THE BERGMAN GAME

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ABSTRACT. Every positive integer may be written uniquely as a base-φ decomposition–that is a legal sum of powers of φ, the golden mean. Guided by earlier work on a two-player game which produces the Zeckendorf Decomposition of an integer (see [1]), we define a related game played on an infinite tuple of non-negative integers which decomposes a positive integer into its base-φ expansion. We call this game the Bergman Game. We prove that the longest possible Bergman game on an initial state S with n summands terminates in $\Theta(n^2)$ time, and we also prove that the shortest possible Bergman game on an initial state terminates in $\Theta(n)$ time. We also show a linear bound on the maximum length of the tuple used throughout the game.

1. INTRODUCTION

1.1. **History and Motivation.** Every integer $n > 0$ can be written uniquely as a sum of non-adjacent Fibonacci numbers $\{F_n : F_1 = 1, F_2 = 2, F_{n+1} = F_n + F_{n-1}\}$, called its Zeckendorf decomposition. For example

$$2021 = 1597 + 377 + 34 + 13 = F_{15} + F_{12} + F_7 + F_5.$$

Previous literature has extensively analyzed generalizations of this theorem to other recurrences using number-theoretic and probabilistic techniques (see [12, 13, 14] and [16, 14, 15, 6, 7] respectively). Earlier work (see [2]) analyzed a combinatorial game played on an infinite one-sided tuple of Fibonacci numbers; starting with n copies of F_1 players alternate by using moves arising from the Fibonacci recurrence which over time consolidate the many original copies of F_1 into a few instances of larger F_i. The winner is the player who moves last and consequently forms the Zeckendorf decomposition of n. For $n > 2$, the second player to move has a winning strategy (though the proof is non-constructive), and all games take on the order of n moves.

Our work takes the Zeckendorf Game and removes its boundary conditions (special moves only allowed at the left edge); this new game–the Bergman Game[1] –is now played on a doubly infinite tape. Playing the Bergman game produces the unique base-φ decomposition of n, where φ is the golden mean. For example:

$$2021 = \varphi^{-16} + \varphi^{-11} + \varphi^{-6} + \varphi^{-3} + \varphi + \varphi^5 + \varphi^{10} + \varphi^{13} + \varphi^{15}.$$

Previous research has extensively studied base-β representations for any real number $\beta > 1$ via Ergodic theory, symbolic dynamics, and algebraic number theory (see [5, 8, 1, 1]).

We further define a Generalized Bergman Game played on any $\beta > 1$ which satisfies the characteristic polynomial of a Non-Increasing Positive Linear Recurrence Sequence (non-increasing PLRS). Our research thus presents a new way of studying these base-β representations using classes of games similar to the Zeckendorf Game. For example, these games provide an elementary proof that any number which is a finite sum of powers of β with non-negative coefficients has a finite base-β decomposition (see [3]). Although we do not prove the complete known results in full generality, the corresponding literature uses ergodic measure theory rather than elementary methods [10]. Our research particularly

[1]Named for George Bergman, who discovered base-φ decompositions in [4].

analyzes the termination time of the Generalized Bergman Game, showing tight bounds on the number of moves in such games. This paper, however, concerns itself principally with the Bergman Game, and we refer to our other work for analogous results about the Generalized Bergman Game [3]. We thus only present the basic definitions of the Generalized Bergman Game.

1.2. Basic Definitions.

Definition 1.1. The Bergman Game is a two player turn-based game played on a doubly infinite tape. A game state of the Bergman Game, denoted by S, is of the form $S = (\dots, S(-1), S(0), S(1), \dots)$, where for all the indices $i \in \mathbb{Z}$, $S(i) \in \mathbb{Z}_{\geq 0}$ and there is only a finite number of i such that $S(i) \neq 0$. We say that S has $S(i)$ "summands" in index i.

The Bergman Game begins with some starting game state, and then two players take turns making moves, choosing either to split or combine. The combine and split moves are defined as follows:

- Combine: If both $S(i-2) \geq 1$ and $S(i-1) \geq 1$, then decrease $S(i-2), S(i-1)$ each by 1 and increase $S(i)$ by 1. E.g.,

$$(1,1,0) \to (0,0,1).$$

- Split: If there exists an i such that $S(i) \geq 2$, then decrease $S(i)$ by 2 and increase $S(i-2)$ by 1 and increase $S(i+1)$ by 1. E.g.,

$$(0,0,2,0) \to (1,0,0,1).$$

The game is played until one player can neither split nor combine, and the last player to make a move wins.

The entire sequence of play (the ordered collection of the game states which appeared over the course of play) is called a "game", (usually) denoted by G.

Notation. *For convenience, we often abbreviate the game state*

$$(\dots, 0, 0, 0, S(a), S(a+1), \dots, S(b), 0, 0, 0, 0, \dots)$$

as the finite tuple

$${}_a(S(a), \dots, S(b))_b$$

where a is the leftmost non-zero valued index and b is the rightmost non-zero valued index. We often also use the notation ${}_aS_b$ to refer to a game state S with leftmost nonzero index a and rightmost nonzero index b. This provides a convenient way to notate shifting a game state; for example ${}_0S_{b-a}$ refers to the game state where we have shifted ${}_aS_b$ to have leftmost nonzero entry at the zeroth index. If there is only one index with a non-zero number of summands, we refer to the game state as ${}_a(n)$.

Example 1.2. We provide an example of two possible Bergman Games which share a common first four moves and then diverge from each other at move five (Figure 1). This shows that the Bergman Game is non-deterministic on the initial game state ${}_0(n)$ in general, as the game may be played down either shown path, each of which gives a different winner.

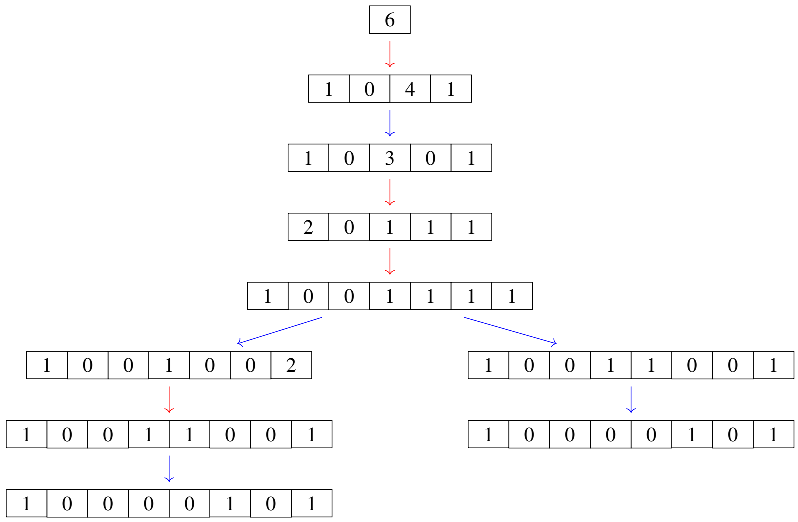

FIGURE 1. A Non-Determistic Bergman Game with initial state ${}_0(6)_0$, Player One wins in the left branch and Player Two wins in the right branch. Red arrows mark splits, and blue combines.

Definition 1.3. We define $|S|$ as the number of summands in S. More formally, $|S| := \sum_j S(j)$. Note that the number of summands throughout a game is a non-increasing monovariant, as the number of summands remains the same with a split move and decreases by one with a combine.

Example 1.4. Let S_i denote the initial game state from Figure 1, and S_f the final state. Then $|S_i| = 6$ and $|S_f| = 3$.

1.3. **Summary of Results.** We derive a number of results concerning the length of the Bergman Game. To do so, we make heavy use of invariant and monovariant quantities associated to game states and moves. In order to make use of these monovariants, we must also bound the window in which a Generalized Bergman Game may take place. We leverage knowledge of base-φ representations to extract information about the final game state and, thereby, about the whole game.

To easily exposit our results, we provide the following two definitions.

Definition 1.5. Define $\mathcal{M}_{\max}(S)$ to be the maximum number of moves in a Bergman Game on a given initial state S. Similarly, let $\mathcal{M}_{\min}(S)$ denote the minimum number of moves in a Bergman Game on a given initial state S.

