GCD OF SUMS OF k CONSECUTIVE SQUARES OF GENERALIZED FIBONACCI NUMBERS

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Abstract. In 2021, Guyer and Mbirika gave two equivalent formulas that computed the greatest common divisor (GCD) of all sums of k consecutive terms in the generalized Fibonacci sequence $(G_n)_{n\geq 0}$ given by the recurrence $G_n = G_{n-1}+G_{n-2}$ for all $n \geq 2$ with integral initial conditions G_0 and G_1. In this current paper, we extend their results to the GCD of all sums of k consecutive squares of these numbers. Denoting these GCD values by the symbol $\mathcal{G}^2_{G_0,G_1}(k)$, we prove $\mathcal{G}^2_{G_0,G_1}(k) = \gcd\left(G_kG_{k+1} - G_0G_1,\ G^2_{k+1} - G^2_1,\ G^2_{k+2} - G^2_2\right)$. Moreover, we provide very tantalizing closed forms in the specific settings of the Fibonacci, Lucas, and generalized Fibonacci numbers. We close with a number of open questions for further research.

1. Introduction and motivation

Appearing in the literature as early as 1901 by Tagiuri [9], the generalized Fibonacci numbers (or so-called *Gibonacci numbers*[1]) are defined by the recurrence

$$G_n = G_{n-1} + G_{n-2} \text{ for all } n \geq 2$$

with initial conditions $G_0, G_1 \in \mathbb{Z}$. In 1963, a problem was proposed by I. D. Ruggles in the inaugural issue of the *Fibonacci Quarterly* on a closed form for the sum of any twenty consecutive Fibonacci numbers [7]. Since then, there has been numerous papers exploring the sums of consecutive Fibonacci or Lucas numbers [5, 10, 11, 3, 2, 8]. This current paper continues this line of research, extending it to the greatest common divisor (GCD) of sums of certain powers of Gibonacci numbers. This current work extends earlier work of Guyer and Mbirika who explored the GCD of the sums of k consecutive Gibonacci numbers, and consequently k consecutive Fibonacci and Lucas numbers [4]. More precisely, given $k \in \mathbb{N}$ they found the exact value of the GCD of an infinite number of finite sums $\sum_{i=1}^{k} G_i\ ,\ \sum_{i=2}^{k+1} G_i\ ,\ \sum_{i=3}^{k+2} G_i\ , \ldots$ thereby computing the GCD of the terms in the sequence $(\sum_{i=1}^{k} G_{n+i})_{n\geq 0}$. For brevity, they used the symbols $\mathcal{F}(k)$, $\mathcal{L}(k)$, and $\mathcal{G}_{G_0,G_1}(k)$, respectively, to denote the three values

$$\gcd\left\{\left(\sum_{i=1}^{k} F_{n+i}\right)_{n\geq 0}\right\},\ \gcd\left\{\left(\sum_{i=1}^{k} L_{n+i}\right)_{n\geq 0}\right\},\ \text{and}\ \gcd\left\{\left(\sum_{i=1}^{k} G_{n+i}\right)_{n\geq 0}\right\},$$

where $(F_n)_{n\geq 0}$, $(L_n)_{n\geq 0}$, and $(G_n)_{n\geq 0}$ denote the Fibonacci, Lucas, and Gibonacci sequences. A main result of Guyer and Mbirika was the formula $\mathcal{G}_{G_0,G_1}(k) = \gcd(G_{k+1}-G_1,\ G_{k+2}-G_2)$ [4, Theorem 15], or an equivalent formula using generalized Pisano periods [4, Theorem 25]. Using either formula yields the values of $\mathcal{F}(k)$, $\mathcal{L}(k)$, and $\mathcal{G}_{G_0,G_1}(k)$ in Table 1.

[1]Thomas Koshy attributes Art Benjamin and Jennifer Quinn for coining this term "Gibonacci" in their 2003 book *Proofs that Really Count: The Art of Combinatorial Proof* [1].

k	$\mathcal{F}(k)$	$\mathcal{L}(k)$	$\mathcal{G}_{G_0,G_1}(k)$
$k \equiv 0, 4, 8 \pmod{12}$	$F_{k/2}$	$5F_{k/2}$	$F_{k/2}$[a] or $5F_{k/2}$[b]
$k \equiv 2, 6, 10 \pmod{12}$	$L_{k/2}$	$L_{k/2}$	$L_{k/2}$
$k \equiv 3, 9 \pmod{12}$	2	2	2[c]
$k \equiv 1, 5, 7, 11 \pmod{12}$	1	1	1[c]

TABLE 1. Closed forms for the values $\mathcal{F}(k)$, $\mathcal{L}(k)$, and $\mathcal{G}_{G_0,G_1}(k)$

[a] This value holds if and only if $\gcd(G_0 + G_2, G_1 + G_3) = 1$.
[b] This value holds if and only if $\gcd(G_0 + G_2, G_1 + G_3) \neq 1$.
[c] These values hold if $G_1^2 - G_0G_1 - G_0^2 = \pm 1$.

