A TRIANGLE WITH SIDES LENGTHS OF A RATIONAL POWER OF THE PLASTIC CONSTANT

KOUICHI NAKAGAWA

Abstract. Using the golden ratio ϕ, a triangle whose side length ratio can be expressed as $1 : \sqrt{\phi} : \phi$ represents a right triangle since the golden ratio has the property of $\phi^2 = \phi + 1$ and therefore satisfies $1^2 + (\sqrt{\phi})^2 = \phi^2$. This triangle is called the Kepler triangle.

As in the case of the Kepler triangle, in this study we determine triangles where the length of the three sides is expressed using only the constant obtained from the linear recurrence sequence (golden ratio, plastic constant, tribonacci constant and supergolden ratio).

1. Introduction

The Fibonacci numbers are defined by the recurrence

$$F_{n+2} = F_{n+1} + F_n$$

with the initial values $F_0 = 0$, $F_1 = 1$, and the limit of the ratios of successive terms converges to

$$\lim_{n\to\infty} \frac{F_n}{F_{n-1}} = \phi = \frac{1+\sqrt{5}}{2} \approx 1.61803\cdots;$$

ϕ is called the *golden ratio*. One property of the golden ratio is $\phi^2 = \phi + 1$; it is also an algebraic integer of degree 2.

By the property of the golden ratio and the Pythagorean theorem, we have

$$\phi^2 = \phi + 1 \iff 1^2 + (\sqrt{\phi})^2 = \phi^2.$$

Since the ratio of the lengths of the three sides is $1 : \sqrt{\phi} : \phi$, this triangle is a right triangle, called the *Kepler triangle*. In addition, the following are known as characteristic triangles. An isosceles triangle whose three side length ratios can be expressed as $\phi : \phi : 1$ is called a *golden triangle*. The three angles are $36^\circ - 72^\circ - 72^\circ$. An isosceles triangle whose three sides length ratio can be expressed as $1 : 1 : \phi$ is called a *golden gnomon*. The three angles are $108^\circ - 36^\circ - 36^\circ$.

From the above, it was found that there are many triangles in which three sides have an integer power of the golden ratio and at least one of the internal angles is an integer degree. Extending these, is there any other triangle whose three sides are rational powers of constants obtained from the linear recurrence sequence and at least one of the internal angles is an integer degree, and how many? I think these questions arise naturally.

1.1. The constant obtained from a linear recurrence sequence.

The following are known as constants obtained from a linear recurrence sequence.

Definition 1.1 (Padovan Sequence)**.** *The Padovan sequence is defined by the recurrence*

$$P_{n+3} = P_{n+1} + P_n$$

with the initial values $P_0 = P_1 = 0$, $P_2 = 1$.

The limit of the ratios of successive terms converges to

$$\lim_{n\to\infty} \frac{P_n}{P_{n-1}} = \rho \approx 1.32471795\cdots,$$

and ρ is called the *plastic constant*. Property of the golden ratio is $\rho^3 = \rho + 1$. It is also an algebraic integer of degree 3.

Definition 1.2 (Tribonacci sequence)**.** *The Tribonacci sequence is defined by the recurrence*

$$T_{n+3} = T_{n+2} + T_{n+1} + T_n$$

with the initial values $T_0 = T_1 = 0$*,* $T_2 = 1$*.*

The limit of the ratios of successive terms converges to

$$\lim_{n\to\infty} \frac{T_n}{T_{n-1}} = t \approx 1.83929\cdots,$$

and t is called the *Tribonacci constant*. A property of this constant is $t^3 = t^2 + t + 1$. It is also an algebraic integer of degree 3.

Definition 1.3 (Narayana's cows sequence)**.** *The Narayana's cows sequence is defined by the recurrence*

$$N_{n+3} = N_{n+2} + N_n$$

with the initial values $N_0 = N_1 = 0$*,* $N_2 = 1$*.*

The limit of the ratios of successive terms converges to

$$\lim_{n\to\infty} \frac{N_n}{N_{n-1}} = \psi \approx 1.46557\cdots,$$

and ψ is called the *supergolden ratio*. A property of this constant is $\psi^3 = \psi^2 + 1$. It is also an algebraic integer of degree 3.