We then define

$$\widehat{\mathcal{M}}_{\max}(n) := \max_{|S|=n} \mathcal{M}_{\max}(S) \tag{1.1}$$

$$\widehat{\mathcal{M}}_{\min}(n) := \max_{|S|=n} \mathcal{M}_{\min}(S). \tag{1.2}$$

Definition 1.6. Let S be some initial game state, and without loss of generality suppose that S has leftmost summand at 0 and rightmost summand at b. We define $\mathcal{L}(S)$ to be the absolute value of the minimum index of the leftmost summand throughout any game played with starting state S.

We then define

$$\widehat{\mathcal{L}}(n, b) := \max_{\substack{{}_0S_b \\ |S|=n}} \mathcal{L}(S). \tag{1.3}$$

For convenience, we also let $\widehat{\mathcal{L}}(n) = \widehat{\mathcal{L}}(n, 0)$.

In order to state the results, we must define asymptotic notation.

Definition 1.7. Consider two functions $f, g : \mathbb{N} \to \mathbb{R}$. We say that $f(n) = O(g(n))$ provided that there is some constant $K > 0$ and some natural number $M \in \mathbb{N}$ so that for every $n \geq M$ we have $|f(n)| \leq Kg(n)$.

We may also write this as $g(n) = \Omega(f(n))$. If we have both that $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$ then we write that $f(n) = \Theta(g(n))$.

With these notions in mind, we may state the main results of the paper.

Theorem 1.8. *We have $\widehat{\mathcal{M}}_{\max}(n) = \Theta(n^2)$. That is, the longest possible Bergman Game for a given non-increasing PLRS with n summands terminates in $\Theta(n^2)$ moves.*

Theorem 1.9. *We have $\widehat{\mathcal{M}}_{\min}(n) = \Theta(n)$. That is, the shortest possible game length which is available from all starting states with n summands is $\Theta(n)$.*

Theorem 1.10. *We have $\widehat{\mathcal{L}}(n, b) = O(n) + O(b)$ and $\widehat{\mathcal{L}}(n) = \Theta(n)$. That is, the maximum that a summand may be moved left throughout the game is at most linear in the number of summands and the length of the initial window. Furthermore, when the summands in the initial state are concentrated in a single index, we know that this quantity is exactly linear in the number of summands.*

Together, these theorems characterize termination times and index ranges of the Bergman Game completely. (The right bound of any game state is trivially logarithmic in the number of chips and width of initial state, as we show later.)

2. Preliminaries

We establish a number of useful quantities related to the Generalized Bergman Game and outline their properties. Crucially, we connect the game to base-ϕ expansions.

Definition 2.1. Given any two game states S, T, define $(S + T)(j) := S(j) + T(j)$. If $S(j) \geq T(j)$ for all j, we also define $(S - T)(j) := S(j) - T(j)$.

Definition 2.2. The value $v(S)$ of a game state S is $\sum_j S(j)\varphi^j$, and the conjugate value $\widetilde{v}(S)$ of a game state S is $\sum_j S(j)\psi^j$, where $\psi = -1/\varphi$.

Lemma 2.3. *The value $v(S)$ and the conjugate value $\widetilde{v}(S)$ are both invariants throughout the course of the Bergman Game. That is, if S is some game state and T is a game state obtained by applying some move to S, then $v(S) = v(T)$ and $\widetilde{v}(S) = \widetilde{v}(T)$.*

Proof. Note that φ and ψ are both roots of the polynomial $x^2 - x - 1$. To show that v and $\widetilde{v}$ are invariants we need only use this property of φ and ψ. For convenience we do the proof for φ.

The sum is clearly invariant under combining moves, because if a combine into index q is performed then we have that

$$v(T) - v(S) = \varphi^q - \varphi^{q-1} - \varphi^{q-2} = \varphi^{q-2}(\varphi^2 - \varphi - 1) = 0 \tag{2.1}$$

Furthermore, this sum is invariant under the split move. To see this, consider that by simple algebraic manipulation φ also satisfies the following equation:

$$2\varphi^2 = \varphi^2 + \varphi + 1 = \varphi^3 + 1 \tag{2.2}$$

Therefore if a split is performed at index q then we have that

$$v(T) - v(S) = \varphi^{q+1} + \varphi^{q-2} - 2\varphi^q = \varphi^{q-2}\left(\varphi^3 + 1 - 2\varphi^2\right) = 0 \tag{2.3}$$

This completes the proof. □

With this result, we can begin to think of game states as base-φ representations of their value. Then we reach the following natural realization, of which we take repeated advantage.

Proposition 2.4. *Given an initial state S for which the Bergman Game terminates, the final state S_f of the game is the unique base-φ expansion of $v(S)$.*

Proof. The final state S_f provides a base-φ representation of $v(S)$ by Lemma 2.3. Note that S_f can contain no entries larger than or equal to two, nor can it contain two consecutive entries both of whose values are greater than zero. Hence S_f is the base-φ expansion by definition (see [4]). □

Thus, the final states of our game are base-φ expansions, and the game actually provides a slow algorithm for computing such expansions exactly given that the game terminates (which we prove in Section 3).

We now quickly state how the number of summands changes for each move, as this is one of our basic tools for deriving bounds.

Lemma 2.5. *A combine decreases $|S|$ by one and a split does not change $|S|$.*

Corollary 2.6. *There are at most $|S|$ combines throughout the course of a game on a given initial state S. Assuming termination (which we prove in Section 3), the number of combines is actually fixed as $|S| - |S_f|$, where S_f is the final state of the game.*

Proof. If $|S| = 0$, then there are trivially no combines. Now note that if $|S| > 0$ then $v(S) > 0$, so by Lemma 2.3 we know that $v(T) > 0$ for any game state T occurring after S. This then implies that $|T| > 0$ as well. We then apply Lemma 2.5 to provide the given bound on the number of combines.

The fact that the number of combines is fixed follows from the fact that our final states are base-φ expansions, which are unique (see [4]). Thus the number of summands in S_f is fixed. □

3. The Bergman Game Terminates

In this section we prove that the Bergman Game is in fact playable.

Proposition 3.1. *Any Bergman Game terminates in a finite number of moves.*

Using Corollary 2.6, it suffices to consider games which consist only of splits in order to prove that the game terminates. To this end, we define a quantity associated to any game state which is a decreasing monovariant in games consisting only of splits. Then, to show that we cannot perform infinitely many splits we bound this monovariant from below by establishing a bound on how far to the left of the initial leftmost summand a particular game can run.

3.1. Defining and Bounding $\mathcal{I}nd(S)$.

Definition 3.2. Let S be some game state. We define the index sum of S, denoted by $\mathcal{I}nd(S)$, to be the quantity $\sum_i iS(i)$.

Lemma 3.3. *A combine into index i increases $\mathcal{I}nd(S)$ by $-i+3$. Likewise, a split at index i decreases $\mathcal{I}nd(T)$ by one.*

Proof. Consider a combine into index i, and let S_1 be the game state before this combine and S_2 be the game state after this combine. In this case we have that

$$\mathcal{I}nd(S_1) - \mathcal{I}nd(S_2) = (i - 1 + i - 2) - i = i - 3.$$

Similarly for splits, we have that

$$\mathcal{I}nd(S_1) - \mathcal{I}nd(S_2) = 2i - (i - 2 + i + 1) = 1.$$

□

Proposition 3.4. *Suppose that given an initial state ${}_0S_b$ that there is a constant $L_S < 0$ such that for any game state T occuring after S we have that the leftmost summand of T is at an index greater than or equal to L_S. Then $\mathcal{I}nd(T) \geq |S| L_S$.*

Proof. Note that $|T| \leq |S|$ by Lemma 2.5. We then have that

$$\mathcal{I}nd(T) = \sum_i iT(i) \geq \sum_i L_S T(i) = L_S |T| \geq L_S |S| .$$

This establishes the result. □

We now prove a bound L_S and thereby prove termination.

3.2. **Establishing a Left Bound via Gaps.** To establish a left bound, we rely on the gaps consisting entirely of zeros between summands in a game state T. At a high level, we prove that given an initial state S, the gaps in any game state T cannot be longer than some constant length. We then space out the summands of T as much as possible in a worst-case analysis to give a left bound.

