |
| $a_n = a_{n-1} + b_{n-1}$ | $(a_n) =$ A005228 | $(b_n) =$ A030124 | [4], 1980 |
| $a_n = b_{n-1} + b_{n-2}$ | $(a_n) =$ A022424 | $(b_n) =$ A055563 | [1], 2007 |

Section 2 gives a definition of linear complementary equations in terms of complementary sequences $(a_n)$ and $(b_n)$ and discusses limiting ratios such as $\lim_{n\to\infty} a_n/a_{n-1}$. Sections 2, 3, and 4 consider linear complementary equations of orders 0, 1, and 2, respectively. Sections 4 and 5 give matrix representations for solution sequences of certain complementary equations of orders 1 and 2. Section 6 introduces the notion of a *system* of complementary equations, using the anti-Fibonacci system as a basis for generalizations to anti-$m$-nacci systems and others. Section 7 gives tables for locating solutions in OEIS of complementary equations, including a few that are not linear. Section 8 gives notes regarding Mathematica codes used to generate solutions to complementary equations.

When dealing with a complementary—or *possibly* complementary—equation, a distinction must be made between having *a priori* solutions, as opposed to using the equation to construct possible solutions. It is easy to create equations that generate pairs $(a_n)$ and $(b_n)$ that start out looking complementary but are not so. Section 3 includes a code that tests a certain class of user-input sequences for complementarity.

## 2. Linear Complementary Equations

By a *linear complementary equation*, we mean an equation of the form

$$a_n = u(a_{n-1}, \ldots, a_{n-k}) + v(b_{m_n}, \ldots, b_0) + f_n \tag{2.1}$$

where

$$u(a_{n-1}, \ldots, a_{n-k}) = u_0 a_{n-1} + u_1 a_{n-2} + \cdots + u_{k-1} a_{n-k} \tag{2.2}$$

and

$$v(b_{m_n}, \ldots, b_0) = v_0 b_{m_n} + v_1 b_{m_n-1} + \cdots + v_{m_n} b_0, \tag{2.3}$$

where $(a_n)$ and $(b_n)$ are complementary, $(a_n)$ and $(b_n)$ are strictly increasing, $(f_n)$ is an integer sequence, and $(m_n)$ is an increasing sequence in $\mathbb{N}$.

The *order* of the equation (2.1) is defined from (2.2) as 0 if $k = 1$ and $u_0 = 0$; otherwise, the order is $k$. Note that an equation of a particular order may be equivalent to an equation of a different order; e.g., the equations $a_n = b_n + b_{n-1} + \cdots + b_0$ and $a_n = a_{n-1} + b_n$ are clearly equivalent.

In this section, we are especially interested in limiting behaviors of $a_n/a_{n-1}$ and $b_n/n$, as given by Theorem 2.1.

**Theorem 2.1.** *Suppose that $(a_n)$ and $(b_n)$ are solution sequences of a complementary equation*

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + D_n$$

*for which the following conditions hold:*

*(i) $c_1, c_2, \ldots, c_k$ are integers, where $k$ is a fixed positive integer, such that the polynomial*

$$x^k - c_1 x^{k-1} - \cdots - c_k$$

*has a (possibly repeated) real root $r > 1$, and all the other roots $z$ (if any) satisfy $|z| < 1$.*

*(ii)* $\lim_{n\to\infty} a_n/a_{n-1}$ *exists;*

*(iii)* $D_n = d_j b_{n-j} + d_{j-1} b_{n-j+1} + \ldots + d_0 b_n + un + v$*, where $u$ and $v$ are integers, $j$ is a fixed nonnegative integer, and the $j+1$ numbers $d_i$ are integers such that $D_n > 0$ for $n \geq 0$;*

*(iv) There exists $m$ in $\mathbb{N}$ such that $b_m < 2m$ and $a_{q+1} - a_q > 1$ for all $q \geq m$.*

*Then*

$$\lim_{n\to\infty} \frac{D_n}{n} = d_0 + d_1 + \cdots + d_j + u \tag{2.4}$$

*and*

$$\lim_{n\to\infty} \frac{a_n}{a_{n-1}} = r. \tag{2.5}$$

Before proving Theorem 2.1, we state and prove three lemmas.

**Lemma 2.2.** *Assuming the hypothesis of Theorem 2.1, $a_n \geq r^n$ for all $n \geq 0$.*

*Proof.* Starting with $a_0 \geq 1$, suppose for arbitrary $h$ that $a_i \geq r^i$ for $i = 0, \ldots, h$. Then

$$\begin{aligned} a_{h+1} &= c_1 a_h + c_2 a_{h-1} + \cdots + c_k a_{h+1-k} + D_{h+1} \\ &\geq c_1 a_h + c_2 a_{h-1} + \cdots + c_k a_{h+1-k} \\ &\geq c_1 r^h + c_2 r^{h-1} + \cdots + c_k r^{h+1-k} \\ &= r^{h+1-k}(c_1 r^{k-1} + c_2 r^{k-2} + \cdots + c_k) \\ &= r^{h+1-k} \cdot r^k \\ &= r^{h+1}. \end{aligned}$$

□

**Lemma 2.3.** *Assuming the hypothesis of Theorem 2.1, there exists $m$ in $\mathbb{N}$ such that $b_n < 2n$ for all $n \geq m$.*

*Proof.* Recalling *(iv)*, assume as an induction hypothesis that $b_h < 2h$ for arbitrary $h \geq m$. Now, either $b_h + 1$ is a term of $(a_n)$, in which case $b_{h+1} = b_h + 2$, or else $b_h + 1$ is not a term of $(a_n)$, in which case $b_{h+1} = b_h + 1$. In both cases, $b_{h+1} \leq 2h + 2 = 2(h+1)$. □