In this current paper, we explore the GCD of sums of **squares** of k consecutive generalized Fibonacci numbers (and in particular, the Fibonacci and Lucas numbers). For brevity, we use the symbols $\mathcal{F}^2(k)$, $\mathcal{L}^2(k)$, and $\mathcal{G}^2_{G_0,G_1}(k)$, respectively, to denote the three values

$$\gcd\left\{\left(\sum_{i=1}^{k} F_{n+i}^2\right)_{n\geq 0}\right\}, \ \gcd\left\{\left(\sum_{i=1}^{k} L_{n+i}^2\right)_{n\geq 0}\right\}, \text{ and } \gcd\left\{\left(\sum_{i=1}^{k} G_{n+i}^2\right)_{n\geq 0}\right\}.$$

This current paper arose when second author Spilker offered a proof of a conjecture, stating $\mathcal{F}^2(k) = F_k$ when k is even, given in the paper of Guyer and Mbirika [4, Question 54], who wrote "We feel this is simply too beautiful a result to not be true." Spilker's proof of this conjecture is a motivation for this current paper, wherein we extend the latter Fibonacci result to the more general setting of $\mathcal{G}^2_{G_0,G_1}(k)$ for all k values. In this paper, we prove the following main results given in Table 2. Note that the value μ (given in Definition 2.3) equals $G_1^2 - G_0G_1 - G_0^2$, and the value g_k equals $\gcd\left(G_{k+1}^2 - G_1^2,\ G_{k+2}^2 - G_2^2\right)$.

k	$\mathcal{F}^2(k)$	$\mathcal{L}^2(k)$	$\mathcal{G}^2_{G_0,G_1}(k)$	Proof in this paper
k even and $5 \nmid \mu$	F_k	$5F_k$	F_k	Theorems 5.2, 5.4, and 4.7, respectively
k even and $5 \mid \mu$	F_k	$5F_k$	$5F_k$	Theorems 5.2, 5.4, and 4.7, respectively
$k \equiv 3 \pmod{6}$	2	2	$\gcd(2\mu, g_k)$	Theorems 5.2, 5.4, and 3.11, respectively
$k \equiv 1, 5 \pmod{6}$	1	1	$\gcd(2\mu, g_k)$	Theorems 5.2, 5.4, and 3.11, respectively

TABLE 2. Closed forms for the values $\mathcal{F}^2(k)$, $\mathcal{L}^2(k)$, and $\mathcal{G}^2_{G_0,G_1}(k)$

Remark 1.1. Compare Tables 1 and 2 and the closed forms for the values $\mathcal{F}(k)$ and $\mathcal{F}^2(k)$, respectively, when k is even. Observe that $\mathcal{F}(k) = F_{k/2}$ when $k \equiv 0 \pmod 4$, whereas $\mathcal{F}(k) = L_{k/2}$ when $k \equiv 2 \pmod 4$. However in this new setting of the GCD of sums of squares of k consecutive Fibonacci numbers, the values $\mathcal{F}^2(k)$ have no dependency on the residue class modulo 4 of k when k is even, so in some sense $\mathcal{F}^2(k)$ is more well behaved than its seemingly simpler counterpart $\mathcal{F}(k)$. Similar phenomena happens for k even when we compare $\mathcal{L}(k)$ and $\mathcal{L}^2(k)$.

The paper is broken down as follows. In Section 2, we provide the necessary definitions used in the subsequent sections. In Section 3, we give a simple formula for the value $\mathcal{G}^2_{G_0,G_1}(k)$. In Section 4, we provide nice closed forms for the value $\mathcal{G}^2_{G_0,G_1}(k)$. Moreover in Section 5, we provide similarly nice closed forms for the values $\mathcal{F}^2(k)$ and $\mathcal{L}^2(k)$, respectively. Finally in Section 6, we end with some tantalizing open questions.