Definition 3.5. Given a game state S, let $g(S)$ denote the maximum gap consisting entirely of zeros between summands.

Remark. Gaps between summands are well studied using probabilistic methods in a variety of integer decompositions, including Zeckendorf Decompositions and PLRS decompositions (see [7, 6, 15]). For example, the gaps between summands in Zeckendorf decompositions displays geometric decay. It is interesting that they are also a natural quantity to prove termination of the Bergman Game.

We now establish two lemmas concerning the maximum gap size and the location of the rightmost summand of a game state.

Lemma 3.6. *A combine may increase the maximum gap size by at most* 2 *and a split cannot increase the quantity* $\max(g(S), 2)$.

This then implies that for a game state T occurring in a game after a state S we have

$$g(T) \leq 2|S| + \max(g(S), 2).$$

Proof. Consider some combining move. In the worst possible case, all of the entries being combined become zeros as below:

$$(1, 0, \ldots, 0, 1, 1, 0, 0, \ldots, 0, 1) \to (1, 0, \ldots, 0, 0, 0, 1, 0, \ldots, 0, 1).$$

There are then two zeros added to the gap on the left side, establishing the first piece of the claim.

A split at index i places a nonzero entry at indices $i-2, i+1$. In the worst possible case, there are no summands at index $i-1$ and only the two used in the split at i, and the split creates a gap of size 2. It may also decreases the size of the gaps to the left of index $i-2$ and to the right of index $i+1$. This worst case analysis establishes the second part of the claim.

To establish the given bound we then just note from Corollary 2.6 that the number of combines is bounded by $|S|$. □

Lemma 3.7. *Suppose that T occurs after some state S in a game. Then the rightmost summand of T is located at an index greater than or equal to the rightmost summand of S.*

Proof. By direct inspection of the combine and split moves, both move at least one summand to the right, and thus neither may decrease the right bound of a game state. The result follows □

Equipped with these, we can now begin our worst-case analysis which provides the left bound $L_S < 0$ needed to prove that the Bergman Game terminates.

Proposition 3.8. *Suppose that some game state T occurs after a game state ${}_0S_b$ in a Bergman Game. Then the leftmost summand of T is located at an index greater than or equal to*

$$b - 2(|S| + 1)(|S| - 1) + \max(g(S), 2).$$

Proof. There are at most $|S|$ summands in T, and the rightmost summand of T is located at or to the right of the index b by Lemma 3.7. Therefore using the bound on the maximum gap size $g(T)$ established in Lemma 3.6 we may analyze the worst case by placing single summands to the left which are spaced out by $2\,|S| + \max(g(S), 2)$ zeros. This gives the bound above. □

With this left bound in hand, we can establish Proposition 3.1 (i.e., that any Bergman Game terminates in a finite number of moves) easily.

Proof of Proposition 3.1. Consider some initial state S, and note that since the moves of our game are translation invariant we may assume that the leftmost summand of S is located at index zero without loss of generality[2].

We know that there are finitely many combines performed in the game by Corollary 2.6. Therefore, it suffices to show that there cannot be infinitely many splits in a row.

We know that the index sum decreases by at least one with each split from Lemma 3.3. Furthermore, there is a constant $L_S < 0$ depending on the initial state S so that the leftmost summand of any game state T occurring after S is at an index greater than or equal to L_S from Proposition 3.8. Therefore $\mathcal{I}nd(T) \geq L_S\,|S|$ by Proposition 3.4.

Putting these results together, there cannot be infinitely many splits or else the index sum would be driven below this bound, which is impossible. □

Remark. Using similar techniques we may establish that a much larger class of games terminates. In particular, the crucial properties used are that all moves either conserve or decrease the number of summands, and that the moves which conserve the number of summands cannot create gaps.

4. Termination of the Bergman Game in $O(|S|^2)$ moves

We now show that given any initial state S, the Bergman Game terminates in $O(|S|^2)$ moves. To do so, we first establish weaker termination results, which allow us to tackle the main result in much greater concision.

4.1. **The Left Bound for the Bergman Game.**

We first establish a better bound on how far a game may run to the left of the leftmost nonzero index of an initial state S. To do this, we must examine the base-φ expansion of $v(S)$, or equivalently the final state S_f achieved by playing the game until it ends (which is guaranteed by Proposition 2.4 and Proposition 3.1).

Proposition 4.1. *Consider a game state S in the Bergman Game whose summands are all in nonnegative indices. Then the final game state S_f (i.e., the base-φ expansion of $v(S)$) has leftmost summand at an index greater than or equal to $-\log_\varphi |S| - 2$. This has been shown previously by Dekking in* [9]*, but we provide a proof here motivated by the game for ease of reading.*

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[2]This is an advantage of working with the Bergman Game instead of the Zeckendorf Game, which is not translation invariant.

Proof. Let a denote the index of the leftmost summand in S_f. Then consider the following chain of inequalities

$$|S| \geq |\widetilde{v}(S)| = |\widetilde{v}(S_f)| = \left|\psi^a + \sum_{j=1}^{\infty} S_f(a+j)\psi^{a+j}\right|$$
$$\geq |\psi|^a \left(1 - \sum_{j=2}^{\infty} S_f(a+j)\,|\psi|^j\right) \geq |\psi|^a \left(1 - \sum_{j=0}^{\infty} |\psi|^{2j+2}\right)$$
$$= \varphi^{-a}\left(1 - \frac{\varphi^{-2}}{1-\varphi^{-2}}\right) = \varphi^{-a-2}.$$

By taking the $\log_\varphi$ on both sides and rearranging we then have that $a \geq -2 - \log_\varphi |S|$. The only nontrivial inequalities above are in the second line. The first inequality in this line follows by pulling out the $|\psi|^a$, applying the reverse triangle inequality, and noting that $S_f(a+1) = 0$ because S_f is a reduced game state. The second inequality in this line also follows by considering what we know because S_f is reduced. Namely, each nonzero entry must be followed by a zero, and all entries are less than or equal to one. Then because $j \mapsto |\psi|^j$ is a decreasing function, the maximal arrangement is $(0,0,1,0,1,0,1,0,\ldots)$, i.e., the sum $\sum_{j=0}^{\infty} |\psi|^{2j+2}$. □

Corollary 4.2. *We conclude that given an initial state S in the Bergman game, the leftmost summand in any state T occurring after S is at an index greater than or equal to $-2\,|S| - \log_\varphi |S| - 2$.*

Proof. Note that a split cannot increase the index of the leftmost summand. There are at most $|S|$ combines throughout the course of the game, and therefore because each combine can bring the index of the leftmost summand in by at most two, the desired bound must follow for every state in order to possibly satisfy the bound on the final base-φ decomposition. □

Having established the left bound, we now write down the right bound.

Proposition 4.3. *The rightmost summand in a game state S is at an index less than or equal to $\log_\varphi v(S)$. Because $v(S) = v(T)$ for any game state T occurring after S, this result also holds for game states T occurring after S.*

Proof. If this were not true, then we would have $v(S) \geq \varphi^{\log_\varphi v(S)+1} > v(S)$, which cannot hold. □

We now prove Theorem 1.9.

Proof. Consider an initial state S with all summands initially at the same index, without loss of generality $S = {}_0(n)$ and $v(S) = n$. By Proposition 4.1 and by the trivial logarithmic right bound on the final state, the final state is contained in a window of indices of logarithmic width $2\log(n) + 2$.

Then the total number of summands in the final state is at most $2\log(n) + 2$, since any more chips would create opportunity for a split by the pigeonhole principle. Since we begin with n chips and each combine removes 1 summand, then we must perform at least $n - 2\log(n) + 2 = \Theta(n)$ combines. The result follows. □

This establishes bounds of $-2\,|S| - \log_\varphi |S| - 2$ and $\log_\varphi v(S)$ where one can play the game, exhibiting that it truly takes place only on a tuple whose length is linear in the number of summands and logarithmic in the value of the game state.

4.2. **The Upper Bound on Termination Time.** We begin by proving that the game terminates quickly for games within a certain index range. We then use this result to show that we can "chunk" apart larger games into non-interacting smaller games. Finally, we show the game terminates from any initial game state in n^2 time where $n := |S|$.