**Lemma 2.4.** *Assuming the hypothesis of Theorem 2.1,* $\lim_{n\to\infty} C(a_h, b_n)/n = 0$*, where $C(x(h), y)$ denotes the number of indices $h$ such that $x(h) < y$.*

*Proof.* By Lemmas 2.2 and 2.3, for $n \geq m$,

$$\begin{aligned} \frac{1}{n} C(a_h, b_n) &\leq \frac{1}{n} C(a_h, 2n) \\ &\leq \frac{1}{n} C(r^h, 2n) \\ &\leq \frac{1}{n} C(h, \log_r 2n) \\ &\leq \frac{1}{n} \log_r 2n, \end{aligned}$$

so that $\lim_{n\to\infty}(1/n)C(a_h, b_n) = 0$. To prove (2.4), it suffices to prove that $\lim_{n\to\infty} b_h/n = 1$ for $h = n-j, n-j+1, \ldots, n$. Now, for these $h$, we have

$$\lim_{n\to\infty}\frac{b_h}{n} = \lim_{n\to\infty}\frac{b_h}{h}\frac{h}{n} = \lim_{n\to\infty}\frac{b_n}{n},$$

so that it suffices to prove that $\lim_{n\to\infty} b_n/n = 1$. To that end, we have

$$b_n = n + C(a_h, b_n) + 1,$$
$$\frac{b_n}{n} = 1 + \frac{1}{n}C(a_h, b_n) + \frac{1}{n},$$

so that $\lim_{n\to\infty} b_n/n = 1$. To prove (2.5), we have

$$\frac{a_n}{a_{n-1}} = c_1 + c_2\frac{a_{n-2}}{a_{n-1}} + \cdots + c_k\frac{a_{n-k}}{a_{n-1}} + \frac{D_n}{a_{n-1}}.$$

Also,

$$\lim_{n\to\infty}\frac{D_n}{a_{n-1}} = \lim_{n\to\infty}\frac{D_n}{n}\frac{n}{a_{n-1}} \le \lim_{n\to\infty}\frac{d_0 + d_1 + \cdots + d_j + u}{n} = 0,$$

so that, letting $t = \lim_{n\to\infty} a_n/a_{n-1}$, we have

$$t = c_1 + \frac{c_2}{t} + \cdots + \frac{c_k}{t^{k-1}},$$

whence $t$ is a zero of the polynomial

$$x^k - c_1x^{k-1} - c_2x^{k-2} - \cdots - c_k = 0$$

in *(i)*. Clearly $t$ is a real number. If $t < 1$, then for large enough $n$, we have $|a_n/a_{n-1} - t| < 1 - t$, but then $a_n < a_{n-1}$, contrary to *(iv)*. Therefore, $t = r$. □

**Example 1.** Let $a_n = 4a_{n-1} - 4a_{n-2} + b_n$, with initial values $a_0 = 1, a_1 = 2$. The polynomial in Theorem 2.1 is $x^2 - 4x + 4 = (x-2)^2$, so that, by Theorem 2.1, $\lim_{n\to\infty} a_n/a_{n-1} = 2$. See Section 8 for a code.

**Example 2.** Let $a_n = 4a_{n-1} - 4a_{n-2} + b_{n-2}$, with initial values $a_0 = 1, a_1 = 2$. The polynomial in Theorem 2.1 is $x^2 - 4x + 4$, as in Example 1, so that $\lim_{n\to\infty} a_n/a_{n-1} = 2$. See Section 8 for two codes.

**Example 3.** Let $a_n = a_{n-2} + a_{n-3} + b_n$, with initial values $a_0 = 1, a_1 = 2, a_2 = 3$. The polynomial in Theorem 2.1 is $x^3 - x - 1$, with roots $-0.6623589\ldots \pm (0.5622795\ldots)i$ and $1.324717957\ldots$. By Theorem 2.1, $\lim_{n\to\infty} a_n/a_{n-1} = 1.324717957\ldots$. It appears that the sequence $(a_n/a_{n-1})$ is strictly decreasing beginning at $n = 18$.

## 3. Linear Complementary Equations of Order 0

Here we consider linear complementary equations of the form

$$a_n = v_0b_n + v_1b_{n-1} + \cdots + v_nb_0.$$

**Lemma 3.1.** *Suppose that $(a_n)$ and $(b_n)$ are complementary sequences. Let $\alpha_*$ and $\alpha^*$ be, respectively, the lower and upper asymptotic density of the set $\{a_n\}$, and let $\beta_*$ and $\beta^*$ be, respectively, the lower and upper asymptotic density of the set $\{b_n\}$. Then*

$$\alpha_* + \beta^* = \alpha^* + \beta_* = 1.$$

*Proof.* We begin with standard definitions: $\alpha_* = \liminf_{n\to\infty} A_n/n$, where $A_n$ is the number of terms $a_k$ that are $\leq n$, and $\beta^* = \limsup_{n\to\infty} B_n/n$, where $B_n$ is the number of terms $b_k$ that are $\geq n$. Suppose that $\alpha_* + \beta^* \neq 1$.

Case 1: $\beta^* < 1 - \alpha_*$. Let $\epsilon > 0$ satisfy $\beta^* < \epsilon < \alpha_*$. Let $n_i$ be a sequence such that $A_{n_i}/n_i \to \alpha_*$. For large enough $i$, we have $\beta^* < \epsilon < 1 - A_{n_i}/n_i$, so that $\beta^* < \epsilon < 1-(1-B_{n_i}/n_i)$, whence $\beta^* < \epsilon < B_{n_i}/n_i$ for infinitely many $n_i$, contrary to the definition of $\beta^*$.