2. Preliminary definitions

Definition 2.1. *The generalized Fibonacci sequence* $(G_n)_{n\geq 0}$ *is defined by the recurrence relation*

$$G_n = G_{n-1} + G_{n-2}$$

for $n \geq 2$ *and arbitrary initial conditions* $G_0, G_1 \in \mathbb{Z}$. *The Fibonacci sequence* $(F_n)_{n\geq 0}$ *is recovered when* $G_0 = 0$ *and* $G_1 = 1$, *and the Lucas sequence* $(L_n)_{n\geq 0}$ *is recovered when* $G_0 = 2$ *and* $G_1 = 1$. *For brevity, we use the term Gibonacci sequence to refer to any generalized Fibonacci sequence.*

Convention 2.2. For reasons explained in Theorem 3.6 and Convention 3.7, it suffices to consider only the Gibonacci sequences with relatively prime initial conditions G_0 and G_1.

Definition 2.3. *The characteristic of the Gibonacci sequence* $(G_n)_{n\geq 0}$ *is denoted* μ *and is defined as* $\mu = G_1^2 - G_0G_1 - G_0^2$.

Remark 2.4. Many authors have differing notation and/or equivalent definitions for this value μ. For instance, Guyer-Mbirika denote this value as D_{G_0,G_1} to highlight this value's dependency on the initial conditions [4]. Moreover, Koshy maintains the symbol μ, but defines it as $G_1^2 + G_1G_2 - G_2^2$ [6]. On the other hand, Vajda reserves no symbol for μ but writes the value as $G_1^2 - G_0G_2$ [12].

Lastly, it is well known that the Fibonacci and Lucas sequences (and more generally any Gibonacci sequence) under a modulus are periodic. The lengths of the periods of these sequences are called Pisano periods. More generally, we have the following definition of the length of the period of a Gibonacci sequence under a modulus.

Definition 2.5. *Let* $m \geq 2$. *The generalized Pisano period,* $\pi_{G_0,G_1}(m)$, *of the Gibonacci sequence* $(G_n)_{n\geq 0}$ *is the smallest positive integer* r *such that*

$$G_r \equiv G_0 \pmod{m} \quad \text{and} \quad G_{r+1} \equiv G_1 \pmod{m}.$$

The value r *is dependent on both the initial conditions* $G_0, G_1 \in \mathbb{Z}$ *and the modulus* m. *In the Fibonacci (respectively, Lucas) setting we denote this period by* $\pi_F(m)$ *(respectively,* $\pi_L(m)$*).*

3. The generalized Fibonacci setting

3.1. A formula for $\mathcal{G}^2_{G_0,G_1}(k)$.

In this subsection, we derive the following formula for $\mathcal{G}^2_{G_0,G_1}(k)$ in Theorem 3.4:

$$\mathcal{G}^2_{G_0,G_1}(k) = \gcd\left(G_kG_{k+1} - G_0G_1,\ G^2_{k+1} - G_1^2,\ G^2_{k+2} - G_2^2\right).$$

Lemma 3.1. *Let* $(a_n)_{n\geq 0}$ *be a sequence of integers. The following identity holds:*

$$\gcd(a_0, a_1, a_2, a_3, \ldots) = \gcd(a_0,\ a_1 - a_0,\ a_2 - a_1,\ a_3 - a_2, \ldots).$$

Proof. Set $d := \gcd(a_0, a_1, a_2, a_3, \ldots)$ and $e := \gcd(a_0,\ a_1 - a_0,\ a_2 - a_1,\ a_3 - a_2, \ldots)$. We will show that $d = e$. By assumption, d divides a_i for all $i \geq 0$, and hence d divides $a_{i+1} - a_i$ for all $i \geq 0$. Hence d divides e and so $d \leq e$. To prove the reverse inequality, observe that e divides both a_0 and $a_1 - a_0$, and hence e divides a_1. Consequently since e divides a_1 and $a_2 - a_1$, then e divides a_2. Continuing inductively in this manner, we see that e must divide a_i for all i. Hence e divides d and so $e \leq d$. □

The next two lemmas are well-known results whose proofs can be found in Vajda [12].