Proposition 4.4. *Let ${}_aS_b$ be some initial game state. Then we know that the Bergman Game played on this initial game state terminates in at most*

$$|S|^2 + 2\,|S| \log_\varphi |S| + 7\,|S| + (b-a)\,|S|$$

moves.

Proof. We begin by shifting S so that its leftmost summand is at zero without loss of generality, since our rules are translation invariant. Thus, we can assume that $a = 0$. We now upper bound $\mathcal{I}nd(S)$ by $b\,|S|$, simply by considering the worst case where all of the summands are located at b.

By Proposition 4.1 we may also lower bound $\mathcal{I}nd(S_f)$, where S_f is the final game state. We have that:

$$\mathcal{I}nd(S_f) \geq -\,|S_f|\,(\log_\varphi n + 2) \geq -n \log_\varphi n - 2n.$$

Then consider how much the given combines may increase $\mathcal{I}nd(T)$. To do this, we must see that the leftmost possible combine is played into the index $-2n - \log_\varphi n$ by Corollary 4.2, since we need a summand two indices to the left to combine into an index. The next combine would then have to occur at $-2n - \log_\varphi n + 2$, and so on until we have all $|S|$ combines in order to satisfy the final left bound of Proposition 4.1. Using Lemma 3.3 we then have that $\mathcal{I}nd(T)$ increases by at most

$$\sum_{r=0}^{n-1} \left(2n + \log_\varphi n - 2r + 3\right) = n^2 + n \log_\varphi n + 4n$$

in $\mathcal{I}nd(T)$ over the course of the whole game. Note then that a split decreases $\mathcal{I}nd(T)$ by one. Therefore in order to not break the lower bound on $\mathcal{I}nd(S_f)$ we have the following maximum total number of splits across the course of the entire game:

$$n^2 + 2n \log_\varphi n + 6n + bn.$$

There are at most n combines throughout the course of the game, and adding this to the contribution from the number of splits gives the bound stated in the proposition. □

Theorem 4.5. *Given any initial state S, the Bergman Game terminates in $8\,|S|^2 + O(|S| \log |S|)$ moves.*

Proof. Fix some Bergman Game G which reduces an initial configuration S to its final state S_f. For ease of reading, let $\ell({}_aT_b) = b - a$ be the length (where we consider ${}_0(n)$ to have length zero) of a game state T with leftmost summand at a and rightmost summand at b.

Now let $T^0 := S_f$. We may break T^0 into chunks $T^{0,1}, \ldots, T^{0,m_0}$. Namely, whenever we see a run of three or more zeros we break off one chunk. An example can be seen below:

$T^{0,1}$		$T^{0,2}$		$T^{0,3}$
1 0 1 0 1	0 0 0 0	1	0 0 0	1 0 0 1 0 0 1

Let $n_{0,j} = \left|T^{0,j}\right|$ and note that $T^0 = \sum T^{0,j}$ so $n_0 := \left|T^0\right| = \sum n_{0,j}$. Since $T^{0,j}$ contains no runs of three or more zeros we know that $\ell(T^{0,j}) \leq 3n_{0,j}$.

Now play the game G in reverse from the final state T^0 using only reverse splits until we get to the last combine. In doing so, note that all the reverse splits played are independent, and could not have interacted. Why? Well a reverse split always moves the leftmost and rightmost bounds of a game state inward if it doesn't change them. Furthermore, we can't perform a reverse split using things from two different chunks, because reverse splits require nonzero values with two indices between then.

Call the game state after performing these reverse splits and one reverse combine T^1. We break T^1 into chunks $T^{1,1}, \ldots, T^{1,m_1}$ by keeping track of where each chunk $T^{0,1}, \ldots, T^{0,m_0}$ went under these game moves, combining two chunks if they become closer than a gap of three zeros.

We then note that $m_1 = m_0$ or $m_1 = m_0 - 1$. Reverse splits drive the bounds of each chunk inwards and cannot merge chunks. Then a reverse combine may only bridge a single gap of at most four zeros between chunks. Similarly, a reverse combine cannot create new chunks, and can only increase the length of a chunk by at most two.

Continue this process to form a sequence of game states $T^0, \ldots, T^K$ where K is the number of combines, and chunks $T^{i,j}$ where $0 \leq i \leq K$ and $1 \leq j \leq m_i$. We then backtrack through any remaining reverse splits to get chunks $S^1, \ldots, S^M$ of the initial state. These chunks all play independently in the game G by the previous remarks.

We now begin to bound the length of the game G. To do this, we bound the total lengths of the chunks in a clever way. A reverse split never increases the length of a chunk, and a reverse combine increases the total length of chunks by at most four. Therefore since there are at most $|S|$ combines we have that

$$\sum_i \ell(S^i) \;\leq\; 4\,|S| + \sum_j \ell(T^{0,j}) \;\leq\; 4\,|S| + 3\sum_j n_{0,j} \;\leq\; 7\,|S|\,.$$

Because the game plays independently on each chunk $S^1, \ldots, S^M$, we may use the bound from Proposition 4.4 to see that the number of moves is bounded above by the following:

$$\sum_i \left|S^i\right|^2 + \ell(|S|^i)\,|S|^i + O(\left|S^i\right| \log \left|S^i\right|).$$

Using the fact that $\sum_i \left|S^i\right| = |S|$ and we may then bound each piece of this easily:

$$\sum_i \left|S^i\right|^2 \leq \left(\sum_i \left|S^i\right|\right)^2 = |S|^2$$
$$\sum_i \ell(S^i)\left|S^i\right| \leq |S| \sum_i \ell(S^i) \leq 7\,|S|^2$$
$$\sum_i \left|S^i\right| \log \left|S^i\right| \leq \log|S| \cdot \sum_i \left|S^i\right| = |S| \log|S|\,.$$

In total, this provides a bound on the length of a game G with initial state S by $8\,|S|^2 + O(|S| \log |S|)$ moves, completing the proof. □

5. Extremal Games

We present an $\Omega(n^2)$ move Bergman Game and thus show that our upper bound on $\widehat{\mathcal{M}}_{\max}(n)$ is tight. We subsequently show that our bound is tight in the constant of the leading term for the Bergman Game. We then present an $O(n)$ Bergman Game, which shows that our lower bound on $\widehat{\mathcal{M}}_{\min}(n)$ is tight.

5.1. **Construction of a $\Omega(|S|^2)$ Bergman Game.** We show that there is a long Generalized Bergman Game. In particular, we show that using a particular strategy for selecting moves, we can make a game which takes $\Omega(n^2)$ moves from the initial state $S := {}_0(n)$ to the final state S_f. This proves the asymptotic tightness of Theorem 4.5, showing that games are in fact $\Theta\left(|S|^2\right)$ as claimed in Theorem 1.8. We start by detailing the strategy employed in this game.

Definition 5.1. The Split Left and Combine Right ($\mathcal{S}_{\mathscr{L}}\mathcal{C}_{\mathscr{R}}$) strategy mandates that each player always takes the split farthest to the left whenever there is a split available, and otherwise takes the farthest combine to the right.

Similarly, we may define the strategies $\mathcal{S}_{\mathscr{L}}\mathcal{C}_{\mathscr{L}}$, $\mathcal{S}_{\mathscr{R}}\mathcal{C}_{\mathscr{L}}$, $\mathcal{S}_{\mathscr{R}}\mathcal{C}_{\mathscr{R}}$ as well as the strategies $\mathcal{C}_{\mathscr{L}}\mathcal{S}_{\mathscr{L}}$, $\mathcal{C}_{\mathscr{L}}\mathcal{S}_{\mathscr{R}}$, $\mathcal{C}_{\mathscr{R}}\mathcal{S}_{\mathscr{R}}$, and $\mathcal{C}_{\mathscr{R}}\mathcal{S}_{\mathscr{L}}$.

Proposition 5.2. *The $\mathcal{S}_{\mathscr{L}}\mathcal{C}_{\mathscr{R}}$ strategy played on any initial state divides the game into two phases. In Phase I, all of the splits are performed, and in Phase II only combines are performed.*

Proof. Suppose that we are in a state S where we cannot split. We show that using the $\mathcal{S}_{\mathscr{L}}\mathcal{C}_{\mathscr{R}}$ strategy we never split again, proving the first claim.