Case 2: $\beta^* > 1 - \alpha_*$. Let $\epsilon > 0$ satisfy $\beta^* > \epsilon > 1 - \alpha_*$. Let $n_j$ be a sequence such that $B_{n_j}/n_j \to \beta^*$. Then for infinitely many $j$,, we have $B_{n_j}/n_j > \epsilon > 1 - \alpha_*$. Consequently, $1 - A_{n_j}/n_j > \epsilon > 1 - \alpha_*$, which implies $A_{n_j}/n_j < 1 - \epsilon < \alpha_*$, contrary to the definition of $\alpha_*$.

Therefore, $\alpha_* + \beta^* = 1$. The method of proof shows also that $\alpha^* + \beta_* = 1$. □

**Theorem 3.2.** *Suppose that $v_j, v_{j+1}, \ldots, v_k, q,$ and $r$ are integers such that $q + v > 1$, where $v = v_j + v_{j+1} + \cdots + v_k$. Suppose that complementary sequences $(a_n)$ and $(b_n)$ satisfy*

$$a_n = qn + r + v_j b_{m_n - j} + v_{j+1} b_{m_n - j - 1} + v_k b_{m_n - k}, \tag{3.1}$$

*where $m_n \geq n$.*

*If $q = 0$, then $\lim_{n\to\infty} a_n/n = 1/(v+1)$ and $\lim_{n\to\infty} b_n/n = v/(v+1)$. If $q \neq 0$, then*

$$\lim_{n\to\infty} \frac{a_n}{n} = \frac{q + v + 1 + \sqrt{(q+v+1)^2 - 4q}}{2q}, \tag{3.2}$$

$$\lim_{n\to\infty} \frac{b_n}{n} = \frac{q + v - 1 + \sqrt{(q+v+1)^2 - 4q}}{2v}. \tag{3.3}$$

*Proof.* Following Pietro Majer [7], let $\alpha_*$ and $\alpha^*$ be as in Lemma 3.1, so that

$$\limsup_{n\to\infty} \frac{a_n}{n} = 1/\alpha_* \ , \qquad \liminf_{n\to\infty} \frac{a_n}{n} = 1/\alpha^*,$$

$$\limsup_{n\to\infty} \frac{b_n}{n} = 1/\beta_* \ , \qquad \liminf_{n\to\infty} \frac{b_n}{n} = 1/\beta^* \ ,$$

where $\alpha_* + \beta^* = \alpha^* + \beta_* = 1$, by Lemma 3.1. By (3.1),

$$\frac{a_n}{n} = q + \frac{r}{n} + v_j \frac{b_{n-j}}{n} + \cdots v_k \frac{b_{n-k}}{n},$$

so that

$$\liminf_{n\to\infty} \frac{a(n)}{n} \geq q + \liminf_{n\to\infty} v_j \frac{b_{n-j}}{n} + \cdots + \liminf_{n\to\infty} v_k \frac{b_{n-k}}{n}, \tag{3.4}$$

whence $q + v/\beta^* \leq 1/\alpha^*$. Substituting $1 - \alpha_*$ for $\beta^*$ then yields

$$q\alpha^*(1 - \alpha_*) + v\alpha^* + \alpha_* \leq 1. \tag{3.5}$$

Similarly,

$$1 \leq q\alpha_*(1 - \alpha^*) + v\alpha_* + \alpha^*, \tag{3.6}$$

so that

$$\begin{gathered} q\alpha^* - q\alpha^*\alpha_* + v\alpha^* + \alpha_* \leq q\alpha_* - q\alpha_*\alpha^* + v\alpha_* + \alpha^* \\ (q + v - 1)\alpha^* \leq (q + v - 1)\alpha_* \\ \alpha^* \leq \alpha_*. \end{gathered}$$

As we also have $\alpha^* \geq \alpha_*$, it follows that $\alpha^* = \alpha_*$, so that $\lim a_n/n$ exists and equals $\alpha^*$, and $\lim b_n/n$ exists and equals $\beta^*$.

If $q = 0$, then (3.5) and (3.6) yield $\alpha^* = 1/(v+1)$ and $\beta^* = v/(v+1)$. If $q \neq 0$, then (3.5) and (3.6) yield

$$q\alpha^*(1-\alpha_*) + v\alpha^* + \alpha_* = 1, \tag{3.7}$$

from which (3.2) and (3.3) follow. □

**Example 4.** For $a_n = b_{n-1} + b_{n-2}$, with $a_0 = 1, a_1 = 1$, we have $\lim_{n\to\infty} a_n/n = 1/3$, and we conjecture that for every $n \geq 0$, the set $\{3n - 1 - a_n : n \geq 0\}$ is simply $\{0, 1, 2\}$; see A022424.

**Example 5.** For $a_n = n + b_{n-1} + b_{n-2}$ with $a_0 = 1, a_1 = 6$, we have $\lim_{n\to\infty} a_n/n = 2 + \sqrt{3}$ and $\lim_{n\to\infty} b_n/n = (1 + \sqrt{3})/2$.