2. Conditions for Special Triangles

2.1. **Condition.** Let $\triangle ABC$ be a triangle with $\angle C = \theta$. Using the law of cosine, we have

$$a^2 + b^2 - 2ab\cos\theta = c^2.$$

Suppose here that the ratio of the sides of the triangle is represented by rational powers of a positive constant $x(\neq 1)$. Let the ratio of the three sides length be expressed as

$$a : b : c = 1 : x^\alpha : x^\beta.$$

Hereafter, such a triangle will be denoted as $(0, \alpha, \beta)_x$. (Since $x^0 = 1$, we used 0.)

At this time, the relational expression of the three sides becomes

$$1 + x^{2\alpha} - 2x^\alpha \cos\theta = x^{2\beta}.$$

Remark 2.1. *Consider* $x = 1$*,* $\triangle ABC$ *is a regular triangle. So* $(0, 0, 0)_x$*. Therefore* x *is any constant.*

Theorem 2.2 (Trigonometric functions and rational numbers [1933, Lehmer])**.** *If* $n > 2$ *and* $(k, n) = 1$*, then* $2\cos 2\pi k/n$ *is an algebraic integer of degree* $\phi(n)/2$*, where* $\phi(n)$ *is the Euler's totient function. For positive* $n \neq 4$*,* $2\sin 2\pi k/n$ *is an algebraic integer of degree* $\phi(n)$*,* $\phi(n)/4$*, or* $\phi(n)/2$ *according as* $(n, 8) < 4$*,* $(n, 8) = 4$*, or* $(n, 8) > 4$*.*

Corollary 2.3 (Algebraic integers of degree 3 or less)**.** *The following can be said about* $0^\circ \leq \theta \leq 180^\circ$*.*

Degree 1: $n = 1, 2, 3, 4, 6$. *In other words,* $\theta = 0°, 60°, 90°, 120°, 180°$.
Degree 2: $n = 5, 8, 10, 12$. *In other words,* $\theta = 30°, 36°, 45°, 72°, 108°, 135°, 144°, 150°$.
Degree 3: $n = 9, 18$. *In other words,* $\theta = 20°, 40°, 80°, 100°, 140°, 160°$.

Remark 2.4. $n = 7, 14$ *is included in degree* 3. *However,* θ *is NOT an integer angle.*

Proposition 2.5. *Substituting the above values of* θ *into* $1 + x^{2\alpha} - 2x^{\alpha}\cos\theta = x^{2\beta}$ *gives the following relational expression.*

$\theta = 0(= 0°)$**:** $x^{2\beta} = (x^{\alpha} - 1)^2 \implies x^{\beta} = x^{\alpha} - 1$.
$\theta = \frac{\pi}{3}(= 60°)$**:** $x^{2\beta} = x^{2\alpha} - x^{\alpha} + 1$.
$\theta = \frac{\pi}{2}(= 90°)$**:** $x^{2\beta} = x^{2\alpha} + 1$.
$\theta = \frac{2\pi}{3}(= 120°)$**:** $x^{2\beta} = x^{2\alpha} + x^{\alpha} + 1$.
$\theta = \pi(= 180°)$**:** $x^{2\beta} = (x^{\alpha} + 1)^2 \implies x^{\beta} = x^{\alpha} + 1$.

2.2. **Trivial case.**

2.2.1. *Triangular condition obvious in* $\theta = \pi$ $(= 180°)$. Since the relational expression is $x^{\beta} = x^{\alpha} + 1$, it is obvious that the triangle $(0, \alpha, \beta)_x$ satisfies the condition.

Example 2.6. $(0, 1, 2)_{\phi}$, $(0, 1, 3)_{\rho}$, $(0, 2, 3)_{\psi}$.

Remark 2.7 ($\theta = 0$ $(= 0°)$)**.** $\angle C = 180° \iff \angle A = \angle B = 0°$. *So we can replace the contents of the above example and change it to* $(\alpha, \beta, 0)_x$ *and* $(0, \beta, \alpha)_x$.