We know that no entry of S is greater than one. In order to split again, we would need to combine to create an entry of size at least two. The only possible way to do this would be to take the move below

$$(1,1,1,0) \to (0,0,2,0).$$

However, the $\mathcal{S}_{\mathscr{L}}\mathcal{C}_{\mathscr{R}}$ strategy mandates that we instead take the move

$$(1,0,0,1).$$

When we take this move, all strings remain less than or equal to one, and so we can never split again, dividing the game played with the $\mathcal{S}_{\mathscr{L}}\mathcal{C}_{\mathscr{R}}$ strategy into Phase I and Phase II. □

Corollary 5.3. *Due to this division into phases, the $\mathcal{S}_{\mathscr{L}}\mathcal{C}_{\mathscr{L}}$ strategy agrees with the $\mathcal{S}_{\mathscr{L}}\mathcal{C}_{\mathscr{R}}$ strategy for all of the moves in Phase I. As a corollary, the $\mathcal{S}_{\mathscr{L}}\mathcal{C}_{\mathscr{L}}$ strategy takes at least as many moves as the $\mathcal{S}_{\mathscr{L}}\mathcal{C}_{\mathscr{R}}$ strategy.*

Proof. Because the $\mathcal{S}_{\mathscr{L}}\mathcal{C}_{\mathscr{L}}$ strategy agrees with the $\mathcal{S}_{\mathscr{L}}\mathcal{C}_{\mathscr{R}}$ strategy when only splits are played, the game proceeds through Phase I with no differences.

Because the number of combines played in a game on an initial state S is constant (see Corollary 2.6), we have the desired claim that the $\mathcal{S}_{\mathscr{L}}\mathcal{C}_{\mathscr{L}}$ strategy takes at least as many moves as the $\mathcal{S}_{\mathscr{L}}\mathcal{C}_{\mathscr{R}}$ strategy for any initial state. In particular, when an extra split is performed in the second phase for the first strategy, it is strictly longer. □

Theorem 5.4. *The $\mathcal{S}_{\mathscr{L}}\mathcal{C}_{\mathscr{R}}$ strategy takes $\Omega(n^2)$ moves from the initial state ${}_0(n)$.*

In particular, the Bergman Game takes at least $\frac{n(n-2\log_\varphi n-1)}{2}$ splits for any n.

Proof. Let T be the state of the game after the completion of Phase I and before the first combine by Proposition 5.2. Note first that all the entries of T are less than or equal to one because we have finished Phase I. Furthermore, the rightmost summand of T is located at an index less than or equal to $\log_\varphi v(T) = \log_\varphi v({}_0(n)) = \log_\varphi n$ by Proposition 4.3. Furthermore $|T| = n$ because splits preserve the number of chips, and we have performed only splits in Phase I.

We now recall the index sum $\mathcal{I}nd(T)$ introduced in Section 3. Note that a split always decreases the index sum by one (see Lemma 3.3). We then know the following upper bound on $\mathcal{I}nd(T)$ given by placing a summand in each index starting at $\log_\varphi n$ and moving left, and placing n summands total:

$$\mathcal{I}nd(T) \leq \sum_{j=-n+\log_\varphi n+1}^{\log_\varphi n} j = \frac{n(2\log_\varphi n - n + 1)}{2}.$$

Therefore since $\mathcal{I}nd({}_0(n)) = 0$, we know that there must have been at least $-\mathcal{I}nd(T)$ splits. Calculating this based on our upper bound on $\mathcal{I}nd(T)$ we see that

$$\#\text{ of splits} \geq \frac{n(n-2\log_\varphi n-1)}{2}.$$

This completes the proof. □

5.2. **Improving the Coefficient on the Dominating Term.** In this subsection, we prove that the coefficients on the dominating term coming from Theorem 4.5 and Theorem 5.4 agree. That is, we can strengthen Theorem 1.8 to the following in this special case.

Proposition 5.5. *Proposition 4.4–which bounds the length of the longest Bergman game played on* ${}_0(n)$ *above by* $n^2 + 2n\log_\varphi n + n$*–is essentially tight.*

More precisely, for $n \geq 20$ *the length of the longest Bergman game played on* ${}_0(n)$ *lies in the interval*

$$\left[n^2 - 6n - 3n\log_\varphi n + 5\log_\varphi n + 2(\log_\varphi n)^2 - 1, n^2 + 2n\log_\varphi n + n\right].$$

Note specifically the agreement in the coefficient of the dominating terms for each side of the interval.

Proposition 5.6. *Corollary 4.2, which tells us that the maximum distance* $\mathcal{L}(n)$ *that the leftmost summand can move to the left in a Bergman Game on* n *summands is bounded above by* $2n + \log_\varphi n + 2$ *is essentially tight.*

More precisely, for $n \geq 20$ *we have that this quantity* $\mathcal{L}(n)$ *lies between* $2n - 3\log_\varphi n - 8$ *and* $2n + \log_\varphi n + 2$*.*

As expected from the methods in Section 5, these propositions are intrinsically linked. In order to prove these statements, we predict the left edge of the game state T occurring after Phase I of the $\mathcal{S}_{\mathscr{L}}\mathcal{C}_{\mathscr{R}}$ strategy. The broad outline of the argument consists of two iterative processes which feed into each other. The first iterative process takes in the position of the leftmost summand and guarantees that the left edge has a certain form with a certain number of zeros. Knowing this number of zeros, we can push the position of the leftmost summand further to the left, and then we run the first process as many times as necessary.

To ease reading, we divide this into a few lemmas.

Lemma 5.7. *Let* T *be some game state consisting entirely of ones and zeros occurring after* ${}_0(n)$ *in the Bergman Game. Further require that the leftmost summand of* T *is at a position to the left of index* $-\log_\varphi n - 5$*. Then the left edge of* T *must have the form* $110\ldots$*.*

Proof. We know that eventually this leftmost summand must be pushed right via a combine, as otherwise we would not satisfy the left bound of $-\log_\varphi n - 2$ on the final state coming from Proposition 4.1. Using a $\mathcal{S}_{\mathscr{L}}\mathcal{C}_{\mathscr{R}}$ strategy on T gives a game consisting of only combines, and so because combines always move summands right, T must already have a combine at the leftmost edge available. That is we know the left edge of T takes the form $11\ldots$.

We now rule out the possibility that three or more ones show up at the left edge in order to finish the proof. Let $T = \underbrace{11\ldots1}_{s \text{ ones}}0T^r$, where T^r is some sub-state of T and $s \geq 3$. First we will perform all possible combines available in the s ones. To see what state then occurs, we case out on whether s is even or odd.

- Suppose that s is odd. For the sake of simplicity, we work out the case where $s = 5$:

$$\begin{matrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 \end{matrix}$$

  Notice that after the first move we are simply taking the available combines for $s = 3$. A simple inductive argument shows that using the $\mathcal{S}_{\mathscr{L}}\mathcal{C}_{\mathscr{R}}$ strategy on these types of states has the following effect:

$$\underbrace{1111\ldots1}_{s \text{ ones}}0 \xmapsto{\mathcal{S}_{\mathscr{L}}\mathcal{C}_{\mathscr{R}}} 100\,\underbrace{101\ldots01}_{(s-1)/2 \text{ ones}}.$$

The leftmost summand is not moved in by doing these moves. Further, doing these moves results in a game state $100101\dots01T^r$ with only zeros or ones. Applying the first piece of this lemma we should have left edge $11\dots$ in this new state. Clearly this is not the case so s cannot be odd.

- Suppose that s is even. For the sake of simplicity, we work out the case where $s = 4$:

$$\begin{matrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{matrix}$$

Notice that after the first move we are simply taking the available combines for $s = 2$. A simple inductive argument shows that using the $\mathcal{S}_{\mathscr{L}}\mathcal{C}_{\mathscr{R}}$ strategy on these types of states has the following effect:

$$\underbrace{111\dots1}_{s \text{ ones}}0 \xmapsto{\mathcal{S}_{\mathscr{L}}\mathcal{C}_{\mathscr{R}}} 00\underbrace{101\dots01}_{s/2 \text{ ones}}.$$

The leftmost summand is moved in by two when we do these moves. Further, doing these moves results in a game state $0010101\dots01T^r$ with only zeros or ones. Applying the first piece of this lemma because $-\log_\varphi n - 3 < -\log_\varphi n - 2$ we should have left edge $11\dots$ in this new state. Clearly this is not the case when $s > 2$.