As mentioned in the Introduction, we include here a code that enables complementarity testing for a certain class of (possibly) complementary equations.

```
mex[list_, start_] := (NestWhile[# + 1 &, start, MemberQ[list, #] &]);
test[sum_, n_] :=
 Module[{a = {}, b = {1}, fVals}, Map[(fVals = sum[#];
     Do[AppendTo[b,
       mex[Flatten[{a, b}], Last[b]]], {Max[fVals] - Length[b] + 1}];
     AppendTo[a, Total[Map[b[[1 + #]] &, fVals]]]) &, Range[n] - 1];
  {If[Position[Differences[Sort[Flatten[{a, b}]]], 0, 1] == {},
    "Complementary so far",
    "Not Complementary " <>
     ToString[Flatten[Split@Sort@a, {2}][[2]]] <> " repeated"], {a, b}}]
(* Following are sample inputs for 100 terms of $(a_n)$ and $(b_n).$ *)
test[{#, 1} &, 100]
test[{#, #} &, 100]
test[{#, #, #} &, 100]
test[{#, Floor[#/2]} &, 100]
test[{#, Floor[#/2] - Floor[#/3]} &, 100]
```

The code above tests five pairs of complementary sequences $(a_n)$ and $(b_n)$, with initial value $b_0 = 1$, and other values determined by the five lines of input that each start with "test". Following are translations of the inputs, along with outputs:

`test[{#, 1} &, 100]`

is the input for $a_n = b_n + b_1$. The output consists of 100 terms of sequences A014601 and A042963, with "Complementary so far". This seems like a good example for showing how the first few values are determined: $b_0 = 1$ by decree, and then $a_0$ must be $b_0 + b_1$, which must exceed 3, so that $b_1$ must be 2, so that $a_0$ must be 3. Then $a_1$ must be $b_1 + b_1 = 4$. Then the requirement that $a_2 = b_2 + b_1$ forces $a_2$ to be at least 7, implying that $b_2 = 5$, $b_3 = 6$, and $a_2 = 7$, and so on.

`test[{#, #} &, 100]`

is the input for $a_n = b_n + b_n$. Output: A036554 and A003159 .

`test[{#, #, #} &, 100]`

is the input for $a_n = 3b_n$. Output: A145204 and A007417.

```
test[{#, Floor[#/2]} &, 100]
```

is the input for $a_n = b_n + b_{\lfloor n/2 \rfloor}$. Output: A304451 and A304452.

```
test[{#, Floor[#/2] - Floor[#/3]} &, 100]
```

is the input for $a_n = b_n + b_{\lfloor n/2 \rfloor} - b_{\lfloor n/3 \rfloor}$. Output: `Not Complementary`, followed by a list of numbers in both sequences: $58, 98, 137, 148, 184$.

## 4. Linear Complementary Equations of Order 1

Here we consider first-order complementary equations of the form

$$a_n = ca_{n-1} + v_0 b_n + v_1 b_{n-1} + \cdots + v_n b_0.$$

Clearly, for $n \geq 2$, the general term $a_n$ can be expressed in terms of $a_0$ and the sequences $(b_n)$ and $(v_n)$. In order to determine this dependence explicitly, we start with

$$a_1 = ca_0 + v_0 b_1 + v_1 b_0, \tag{4.1}$$

and continue inductively,

$$\begin{aligned} a_n =& c^n a_0 + v_0 b_n + (cv_0 + v_1) b_{n-1} + (c^2 v_0 + c_1 v_1 + v_2) b_{n-2} \\ &+ \cdots + (c^{n-2} v_0 + c^{n-3} v_1 + \cdots + v_{n-2}) b_2 \\ &+ \cdots + (c^{n-1} v_0 + c^{n-2} v_1 + \cdots + v_{n-1}) b_1 \\ &+ \cdots + (c^{n-1} v_1 + c^{n-2} v_2 + \cdots + v_n) b_0. \end{aligned}$$

This result can be written as a sum:

$$a_n = c^n a_0 + y_{0n} + y_{1n} + \cdots + y_{nn},$$

where $y_{in}$, for $n \geq 1$ and $i = 0, 1, \ldots, n$ are given by the matrix equation $Y = BVC$, where

$$Y = (y_{0n}, y_{1n}, \ldots, y_{nn})^{tr},$$

$$B = \begin{pmatrix} b_0 & 0 & 0 & \cdots & 0 \\ 0 & b_1 & 0 & \cdots & 0 \\ 0 & 0 & b_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & b_n \end{pmatrix}$$

$$V = \begin{pmatrix} v_1 & v_2 & v_3 & v_4 & \cdots & v_{n-1} & v_n \\ v_0 & v_1 & v_2 & v_3 & \cdots & v_{n-2} & v_{n-1} \\ 0 & v_0 & v_1 & v_2 & \cdots & v_{n-3} & v_{n-2} \\ 0 & 0 & v_0 & v_1 & \cdots & v_{n-4} & v_{n-3} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & v_0 \end{pmatrix}$$

$$C = (c^{n-1}, c^{n-2}, \ldots c, 1, 1)^{tr}.$$

Note the dimensions of the matrices in the product $BVC : (n+1, n+1) \times (n+1, n) \times (n, 1)$.

**Example 6.** If $a_n = a_{n-1} + b_n$, with initial value $a_0 = 1$, then $(a_n) = (1, 4, 9, 15, 22, 30, \ldots) =$ A022443.