Lemma 3.2. *For all $k \geq 1$, we have*

$$\sum_{i=1}^{k} G_i^2 = G_k G_{k+1} - G_0 G_1.$$

Proof. See Identity (44) of Vajda [12, p. 43]. □

Lemma 3.3. *For all $m, n \in \mathbb{Z}$, we have*

$$G_{m+n} = G_{m+1} F_n + G_m F_{n-1}.$$

Proof. See Identity (8) of Vajda [12, pp. 24–25]. □

We are now ready to prove the main theorem of this subsection, namely a formula for the value $\mathcal{G}^2_{G_0,G_1}(k)$.

Theorem 3.4. *For all $k \geq 1$, we have*

$$\mathcal{G}^2_{G_0,G_1}(k) = \gcd\left(G_k G_{k+1} - G_0 G_1,\ G_{k+1}^2 - G_1^2,\ G_{k+2}^2 - G_2^2\right).$$

Proof. Fix $k \geq 1$ and for $n \geq 0$, set $S_n := \sum_{i=1}^{k} G_{n+i}^2$ and $H_n := S_{n+1} - S_n$. Hence we have

$$\mathcal{G}^2_{G_0,G_1}(k) = \gcd\left(S_0, S_1, S_2, S_3, \ldots\right) = \gcd\left(S_0, H_0, H_1, H_2, \ldots\right),$$

where the second equality holds by Lemma 3.1. We will show that for all $n \geq 2$, the value H_n is a linear combination of S_0, H_0, and H_1, and hence the last equality above coincides with $\gcd(S_0, H_0, H_1)$, and the result follows. To that end, we first prove the following two equivalent expressions for S_n and H_n, respectively:

$$S_n = G_{n+k} G_{n+k+1} - G_n G_{n+1} \tag{3.1}$$

$$H_n = G_{n+(k+1)}^2 - G_{n+1}^2. \tag{3.2}$$

Identity (3.1) holds by the sequence of equalities

$$\begin{aligned}
S_n &= \sum_{i=1}^{k} G_{n+i}^2 \\
&= \sum_{i=1}^{n+k} G_i^2 - \sum_{i=1}^{n} G_i^2 \\
&= (G_{n+k} G_{n+k+1} - G_0 G_1) - (G_n G_{n+1} - G_0 G_1) && \text{by Lemma 3.2} \\
&= G_{n+k} G_{n+k+1} - G_n G_{n+1}.
\end{aligned}$$

Moreover, Identity (3.2) holds by the sequence of equalities

$$\begin{aligned}
H_n &= S_{n+1} - S_n \\
&= \sum_{i=1}^{k} G_{n+1+i}^2 - \sum_{i=1}^{k} G_{n+i}^2 \\
&= (G_{n+2}^2 + G_{n+3}^2 + \cdots + G_{n+(k+1)}^2) - (G_{n+1}^2 + G_{n+2}^2 + \cdots + G_{n+k}^2) \\
&= G_{n+(k+1)}^2 - G_{n+1}^2.
\end{aligned}$$

Applying Lemma 3.3 to both $G_{n+(k+1)}$ and G_{n+1} in the latter equality and expanding the squared terms, we have

$$\begin{aligned} H_n &= (G_{k+2}F_n + G_{k+1}F_{n-1})^2 - (G_2F_n + G_1F_{n-1})^2 \\ &= \left(G_{k+2}^2F_n^2 + G_{k+1}^2F_{n-1}^2 + 2(G_{k+1}G_{k+2})(F_{n-1}F_n)\right) \\ &\qquad\qquad - \left(G_2^2F_n^2 + G_1^2F_{n-1}^2 + 2(G_1G_2)(F_{n-1}F_n)\right) \\ &= \left(G_{k+2}^2 - G_2^2\right)\cdot F_n^2 + \left(G_{k+1}^2 - G_1^2\right)\cdot F_{n-1}^2 \\ &\qquad\qquad + 2\left(G_{k+1}G_{k+2} - G_1G_2\right)\cdot(F_{n-1}F_n) \\ &= H_1F_n^2 + H_0F_{n-1}^2 + 2S_1(F_{n-1}F_n). \end{aligned}$$

where the last equality holds by Identities (3.1) and (3.2). Lastly, since $H_0 = S_1 - S_0$ implies $S_1 = H_0 + S_0$, we can rewrite the last equality as

$$H_n = S_0(2F_{n-1}F_n) + H_0(F_{n-1}^2 + 2F_{n-1}F_n) + H_1F_n^2,$$

and hence for $n \geq 2$, we see that H_n is an integer linear combination of S_0, H_0, and H_1, as desired. We conclude that

$$\begin{aligned} \mathcal{G}^2_{G_0,G_1}(k) &= \gcd\left(S_0, H_0, H_1, H_2, \ldots\right) \\ &= \gcd\left(S_0, H_0, H_1\right) \\ &= \gcd\left(G_kG_{k+1} - G_0G_1,\ G_{k+1}^2 - G_1^2,\ G_{k+2}^2 - G_2^2\right), \end{aligned}$$

where the last equality holds by Identities (3.1) and (3.2). □

Next we show that it is sufficient to explore only the Gibonacci sequences which have relatively prime initial values