This shows that the left edge must be exactly $110\dots$ as claimed. □

Lemma 5.8. *Let T be some game state consisting entirely of ones and zeros occurring after $_0(n)$ in the Bergman Game. Further require that the leftmost summand of T is at a position to the left of index $-L \leq -\log_\varphi n - 3$. Then the left edge of T must have the form*

$$\underbrace{110101\dots10}_{z_L \text{ zeros}}\dots,$$

where $z_L = \left\lfloor \frac{L - \log_\varphi n - 3}{2} \right\rfloor$. Furthermore all these zeros occur in negative indices because $-L + 2z_L \leq -\log_\varphi n - 3 < 0$.

Proof. We essentially apply Lemma 5.7 z_L times. Namely if $1 \leq z_L$, then this implies that $2 \leq L - \log_\varphi n - 3$, so there is a summand at a position left of index $-L \leq -\log_\varphi n - 5$. We apply Lemma 5.7 and write $T = 110T^r$, then taking the available combine move to get $T' = 001T^r$. Note that T' consists entirely of ones and zeros and has leftmost summand left of $-L+2$. If $-L+2 \leq -\log_\varphi n - 5$ we again apply Lemma 5.7 to predict that $T' = 00110\dots = 001T^r$, giving that T^r has left edge $10\dots$. We're then able to see that T has left edge $11010\dots$.

Running this argument for all $z \in \mathbb{N}$ with $-L + 2z \leq -\log_\varphi n - 5$ then gives the desired result. □

Lemma 5.9. *Let T be some game state consisting entirely of ones and zeros occuring after $_0(n)$ in the Bergman Game such that $|T| = n$. Further require that the leftmost summand of T is at a position to the left of index $-L \leq -\log_\varphi n - 3$. Then there is a summand to the left of index $-n - z_L + \log_\varphi n + 1$.*

Proof. Notice that there are at most $\log_\varphi n + 1$ summands in non-negative indices because of the right bound of $\log_\varphi n$ proved in Proposition 4.3. Applying Lemma 5.8 we know that the left edge of T has the form

$$\underbrace{110101\dots10}_{z_L \text{ zeros}}\dots.$$

All of these zeros occur in negative indices. There are then $n - \log_\varphi n - 1$ summands to be distributed in the negative indices and there are at least z_L zeros in these indices. By the pigeonhole principle there is a summand to the left of index $-n - z_L + \log_\varphi n + 1$ as desired. □

Proposition 5.10. *Pick $n \geq 20$. Then we have that the left edge of the game state T achieved by playing Phase I of the $\mathcal{S}_{\mathscr{L}}\mathcal{C}_{\mathscr{R}}$ strategy on ${}_0(n)$ has the form:*

$$\underbrace{1101\ldots10}_{z \text{ zeros}}\ldots$$

where each zero is in a position left of index $-\log_\varphi n - 3$ and $z := n - 2\log_\varphi n - 7$.

Proof. Let $L^{(1)} := n - \log_\varphi n - 1$. Then in the spirit of Lemma 5.9 recursively define

$$z^{(j)} := \frac{L^{(j)} - \log_\varphi n - 5}{2} \qquad (j \geq 1)$$
$$L^{(j)} := n + z^{(j-1)} - \log_\varphi n - 1. \qquad (j \geq 2)$$

The key here is that $z^{(j)} \leq z_{L^{(j)}}$ by the definition of the latter term in Lemma 5.8. We then define $L^{(j)}$ in terms of $z^{(j-1)}$ using the result of Lemma 5.9, motivated by the idea that we will always have a summand to the left of $L^{(j)}$.

We now explicitly compute that

$$\begin{aligned}
L^{(j)} &= n + \frac{L^{(j-1)}}{2} - \frac{3\log_\varphi n}{2} - \frac{7}{2} \\
L^{(j)} - L^{(j-1)} &= \frac{L^{(j-1)} - L^{(j-2)}}{2} \\
L^{(J)} - L^{(1)} &= \sum_{j=0}^{J-2} L^{(j+2)} - L^{(j+1)} = \left(2 - \frac{1}{2^{J-2}}\right)\left(L^{(2)} - L^{(1)}\right) \\
L^{(J)} &= \left(2 - \frac{1}{2^{J-2}}\right)\left(\frac{n}{2} - \log_\varphi n - 3\right) + n - \log_\varphi n - 1 \\
&= \left(2 - \frac{1}{2^{J-1}}\right) n - \left(3 - \frac{1}{2^{J-2}}\right)\log_\varphi n - \left(7 - \frac{3}{2^{J-2}}\right).
\end{aligned}$$

Notice that for $n \geq 20$ we have for all j that $L^{(j)} \geq \log_\varphi n + 3$. Thus, we can apply Lemmas 5.8 and 5.9 repeatedly as many times as we wish to obtain the desired left edge and a summand to the left of $L^{(j)}$ for every j. As $j \to \infty$ we have that $L^{(j)}$ approaches $2n - 3\log_\varphi n - 7$ monotonically for $n \geq 20$. Likewise, $z^{(j)}$ approaches $n - 2\log_\varphi n - 6$ monotonically for $n \geq 20$ as $j \to \infty$.

Therefore since the true number of zeros must be an integer, we know that there must be at least $n - 2\log_\varphi n - 7$ zeros as claimed, and they must be arranged as above by the specific claims about the left edge made in Lemma 5.8. □

We can then immediately conclude from the pigeonhole principle that there is a summand in T to the left of index $-2n + 3\log_\varphi n + 8$, confirming Proposition 5.6.

Furthermore, we can also immediately conclude Proposition 5.5 using the techniques of Section 5. Namely, we know that in T there are summands $110101\ldots01$ at indices to the left of $-2n+3\log_\varphi n+8$ and $-2n + 3\log_\varphi n + 7 + 2j$ for $1 \leq j \leq n - 2\log_\varphi n - 7$. This implies that $-\mathcal{I}nd(T)$ is bounded below by

$$2n - 3\log_\varphi n - 8 + \sum_{j=1}^{n-2\log_\varphi n-7} \left(2n - 3\log_\varphi n - 7 - 2j\right) - \log_\varphi n.$$

Evaluating this sum gives

$$-\mathcal{I}nd(T) \geq n^2 - 6n - 3n\log_\varphi n + 5\log_\varphi n + 2(\log_\varphi n)^2 - 1.$$

We know that each split decreases the index sum by exactly one in the Bergman Game, and so this tells us that there are at least this many splits, implying Proposition 5.5.

5.3. **Construction of an $O(|S|)$ Bergman Game.** Here, we construct a $O(|S|)$ Bergman Game given arbitrary initial state. We thus establish a tight bound $\Theta(n)$ on the shortest number of moves which can ensure completion of a game with $n = |S|$ summands arranged in any initial state, for all n, and thus we prove Theorem 1.9. We play this game via the following strategy.

Definition 5.11. The Quick Termination ($\mathcal{QT}$) strategy instructs the players to choose moves in the following order of priority.

(1) Combine.
(2) Split anywhere with more than two summands.
(3) Split at the 2 in a configuration $(\dots, a_1, a_2, 0, 2, 0, a_3, \dots)$, $a_1 > 0$ or $a_3 > 0$.
(4) Split at the 2 in a configuration $(\dots, 0, a_2, 0, 2, 0, 0, \dots)$, $a_2 \geq 2$.
(5) Split at the rightmost possible index.

Proposition 5.12. *The $\mathcal{QT}$ strategy always produces a game of length at most $4\,|S|$ moves from any starting state with $|S|$ summands.*

Proof. Let $n = |S|$. We may only combine at most n times. Furthermore, when we take (2) or (3) in Definition 5.11, we immediately create opportunity for a combine. Additionally, taking (4) creates an index with three summands, at which we then split (3) and subsequently combine. Therefore, doing either (1), (2), (3), or (4) causes us to combine within the next three moves. As such, we may only make $3n$ total of these moves.