**Example 7.** If $a_n = 2a_{n-1} + b_n + b_{n-1}$, with initial value $a_0 = 1$, then

$$(a_n) = (1, 7, 21, 51, 113, 240, 497, \ldots).$$

## 5. Linear Complementary Equations of Order 2

Here we consider second-order complementary equations of the form

$$a_n = ca_{n-1} + da_{n-2} + v_0 b_n + v_1 b_{n-1} + \cdots + v_n b_0.$$

In order to obtain a matrix representation for $a_n$ that depends on $a_0$ and $a_1$, we call upon the generalized $(c, d)$-Fibonacci polynomials $U_n = U_n(c, d)$, defined as follows: $U_0 = 0$, $U_1 = 1$, and $U_n = cU_{n-1} + dU_{n-2}$. We have

$$\begin{aligned} a_3 &= ca_2 + da_1 + v_0 b_2 + v_1 b_1 + v_2 b_0) + da_1 + v_0 b_3 + x_1 b_2 + v_2 b_1 + v_3 b_0 \\ &= a_1 U_3 + a_0 dU_2 + v_0 b_3 + (v_0 U_2 + v_1 U_1) b_2 + (v_1 U_2 + v_2 U_1) b_1 + (v_2 U_2 + v_3 U_1) b_0. \end{aligned}$$

Continuing, we reach a representation for $a_n$ as a linear combination, similar to that in the preceding section:

$$a_n = a_1 U_n + a_0 dU_{n-1} + y_{0n} + y_{1n} + \cdots + y_{nn},$$

where $Y$ and $B$ are as in Section 4, $Y = BVU$, where

$$V = \begin{pmatrix} v_2 & v_3 & v_4 & \cdots & v_n \\ v_1 & v_2 & v_3 & \cdots & v_{n-1} \\ v_0 & v_1 & v_2 & \cdots & v_{n-2} \\ 0 & v_0 & v_1 & \cdots & v_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & v_0 \end{pmatrix}$$

and

$$U = (U_{n-1}, U_{n-2}, \ldots, U_2, U_1)^{tr}.$$

Note the dimensions of the matrices in the product $BVU : (n+1, n+1) \times (n+1, n-1) \times (n-1, 1)$.

**Example 8.** If $a_n = a_{n-1} + a_{n-2} + b_n$, with initial values $a_0 = 1, a_1 = 2$, then $(a_n) = (1, 2, 8, 16, 31, 56, \ldots) =$ A295949. Here, $(a_n/a_{n-1})$ converges rapidly to the golden ratio.

**Example 9.** If $a_n = a_{n-1} + a_{n-2} + b_0 + b_1 + \cdots + b_{n-1}$, with initial values $a_0 = 1, a_1 = 2, a_2 = 3$, then $(a_n) = (1, 2, 10, 24, 52, 101, \ldots) =$ A295053.

For many choices of $(v_n)$, the sum

$$V_n = v_0 b_n + v_1 b_{n-1} + \cdots + v_n b_0$$

stays small enough that $\lim_{n\to\infty} V_n/a_{n-1} = 0$. In such a case, we find from

$$a_n/a_{n-1} = c + da_{n-2}/a_{n-1} + V_n/a_{n-1}$$

and $r = \lim_{n\to\infty} a_n/a_{n-1}$, that $r = c + d/r$, so that $r = (c + \sqrt{c^2 + 4d})/2$. The closeness of $a_n/a_{n-1}$ to $r$ and of $a_n$ to $r^n$ are measured by the ratio-sum,

$$\sum_{n=1}^{\infty} |a_n/a_{n-1} - r|$$

and the limiting power ratio, $\lim_{n\to\infty} a_n/r^n$, respectively. Table 4 shows a few examples for the complementary equation $a_n = a_{n-1} + a_{n-2} + b_n$. The first column gives initial values in abbreviated form; e.g. "$1, 3; 2, 4, 5$" means that $a_0 = 1, a_1 = 3; b_0 = 2, b_1 = 4, b_2 = 5$.

Table 3. Ratio-sums and limiting power ratios

| initial values | ratio-sum | limiting power ratio |
|---|---|---|
| $1,2;3,4,5$ | $3.975789\ldots$ | $6.136385\ldots$ |
| $1,3;2,4,5$ | $6.210327\ldots$ | $6.514710\ldots$ |
| $1,4;2,3,5$ | $4.289969\ldots$ | $6.920208\ldots$ |
| $1,5;2,3,4$ | $4.845853\ldots$ | $7.090700\ldots$ |
| $2,3;1,4,5$ | $2.571971\ldots$ | $6.749918\ldots$ |
| $2,4;1,3,5$ | $2.427179\ldots$ | $7.171351\ldots$ |
| $3,4;1,2,5$ | $1.916978\ldots$ | $7.432138\ldots$ |

If $r = 1$, the hypothesis of Theorem 2.1 fails, although the conclusion appears to be valid. Example 10 shows what can happen.

**Example 10.** If $a_n = 2a_{n-1} - a_{n-2} + b_n$ with initial values $a_0 = 1, a_1 = 2$, then $(a_n) = (1, 2, 8, 20, 39, 67, \ldots) =$ A305129. Here, the ratio-sum and limiting power ratio are both infinite. We conjecture that the 3rd difference sequence of $(a_n)$ consists entirely of 1s and 2s.

## 6. Anti-Fibonacci system of complementary equations and generalizations

The anti-Fibonacci sequence, A075326, is the solution $(c_n)$ for the following system of three complementary equations: $a_n = \text{mex}$, $b_n = \text{mex}$, $c_n = a_n + b_n$; specifically,

$$a_n = \text{mex}(\{a_0, \ldots, a_{n-1}, b_0, \ldots, b_{n-1}, c_0, \ldots, c_{n-1}\}),$$
$$b_n = \text{mex}(\{a_0, \ldots, a_{n-1}, a_n, b_0, \ldots, b_{n-1}, c_0, \ldots, c_{n-1}\}).$$

The sequences $(a_n), (b_n), (c_n)$ partition $\mathbb{N}$ and start out as in Table 4. (Each of the three equations is a complementary equation in the sense that its solution sequence, as a set, is complementary to the union of solution sets of the other two equations.)