Lemma 3.5. *For all $n \in \mathbb{Z}$, the values $\gcd(G_{n+1}, G_{n+2})$ and $\gcd(G_n, G_{n+1})$ coincide. In particular, $\gcd(G_0, G_1) = \gcd(G_n, G_{n+1})$ for all $n \in \mathbb{Z}$.*

Proof. See Lemma 18 of Guyer-Mbirika [4]. □

Theorem 3.6. *Fix $G_0, G_1 \in \mathbb{Z}$ and set $d := \gcd(G_0, G_1)$. Consider the two Gibonacci sequences $(G_n)_{n\geq 0}$ and $(G'_n)_{n\geq 0}$, where $(G'_n)_{n=0}^{\infty}$ is generated by the relatively prime initial conditions $G'_0 = \frac{G_0}{d}$ and $G'_1 = \frac{G_1}{d}$. Then the following equality holds:*

$$\mathcal{G}^2_{G_0,G_1}(k) = d^2 \cdot \mathcal{G}^2_{G'_0,G'_1}(k).$$

Proof. Set $d := \gcd(G_0, G_1)$. By Lemma 3.5, we have $\gcd(G_{k+1}, G_{k+2}) = \gcd(G_0, G_1) = d$ for all $k \in \mathbb{Z}$. By Theorem 3.4, we have $\mathcal{G}^2_{G_0,G_1}(k) = \gcd\left(G_kG_{k+1} - G_0G_1,\ G_{k+1}^2 - G_1^2,\ G_{k+2}^2 - G_2^2\right)$. Moreover, since d divides G_0 and G_1, then d divides every term in the sequence $(G_n)_{n\geq 0}$. In particular, $\frac{G_kG_{k+1}-G_0G_1}{d^2}$, $\frac{G_{k+1}^2-G_1^2}{d^2}$, and $\frac{G_{k+2}^2-G_2^2}{d^2}$ are integers. Observe the sequence of equalities

$$\begin{aligned} \mathcal{G}^2_{G_0,G_1}(k) &= \gcd\left(G_kG_{k+1} - G_0G_1,\ G_{k+1}^2 - G_1^2,\ G_{k+2}^2 - G_2^2\right) \\ &= \gcd\left(d^2\cdot\frac{G_kG_{k+1} - G_0G_1}{d^2},\ d^2\cdot\frac{G_{k+1}^2 - G_1^2}{d^2},\ d^2\cdot\frac{G_{k+2}^2 - G_2^2}{d^2}\right) \\ &= d^2\cdot\gcd\left(\frac{G_kG_{k+1} - G_0G_1}{d^2},\ \frac{G_{k+1}^2 - G_1^2}{d^2},\ \frac{G_{k+2}^2 - G_2^2}{d^2}\right). \end{aligned}$$

However, by Theorem 3.4, the value $\gcd\left(\frac{G_kG_{k+1}-G_0G_1}{d^2},\ \frac{G_{k+1}^2-G_1^2}{d^2},\ \frac{G_{k+2}^2-G_2^2}{d^2}\right)$ is the GCD of the sum of k consecutive squares of Gibonacci numbers in the new sequence $(G'_n)_{n=0}^{\infty}$ generated by the initial values $G'_0 = \frac{G_0}{d}$ and $G'_1 = \frac{G_1}{d}$. Clearly G'_0 and G'_1 are relatively prime. In particular, we have $\mathcal{G}^2_{G_0,G_1}(k) = d^2 \cdot \mathcal{G}^2_{G'_0,G'_1}(k)$, as desired. □

Convention 3.7. In order to give a complete classification of the GCD of every sum of k consecutive squares of Gibonacci numbers, as a consequence of Theorem 3.6, we need only to consider Gibonacci sequences with relatively prime initial values.