We now bound the number of (5) moves. By construction, the game state S which precedes a (5) move contains only 0's, 1's, and 2's and no adjacent summands, and every 2 is surrounded by a neighborhood $(\dots, 0, a_2, 0, 2, 0, 0, \dots)$, where $a_2 < 2$. Then the neighborhood of any 2 will look like $(a_1, 0, 0, 1, 0, 1, 0, \dots, 1, 0, 1, 0, 2, 0, 0)$. We call these configurations A_k, where k is the number of non-zero indices in the configuration, excluding a_1. Call the configuration $(1, 0, 1, 0, \dots, 1, 0, 1) = B_k$, where again k refers to the number of non-zero indices. In general then, any game state just prior to making a (5) move is a finite string of disjoint A_k's and B_k's, padded with possible intervening zeros.

By Definition 5.11, we first split at the right-most possible index when we make a (5) move, which is also the 2 of the right-most A_k. We do not create an opportunity for moves (1)-(4) but rather create an A_{k-1} two indices to the left of the original A_k, so inductively, we make k splits in a row from right to left and convert our A_k into a state $(a_1, 1, 0, 0, 1, 0, 1, 0, \dots, 1, 0, 1, 0, 1, 0)$. Regardless of the value of a_1, we may perform no more (5) moves in this region of S; depending on a_1, we may play one more combine, and then we will have no more moves in this region. Therefore, playing on each A_k provides at most k consecutive (5) moves. Furthermore, these moves do not alter any part of S to the right of A_k. Therefore, once we play our k splits on an A_k, we never play on any indices to the right of its left edge for the remainder of the game.

Then the total number of (5) moves is exactly the total number of split moves played on all the A_k which ever form. Each A_k gives k (5) moves, which is less than $k + 1$ the number of summands in an A_k. Then the total number of option (5) moves is less than the total number of summands in all the A_k, which is at most n. We have $3n$ moves from options (1)-(4) and n from (5), giving a linear bound of $4n$ total moves from any initial state. □

We therefore deduce that $\hat{B}'(n) = O(n)$. Together with Theorem 1.9, Theorem 1.9 follows: $\hat{B}'(n) = \Theta(n)$.

We may generalize nearly all of our results about the Bergman Game to a much larger class of games, called Generalized Bergman Games. We define these games here and mention major results, but we refer to our extended paper [3] for a fuller exposition and proofs.

6. The Generalized Bergman Game

Definition 6.1. We say a sequence is a Positive Linear Recurrence Sequence (PLRS) if it is given by a linear recurrence with characteristic polynomial $x^k - c_1x^{k-1} - \cdots - c_k$ for some c_i with $c_1, c_k > 0$ and $k \geq 2$. We say it is non-increasing (non-increasing PLRS) if $c_1 \geq c_2 \geq \ldots \geq c_k > 0$. For convenience if $j > k$ we let $c_j = 0$.

Definition 6.2. The Bergman Game derives its moves from the Fibonacci recurrence relation. We can generalize it to an arbitrary non-increasing PLRS of depth at least two, which yields the Generalized Bergman Game (GBG). The Gneralized Bergman Game is also played on states which are doubly infinite tuples $S = (\ldots, S(-1), S(0), S(1), \ldots)$ of non-negative integers.

On a given non-increasing PLRS with characteristic polynomial $x^k - c_1x^{k-1} - \cdots - c_k$ the combine and split moves are given as follows.

- Combine: if there is an i such that $S(i+j) \geq c_{k-j}$ for each $j \in \{0, 1, 2, \ldots, k\}$ then decrease each $S(i+j)$ by c_{k-j} and increase $S(i+k+1)$ by 1. E.g.,

$$(c_k, \ldots, c_1, 0) \to (\underbrace{0, ...0}_{k}, 1).$$

- Split of type $1 \leq p \leq k-1$: If there is an i such that $S(i) \geq c_p + 1$ and $S(i+j) \geq c_{p-j}$ for $j \in \{1, \ldots, p-1\}$, decrease $S(i+j)$ by c_{p-j}, and then decrease $S(i)$ by $c_p + 1$ and increase $S(i+p)$ by 1 and increase $S(i-j)$ each by $d_{p,j}$ for $j \in \{1, 2, 3, \ldots, k\}$, where $d_{p,j} := c_j - c_{j+p}$. If $j + p > k$ then $d_{p,j} = c_j$. E.g. a type one split has the form,

$$(\underbrace{0, \ldots, 0}_{k}, c_1 + 1, 0) \to (d_{1,k}, d_{1,k-1}, \ldots, d_{1,1}, 0, 1).$$

A split of type p has the form

$$(0, \ldots, 0, c_p + 1, c_{p-1}, \ldots, c_1, 0) \to (d_{p,k}, d_{p,k-1}, \ldots, d_{p,1}, 0, 0, \ldots, 0, 1).$$

Note that we can think of this as the combination of a "reverse combine" at index i, and then a combine performed into index $i+p$. This is illustrated below:

$$\begin{aligned}(0, \ldots, 0, c_p + 1, c_{p-1}, \ldots, c_1, 0) &\to (c_k, \ldots, c_1, c_p, c_{p-1}, \ldots, c_1, 0)\\ &\to (d_{p,k}, d_{p,k-1}, \ldots, d_{p,1}, 0, 0, \ldots, 0, 1).\end{aligned}$$

- Split Restriction: If a player may perform a split of type p and a split of type $p' > p$ at index i, then the player must perform the split of type p rather than the split of type p'. This condition makes the Generalized Bergman Game on the (c, k)-binacci reccurences (those where $c_i = c_j$ for all $1 \leq i, j \leq k$) more accurately reflect a Generalized Zeckendorf Game (see [1]) by disallowing moves like the following type two split for $c_1 = c_2 = c_3 = 1$:

$$(0, 0, 0, 2, 1, 0) \to (1, 1, 0, 0, 0, 1).$$

In a proper Generalized Zeckendorf Game where we're far from the boundary, players should instead take the type one splits displayed below:

$$(0, 0, 0, 2, 1, 0) \to (1, 0, 0, 0, 2, 0) \to (1, 1, 0, 0, 0, 1).$$

> The reader might then ask why we do not allow only splits of type 1. This essentially rests on the fact that we wish for the final states of the Generalized Bergman Game to be base-β expansions for β the dominating root of the given recurrence.

Once again, players take turns making moves from some starting state S, and the game ends when there are no more moves available to play.

6.1. **Generalized Results.**

Theorem 6.3. *We have $\widehat{\mathcal{M}}_{\max}(n) = \Theta(n^2)$. That is, the longest possible Generalized Bergman Game for a given non-increasing PLRS with n summands terminates in $\Theta(n^2)$ moves.*

Theorem 6.4. *We have $\widehat{\mathcal{M}}_{\min}(n) = \Omega(n)$. That is, the shortest possible game length which is available from all starting states with n summands is $\Omega(n)$ on any non-increasing PLRS.*

Theorem 6.5. *We have $\widehat{\mathcal{L}}(n, b) = O(n) + O(b)$ and $\widehat{\mathcal{L}}(n) = \Theta(n)$. That is, the maximum that a summand may be moved left throughout the game is at most linear in the number of summands and the length of the initial window. Furthermore, when the summands in initial state are concentrated in a single index, we know that this quantity is exactly linear in the number of summands.*

All proofs are to be found in [3]. Together, these theorems nearly characterize termination times and index ranges of Generalized Bergman Games completely, with the exception of an $O(n)$ upper bound for $\widehat{\mathcal{M}}_{\min}(n)$ in the general case.

Remark. In general, we may not rely on the elementary arguments of this paper in the general case. Instead, we leverage several results in algebraic number theory from both [11] and [10] to costruct alternative arguments. However, we still have not recovered agreement in the dominating coefficient of termination time, nor do we construct an $O(|S|)$ game from arbitrary game state.

7. Future Work

For game-theoretic concerns, we are interested in showing that a winning strategy exists for either player–either for the Bergman Game or in general, although we expect a solution to be non-constructive (as in the Zeckendorf Game, [2]) and any winning strategy algorithmically hard.