Table 4. Sequences $(a_n), (b_n), (c_n)$

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| $a_n$ | 1 | 4 | 6 | 8 | 11 | 14 | 16 | 19 | 21 |
| $b_n$ | 2 | 5 | 7 | 10 | 12 | 15 | 17 | 20 | 22 |
| $c_n$ | 3 | 9 | 13 | 18 | 23 | 29 | 33 | 39 | 43 |

Another, perhaps surprising, way to generate the anti-Fibonacci sequence $(c_n)$ stems from observing that its difference sequence, specifically

$$(\Delta c_n) = (6, 4, 5, 5, 6, 4, 6, 4, 6, 4, 5, 5, 6, 4, 5, 5, \ldots),$$

regarded as a word, consists of concatenated blocks 55 and 64. Substituting 0 for 55 and 1 for 64 gives

$$A035263 : 101110101011101110111 \cdots,$$

which appears to be the fixed word of the morphism $\{0 \to 11, 1 \to 10\}$ applied to 1 :

$$1 \to 10 \to 1011 \to 10111010 \to 1011101010111011 \to \cdots.$$

Reversing the procedure, we can generate $(c_n)$ as in the following Mathematica code:

```
Accumulate[
 Prepend[Flatten[
   Nest[Flatten[# /. {0 -> {1, 1}, 1 -> {1, 0}}] &, {0}, 7] /.
    Thread[{0, 1} -> {{5, 5}, {6, 4}}]], 3]]
```

We confirmed that this code generates at least the first 16,000 terms of the sequence $(c_n)$.

The anti-Fibonacci system generalizes to an anti-tribonacci system (c.f. A265389), in the obvious manner: $a_n = \text{mex}$, $b_n = \text{mex}$, $c_n = \text{mex}$, $d_n = a_n + b_n + c_n$. These four sequences partition $\mathbb{N}$, and the anti-tribonacci sequence,

$$(d_n) = (6, 16, 27, 36, 46, 57, 66, 75, 87, 96, 101, \ldots)$$

can be generated from its difference sequence,

$$\Delta(d_n) = (10, 11, 9, 10, 11, 9, 9, 12, 9, 10, 11, 9, \ldots). \tag{6.1}$$

To see how this works, substitute

$$9 \to 1, 10 \to 2, 11 \to 3, 12 \to 4$$

into (6.1) and write the result as the word

$$231231141231231141231321131 \cdots . \tag{6.2}$$

Then put $A = 114, B = 123, C = 132$, so that (6.2) gives

$$23BABBABCABBABBABCABBA \cdots .$$

Next, substitute $23 \to 1, A \to 0, B \to 1, C \to 2$ to get

$$110110120110110120110100 1 \cdots ,$$

which, we conjecture, is the fixed point of the morphism

$$\{0 \to 120, 1 \to 110, 2 \to 100\}$$

applied to 1.

This procedure for generating (the first 1,594,232 terms of) the fixed point is performed by the following code:

```
Nest[Flatten[# /. {0 -> {1, 2, 0}, 1 -> {1, 1, 0},
                    2 -> {1, 0, 0}}] &, {0}, 13];
```

For a generalization to the anti-tetranacci sequence, see A299405, and to push even further in this direction, for $m \geq 2$, define the anti-$m$-nacci sequence $(a_m(n))$, as follows: let $a_i(n) = \text{mex}$ for $i = 1, 2, \ldots, m-1$, and let

$$a_m(n) = a_1(n) + a_2(n) + \cdots + a_{m-1}(n).$$

Is $(a_m(n))$ the fixed point of a morphism? The conjectured answer is yes! We take $m = 7$ to illustrate a procedure, starting with

$$(a_7(n)) = (1, 1, 2, 7, 1, 1, 1, 1, 1, 3, 6, 1, 1, 1, 1, 1, 4, 5, 1, 1, 1, 1, 1, 4, 5, 1, \ldots). \tag{6.3}$$

Partition $(a_7(n))$ into consecutive blocks of length 7, as indicated by Table 5.

Table 5. Blocks of length 7

| | |
|---|---|
| (1,1,2,7,1,1,1) | (1,1,3,6,1,1,1) |
| (1,1,4,5,1,1,1) | (1,1,4,5,1,1,1) |
| (1,1,1,6,3,1,1) | (1,1,1,7,2,1,1) |
| (1,1,1,6,3,1,1) | (1,1,1,7,2,1,1) |
| (1,1,1,8,1,1,1) | (1,1,2,7,1,1,1) |
| (1,1,3,6,1,1,1) | (1,1,4,5,1,1,1) |
| (1,1,3,6,1,1,1) | (1,1,1,6,3,1,1) |
| (1,1,1,7,2,1,1) | (1,1,1,8,1,1,1) |

Delete duplicates and sort lexicographically, getting

$$(1,1,1,6,3,1,1),(1,1,1,7,2,1,1),(1,1,1,8,1,1,1),(1,1,2,7,1,1,1),$$

$$(1,1,3,6,1,1,1),(1,1,4,5,1,1,1)$$

from what is shown in Table 5, and seeking further, find the seventh block, $(1,1,5,4,1,1,1)$. Next, number the blocks as $0,1,2,3,4,5,6$, and substitute them into (6.3) to get

$$(3,4,5,5,0,1,2,3,4,5,4,0,1,2,3,4,5,3,0,1,2,3,4,5,3,0,1,2,3,4,5,6,\ldots), \tag{6.4}$$

which is the sequence we wish to generate using a morphism. Partition (6.4) into blocks of length 7; again, delete duplicates and sort into descending order 7 distinct blocks, obtaining

$$(3,4,5,6,2,1,2),(3,4,5,6,1,1,2),(3,4,5,6,0,1,2),(3,4,5,5,0,1,2),$$

$$(3,4,5,4,0,1,2),(3,4,5,3,0,1,2),(3,4,5,1,1,4,5).$$

Numbering these as $0,1,\ldots,6$, we concatenate blocks as follows:

$$(\text{block } 3)(\text{block } 4)(\text{block } 5)(\text{block } 5)(\text{block } 0)\cdots.$$