2.2.2. *Triangular condition obvious in* $\theta = \frac{\pi}{2}$ $(= 90°)$. Since the relational expression is $x^{2\beta} = x^{2\alpha} + 1$, it is obvious that the triangle $(0, \frac{\alpha}{2}, \frac{\beta}{2})_x$ satisfies the condition.

Example 2.8. $\left(0, \frac{1}{2}, 1\right)_{\phi}$, $\left(0, \frac{1}{2}, \frac{3}{2}\right)_{\rho}$, $\left(0, 1, \frac{3}{2}\right)_{\psi}$.

2.2.3. *Triangular condition obvious in* $\theta = \frac{2\pi}{3}$ $(= 120°)$. Since the relational expression is $x^{2\beta} = x^{2\alpha} + x^{\alpha} + 1$, it is obvious that the triangle $(0, 1, \frac{3}{2})_t$ satisfies the condition.

2.3. **Non-trivial case.**

2.3.1. *Triangle* $(0, 2, \frac{5}{2})_x$ *in* $\theta = \dfrac{\pi}{2}$ $(= 90°)$. In this case it satisfies

$$x^5 = x^4 + 1 \iff \left(x^2 - x + 1\right)\left(x^3 - x - 1\right) = 0.$$

Since $x^2 - x + 1 = (x - \frac{1}{2})^2 + \frac{3}{4} > 0$, we have

$$x^5 = x^4 + 1 \implies x^3 - x - 1 = 0.$$

If this triangle is $\left(0, 2, \frac{5}{2}\right)_{\rho}$, since the property that the plastic constant satisfies is

$$\rho^3 = \rho + 1,$$

so this triangle obviously satisfies the condition.

Remark 2.9 ($\theta = \pi$ $(= 180°)$)**.** $(0, 4, 5)_{\rho}$ *also satisfies the condition.*

2.3.2. *Triangle* $(0,2,\frac{3}{2})_x$ *in* $\theta = \dfrac{\pi}{3}$ $(=60°)$. In this case it satisfies

$$x^3 = x^4 - x^2 + 1 \iff (x-1)\left(x^3 - x - 1\right) = 0.$$

Since $x \neq 1$, we have

$$x^3 = x^4 - x^2 + 1 \implies x^3 - x - 1 = 0.$$

If this triangle is $\left(0,2,\frac{3}{2}\right)_\rho$, since the property that the plastic constant satisfies is

$$\rho^3 = \rho + 1,$$

so this triangle obviously satisfies the condition.

2.3.3. *Triangle* $(0,3,\frac{5}{2})_x$ *in* $\theta = \dfrac{\pi}{3}$ $(=60°)$. In this case it satisfies

$$x^5 = x^6 - x^3 + 1 \iff (x-1)\left(x^2+1\right)\left(x^3 - x - 1\right) = 0.$$

Since $x \neq 1$ and $x^2 + 1 > 0$, we have

$$x^5 = x^6 - x^3 + 1 \implies x^3 - x - 1 = 0.$$

If this triangle is $\left(0,3,\frac{5}{2}\right)_\rho$, since the property that the plastic constant satisfies is

$$\rho^3 = \rho + 1,$$

so this triangle obviously satisfies the condition.

2.3.4. *Triangle* $(0,1,\frac{5}{2})_x$ *in* $\theta = \dfrac{2\pi}{3}$ $(=120°)$. In this case it satisfies

$$x^5 = x^2 + x + 1 \iff \left(x^2+1\right)\left(x^3 - x - 1\right) = 0.$$

Since $x^2 + 1 > 0$, we have

$$x^5 = x^2 + x + 1 \implies x^3 - x - 1 = 0.$$

If this triangle is $\left(0,1,\frac{5}{2}\right)_\rho$, since the property that the plastic constant satisfies is

$$\rho^3 = \rho + 1,$$

so this triangle obviously satisfies the condition.