The behavior of games wherein each player makes one of all of their available moves with uniform probability at each term is currently completely unexplored. Through numerical analysis, we have been able to make conjectures concerning their behavior. However, there are no concretely proven results. This closely reflects the state of study for random Zeckendorf games (see [2]). Our primary conjecture concerning game length says that as the number of summands grows the distribution of the number of moves used in a random game approaches a Gaussian (see Conjecture A.2). We expect that the current techniques for proving things about these games will not easily apply to random games. The numerics for game length are explored in detail in Appendix A.

Future work can also be done in extending our results about the Bergman Game to the Generalized Bergman Game. For more details, see the future work section of [3].

Appendix A. Numerics of Random Games

Similarly to the numerics collected on the Zeckendorf Game in [2], we can consider the length of a random game played from an initial starting state ${}_0(n)$. For ease of coding, we only explored such numerics for the Bergman Game. However, we expect that similar numerical analysis of Generalized Bergman Games would yield similar results.

Remark. We choose to take the initial starting state to be ${}_0(n)$ so that we have a natural one-dimensional parameter to vary, as the limiting behavior if of the most interest here.

For clarity, we define what we mean by a random game.

Definition A.1. To play random game on an initial state S, list every available move from this initial state, and play one with a uniform probability. Continue this process from each new state created to generate a tree with probabilities attached to each edge. The probability of achieving a certain game (i.e., a path down the tree) is the product of all the probabilities associated to each edge.

We then form the following conjecture based on the data collected from our C++ code, which can be found at

`https://github.com/FayeAlephNil/public-bergman-code`

For any questions about the code, please email Faye Jackson (alephnil@umich.edu).

Conjecture A.2. *The distribution $\mathcal{L}_n$ of the length of a random game payed from the initial state $_0(n)$ converges to a Gaussian with linear mean $\mu_n \approx 2n + O(1)$.*

This conjecture is supported by Figures 2 to 4.

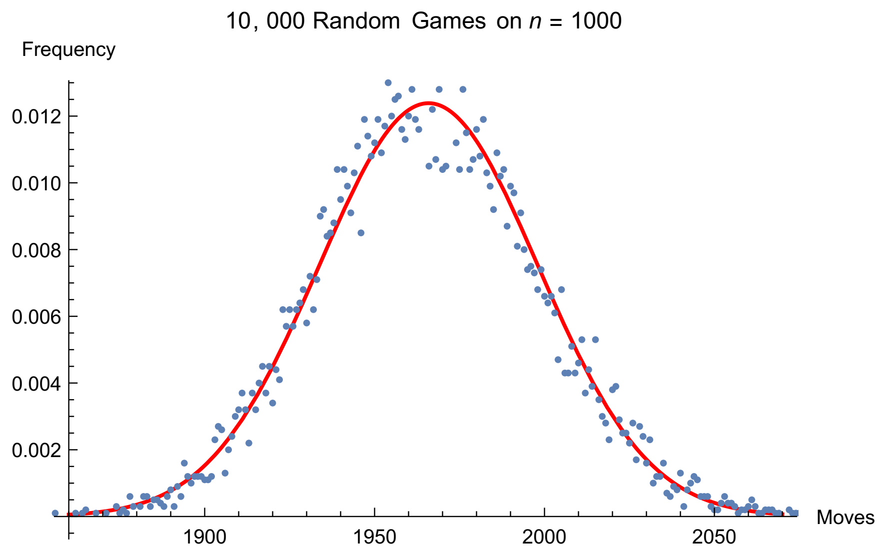

FIGURE 2. Frequency Graph of the Number of Moves in 10,000 simulations of the Bergman Game with random moves when $n = 1000$ with the best fit Gaussian overlaid in red.

The conjecture is also supported by the moments of the statistical sample from random games. After renormalizing to have mean zero and variance one the 20,000 random trials on $n = 2000$ give the following table of moments:

Moments	3	4	5	6	7	8
Data	0.08	3.03	0.90	15.74	12.19	120.50
Gaussian	0	3	0	15	0	105

Given the close agreement with the moments of the Gaussian, we have confidence that the first statement of the conjecture holds. For the second statement in the conjecture, numerics show that the best fit line to the average number of moves in a random game has linear coefficient 1.98 with coefficient of determination 0.999991, indicating a near perfect fit. In fact, we neglected to plot the best fit line in Figure 4 because it was too difficult to see at the same time as the data.

Despite the strong numerical evidence, we expect that this conjecture is rather difficult to prove. Intuitively the length of a random game matches a Gaussian because so many independent random

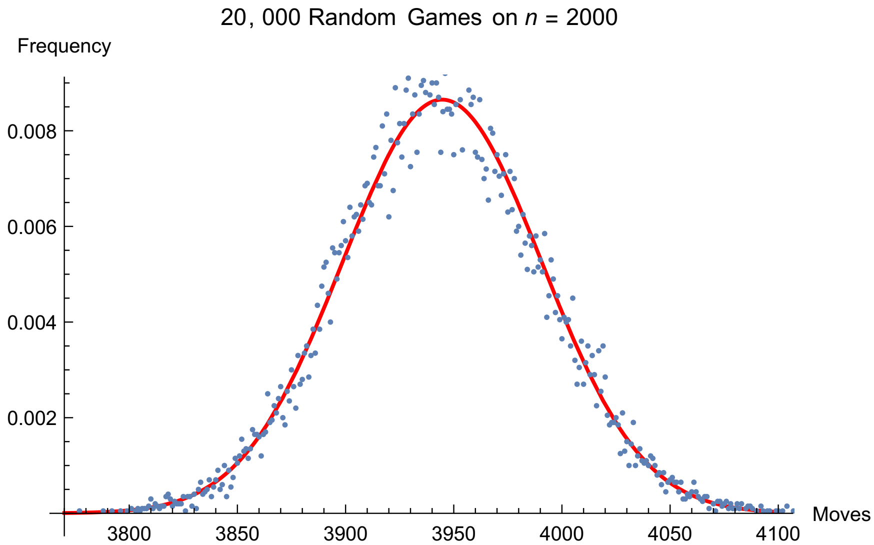

FIGURE 3. Frequency Graph of the Number of Moves in 20,000 simulations of the Bergman Game with random moves when $n = 2000$ with the best fit Gaussian overlaid in red.

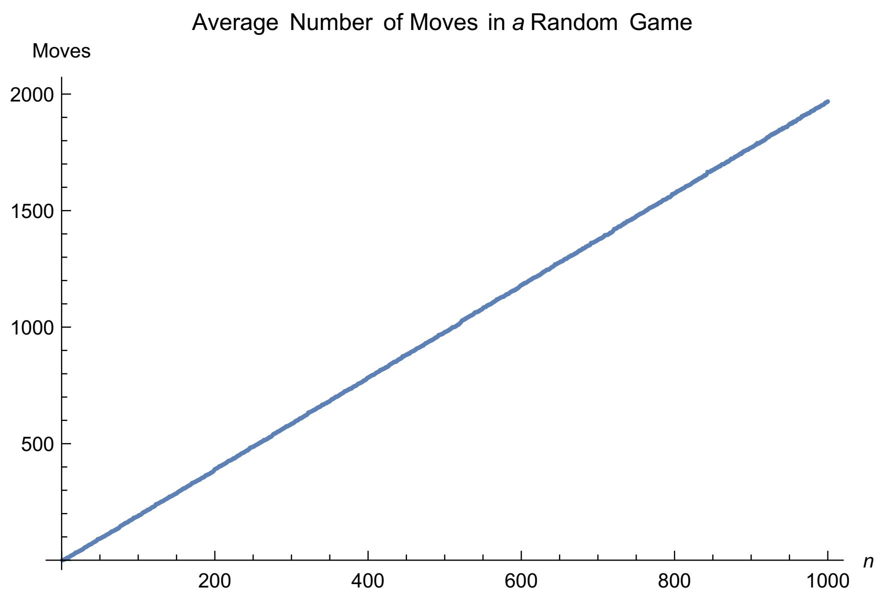

FIGURE 4. Graph of the average number of moves in random Bergman Games with initial state $n \in [1, 1000]$, averaging over 1,000 trials.

choices are performed as we traverse the tree. However, there is no clear recursive structure (i.e. a martingale interpreation) nor a way to express the process as a sum of binomial random variables in order to exploit a Central Limit Theorem and thereby prove the result. This puts the problem beyond the reach of our current techniques.

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