This concatenation yields (6.4), showing that it is the fixed point of the morphism. Following is a code for this morphism:

```
morph=Take[Nest[Flatten[#/.Thread[Range[n]-1->subs]]&,{0},9],100]
```

Now putting the procedure together for arbitrary $n \geq 2:$

```
stringPartition[s_,n_]:=StringCases[s,Repeated[_,{n}]]

Clear[f];
m=7;
f[n_]:=Block[{a={},r=Range@n,s},Do[If[Length@r>m+1,s=Total@Take[r,m];
AppendTo[a,s];r=Drop[#,m]&@DeleteCases[r,x_/;x==s],Break[]],{k,n}];a];
d[m]=f@10000;

diffs[m]=Differences[d[m]]-m^2+1;
replacements=Union[stringPartition[StringJoin[Map[ToString,diffs[m]]],m]];
str=StringReplace[StringJoin[Map[ToString,diffs[m]]],
Table[replacements[[nn]]->FromCharacterCode[47+nn],{nn,m}]];

subs=ToExpression[Characters[Reverse[Union[stringPartition[str,m]]]]]

morph=Take[Nest[Flatten[#/.Thread[Range[m]-1->subs]]&,{0},8],200]

result=Accumulate[Prepend[Flatten[morph/.Thread[Range[m]-1
->ToExpression[Characters[replacements]]]]+m^2-1,1/2 m (1+m)]]
Take[d[m],#]-Take[result,#]&[Min[Length[d[m]],Length[result]]]
(*should be a bunch of zeros*)
```

Explicitly, for $m = 7$,

```
morph = Take[Nest[Flatten[#/.Thread[{0,1,2,3,4,5,6}-> {{3,4,5,6,2,1,2},
{3,4,5,6,1,1,2},{3,4,5,6,0,1,2},{3,4,5,5,0,1,2},{3,4,5,4,0,1,2},
{3,4,5,3,0,1,2},{3,4,5,1,1,4,5}}]] &,{0},8],200]
Accumulate[Prepend[Flatten[morph/.Thread[{0,1,2,3,4,5,6}->{{1,1,1,6,3,1,1},
{1,1,1,7,2,1,1},{1,1,1,8,1,1,1},{1,1,2,7,1,1,1},{1,1,3,6,1,1,1},
{1,1,4,5,1,1,1},{1,1,5,4,1,1,1}}]]+48,28]]
```

**Example 11.** A second kind of generalization of the anti-Fibonacci system is represented by the partition of $\mathbb{N}$ into 3 sequences defined by $a_n =$ least new $k$, $b_n =$ least new $k \geq a_n + n$, and $c_n = a_n + b_n$; see A298870.

Table 6. Sequences $(a_n), (b_n), (c_n)$

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| $a_n$ | 1 | 4 | 6 | 8 | 11 | 14 | 15 | 17 | 19 |
| $b_n$ | 2 | 5 | 7 | 10 | 12 | 16 | 20 | 22 | 25 |
| $c_n$ | 3 | 9 | 13 | 18 | 23 | 30 | 35 | 39 | 44 |

Let $x = \lim_{n\to\infty} a_n/n$. Then $x = \lim_{n\to\infty} b_n/n = x+1$, $x = \lim_{n\to\infty} c_n/n = 2x+1$, and

$$\frac{1}{x} + \frac{1}{x+1} + \frac{1}{2x+1} = 1,$$

with positive solution

$$x = \frac{1}{3} + \frac{2\sqrt{7}}{3} \cos \frac{1}{3} \tan^{-1} \frac{3\sqrt{111}}{67} = 2.078162587\ldots$$

A third kind of generalization of the anti-Fibonacci system comes from replacing $a_n + b_n$; e.g., for $2a_n + b_n$, see A304500. In the next example, we take $c_n = a_n + b_n + n$ :

**Example 12.** Here, the system consists of three sequences that partition $\mathbb{N}$: $a_n =$ mex, $b_n =$ mex, $c_n = a_n + b_n + n$.

Table 7. Sequences $(a_n), (b_n), (c_n)$

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| $a_n$ | 1 | 4 | 6 | 8 | 11 | 13 | 16 | 18 | 21 |
| $b_n$ | 2 | 5 | 7 | 9 | 12 | 14 | 17 | 19 | 22 |
| $c_n$ | 3 | 10 | 15 | 20 | 27 | 32 | 39 | 44 | 51 |

Let

$$x = \lim_{n\to\infty} a_n/n, \quad y = \lim_{n\to\infty} b_n/n, \quad z = \lim_{n\to\infty} c_n/n.$$

It is easy to prove that $x = y = 1 + \sqrt{2}$ and $z = 3 + 2\sqrt{2}$, and we conjecture that

$$\frac{\sqrt{2}}{2} < a_n - nx < 1 + \frac{\sqrt{2}}{2} < b_n - nx < 2 + \frac{\sqrt{2}}{2}$$

and

$$1 + \sqrt{2} < c_n - nz < 3 + \sqrt{2}$$

for all $n$.