2.3.5. *Triangle* $(0,1,2)_x$ *in* $\theta = \dfrac{2\pi}{3}$ $(=120°)$. In this case it satisfies

$$x^4 = x^2 + x + 1 \iff (x+1)\left(x^3 - x^2 - 1\right) = 0.$$

Since $x + 1 > 0$, we have

$$x^4 = x^2 + x + 1 \implies x^3 - x^2 - 1 = 0.$$

If this triangle is $(0,1,2)_\psi$, since the property that the supergolden ratio satisfies is

$$\psi^3 = \psi^2 + 1,$$

so this triangle obviously satisfies the condition.

The following results were obtained for the non-trivial cases.

θ	Ratio	Constant
$\frac{\pi}{2}(=90°)$	$\left(0,2,\frac{5}{2}\right)_\rho$	Plastic Constant
$\frac{\pi}{3}(=60°)$	$\left(0,2,\frac{3}{2}\right)_\rho$	Plastic Constant
$\frac{\pi}{3}(=60°)$	$\left(0,3,\frac{5}{2}\right)_\rho$	Plastic Constant
$\frac{2\pi}{3}(=120°)$	$\left(0,1,\frac{5}{2}\right)_\rho$	Plastic Constant
$\frac{2\pi}{3}(=120°)$	$(0,1,2)_\psi$	Supergolden Ratio
$\pi(=180°)$	$(0,4,5)_\rho$	Plastic Constant

2.4. Other non-trivial cases (golden triangle and golden gnomon).

Proposition 2.10. *The following results were obtained for* $\theta = \frac{\pi}{5}, \frac{2\pi}{5}, \frac{3\pi}{5}$ *related to the golden ratio.*

$\theta = \frac{\pi}{5}(=36°)$**:** $(0,0,-1)_\phi$, $(0,1,0)_\phi$.
$\theta = \frac{2\pi}{5}(=72°)$**:** $(0,1,1)_\phi$.
$\theta = \frac{3\pi}{5}(=108°)$**:** $(0,0,1)_\phi$.

Remark 2.11. $\theta = 4\pi/5(=144°)$ *is NOT a triangle that satisfies the condition because the triangle* $(0,\alpha,\beta)_\phi$ *satisfies*

$$1 + x^{2\alpha} - x^{\alpha+1} = x^{2\beta} \iff (L_{2\alpha} + L_{\alpha+1} - L_{2\beta} + 2) + \sqrt{5}(F_{2\alpha} + F_{\alpha+1} - F_{2\beta}) = 0,$$

and $L_{2\alpha} + L_{\alpha+1} - L_{2\beta} + 2 \neq 0$. *So this Diophantine equation has no solution.*

Remark 2.12. *These match the special name triangles:*

- *Triangle* $(0,0,-1)_\phi$ *and* $(0,1,1)_\phi$ *are golden triangles.*
- *Triangle* $(0,1,0)_\phi$ *and* $(0,0,1)_\phi$ *are golden gnomons.*

3. Summary

- We have determined all triangles where at least one angle is an integer and the ratio of the side lengths is a rational power of the golden ratio, the plastic constant, the tribonacci constant, or the supergolden ratio.
- We find that many non-trivial triangles use the plastic constant.

4. Future work

If we do not add the condition of integer angles, we can construct many triangles with side lengths a rational powers of a constant. Can we find the ratio of the side lengths of the triangle where at least one angle is an integer, using only the value of a rational power of a constant obtained from a linear recurrence sequence?

References

[1] T. Koshy, *Fibonacci and Lucas Numbers with Applications* (2nd ed.), John Wiley & Sons, 2017.
[2] I. M. Niven, *Irrational Numbers*, Wiley, New York, 1956. pp.37–41.
[3] D. H. Lehmer, A note on trigonometric algebraic numbers, *Amer. Math. Monthly* **40** (1933), 165–166.

MSC2010: 11B37, 11B39, 51M04

Graduate School of Science and Engineering, Saitama University, 255, Shimo-Okubo, Sakura-Ku, Saitama, Japan
E-mail address: k-nakagawa@h6.dion.ne.jp