## 7. Examples of complementary equations and systems in OEIS

In OEIS, entries for certain sequences related to complementary equations include, in the Comments section, a guide to related sequences. We conclude this section with a list of such entries in Table 8 and a similar list in Table 9 for systems of complementary equations, as introduced in Section 6:

Table 8. OEIS complementary equations with guides

| | |
|---|---|
| A022424 | $a_n = b_{n-1} + b_{n-2}$ |
| A022940 | $a_n = a_{n-1} + b_{n-2}$ |
| A293076 | $a_n = a_{n-1} + a_{n-2} + b_{n-2} + 2n$ |
| A293358 | $a_n = a_{n-1} + a_{n-2} + b_{n-1}$ |
| A293765 | $a_n = a_{n-1} + a_{n-2} + b_{n-1} + 2$ |
| A294532 | $a_n = a_{n-1} + a_{n-2} + b_{n-2}$ |
| A294414 | $a_n = a_{n-1} + a_{n-2} - b_{n-1} + b_{n-2}$ |
| A294476 | $a_n = a_{n-2} + b_{n-1} + 1$ |
| A294860 | $a_n = a_{n-2} + b_{n-2}$ |
| A295053 | $a_n = a_{n-1} + a_{n-2} + b_0 + b_{n-1} + \cdots + b_{n-1}$ |
| A295357 | $a_n = a_{n-1} + a_{n-2} + b_{n-1} + b_{n-2} + b_{n-3}$ |
| A295613 | $a_n = 2a_{n-1} - a_{n-3} + b_{n-1}$ |
| A295862 | $a_n = a_{n-1} + a_{n-2} + b_n$ |
| A296000 | $a_n = a_0 b_{n-2} + a_1 b_{n-2} + \cdots + a_{n-2} b_0$ |
| A296245 | $a_n = a_{n-1} + a_{n-2} + b_n^2$ |
| A297800 | $a_n = a_1 b_n - a_0 b_{n-1} + 2n$ |
| A297830 | $a_n = a_1 b_{n-1} - a_0 b_{n-2} + 2n$ |
| A304799 | $a_n = b_n + b_{2n}$ |

Table 9. OEIS systems with guides

| entry | $a_n$ | $b_n$ | $c_n$ |
|---|---|---|---|
| A297469 | mex | $a_n + c_{n-1}$ | mex |
| A298868 | mex | mex $\geq a_n + n$ | $a_n + b_n$ |
| A299634 | mex $\geq 2b_{n-1}$ | mex | $a_n + b_n$ |

## 8. Notes regarding Mathematica codes

This section shows Mathematica (version $\geq 7$) codes used to generate complementary sequences discussed in this article. These may prove useful for further research.

Code for Example 1.

```
a[0] = 1; a[1] = 2; b[0] = 3; b[1] = 4; b[2] = 5;
a[n_] := a[n] = 4 a[n - 1] - 4 a[n - 2] + b[n];
j = 1; While[j < 12, k = a[j] - j - 1;
 While[k < a[j + 1] - j + 1, b[k] = j + k + 2; k++]; j++];
Table[a[n], {n, 0, 30}]
Table[b[n], {n, 0, 30}]
k (* k = number of terms of a( ) and b( ) that are computed *)
Column[N[Table[a[n]/a[n - 1], {n, 1, 100}], 10 ]]
```

A first code for Example 2.

```
a[0] = 1; a[1] = 2; b[0] = 3;
a[n_] := a[n] = 4 a[n - 1] - 4 a[n - 2] + b[n - 2];
j = 1; While[j < 12, k = a[j] - j - 1;
 While[k < a[j + 1] - j + 1, b[k] = j + k + 2; k++]; j++];
Table[a[n], {n, 0, 30}]
```

A second code for Example 2. This code uses the mex function and displays several terms of $(n)$, $(a_n)$, and $(b_n)$.

```
mex[list_, start_] := (NestWhile[# + 1 &, start, MemberQ[list, #] &]);
a = {1, 2}; b = {};
Do[AppendTo[b, mex[Flatten[{a, b}], If[b == {}, 1, Last[b]]]];
  AppendTo[a, 4 a[[-1]] - 4 a[[-2]] + Last[b]], {20}];
Grid[{Join[{"n"}, Range[0, Length[b] - 1]],
  Join[{"a(n)"}, Take[a, Length[b]]], Join[{"b(n)"}, b]},
 Alignment -> ".",
 Dividers -> {{2 -> Red, -1 -> Blue}, {2 -> Red, -1 -> Blue}}]
```

Code for Example 12. The following code generates the first $z$ terms of the three sequences in Example 12. In Mathematica terminology, they are "lists" denoted by a, b, c, for which the indexing starts at 1.

```
z=1000;w=100;
mex[list_,start_]:=(NestWhile[#+1&,start,MemberQ[list,#]&]);
a={};b={};c={};
Do[AppendTo[a,mex[Flatten[{a,b,c}],If[Length[a]==0,1,Last[a]]]];
AppendTo[b,mex[Flatten[{a,b,c}],Last[a]]];
AppendTo[c,Last[a]+Last[b]+Length[a]-1],{z}];
Take[a,w]
Take[b,w]
Take[c,w]
Map[N[a[[#]]-(#-1)*(1+Sqrt[2])]&,Range[w]]
Map[N[b[[#]]-(#-1)*(1+Sqrt[2])]&,Range[w]]
Map[N[c[[#]]-(#-1)*(3+2 Sqrt[2])]&,Range[w]]
```

## References


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