3.2. **Simplified formulas for $\mathcal{G}^2_{G_0,G_1}(k)$ when k is even versus odd.** In this subsection, we reveal that the formula for $\mathcal{G}^2_{G_0,G_1}(k)$ given in Theorem 3.4 in the previous subsection can be simplified further if we consider the two cases of the parity of k as follows:

$$\mathcal{G}^2_{G_0,G_1}(k) = \begin{cases} \gcd\left(G_{k+1}^2 - G_1^2,\ G_{k+2}^2 - G_2^2\right), & \text{if } k \text{ is even},\\ \gcd\left(2\mu,\ G_{k+1}^2 - G_1^2,\ G_{k+2}^2 - G_2^2\right), & \text{if } k \text{ is odd},\end{cases}$$

where the characteristic μ is defined as $G_1^2 - G_0G_1 - G_0^2$ (recall Definition 2.3 and Remark 2.4). Pivotal in the proof of the formulas above is the use of the following lemma known as Cassini's identity for generalized Fibonacci sequences.

Lemma 3.8 (Generalized Cassini's Identity)**.** *Let $n \geq 1$ be given. Then the following identity holds: $G_{n+1}G_{n-1} - G_n^2 = (-1)^n \cdot \mu$, where $\mu = G_1^2 - G_0G_1 - G_0^2$.*

Proof. See Identity (28) of Vajda [12, p. 32]. □

Lemma 3.9. *Let $D_{G_n,G_{n+1}}$ denote the value $G_{n+1}^2 - G_nG_{n+1} - G_n^2$. Then the following holds:*

$$D_{G_n,G_{n+1}} = (-1)^n \cdot \mu \tag{3.3}$$

for all $n \geq 0$, where $\mu = D_{G_0,G_1}$ (i.e., the characteristic $G_1^2 - G_0G_1 - G_0^2$). In particular, we have $|D_{G_n,G_{n+1}}| = |\mu|$ for all $n \geq 0$.

Proof. See Lemma 34 of Guyer-Mbirika [4]. □

Lemma 3.10. *For all $k \geq 1$, we have*

$$\left(G_k^2 - G_0^2\right) - 3\left(G_{k+1}^2 - G_1^2\right) + \left(G_{k+2}^2 - G_2^2\right) = \begin{cases} 0, & \text{if } k \text{ is even},\\ 4\mu, & \text{if } k \text{ is odd},\end{cases}$$

where $\mu = G_1^2 - G_0G_1 - G_0^2$.

Proof. For ease of notation, set $M_i := G_{k+i}^2 - G_i^2$. Hence it suffices to show the following:

$$M_0 - 3M_1 + M_2 = \begin{cases} 0, & \text{if } k \text{ is even},\\ 4\mu, & \text{if } k \text{ is odd}.\end{cases}$$

Since $G_k = G_{k+2} - G_{k+1}$ and $G_0 = G_2 - G_1$, we have the sequence of equalities

$$\begin{aligned} M_0 &= G_k^2 - G_0^2\\ &= (G_{k+2} - G_{k+1})^2 - (G_2 - G_1)^2\\ &= \left(G_{k+2}^2 - 2G_{k+1}G_{k+2} + G_{k+1}^2\right) - \left(G_2^2 - 2G_1G_2 + G_1^2\right)\end{aligned}$$

$$= \left(G_{k+2}^2 - G_2^2\right) + \left(G_{k+1}^2 - G_1^2\right) - 2\left(G_{k+1}G_{k+2} - G_1G_2\right)$$
$$= M_2 + M_1 - 2\left(G_{k+1}G_{k+2} - G_1G_2\right).$$

Thus it follows that

$$M_0 - 3M_1 + M_2 = 2M_2 - 2M_1 - 2\left(G_{k+1}G_{k+2} - G_1G_2\right). \tag{3.4}$$

Recalling from Lemma 3.10 that $D_{G_n,G_{n+1}} = G_{n+1}^2 - G_nG_{n+1} - G_n^2$, the right side of Equation (3.4) decomposes as

$$\begin{aligned}
2M_2 - 2M_1 &- 2\left(G_{k+1}G_{k+2} - G_1G_2\right)\\
&= 2\left(G_{k+2}^2 - G_2^2\right) - 2\left(G_{k+1}^2 - G_1^2\right) - 2\left(G_{k+1}G_{k+2} - G_1G_2\right)\\
&= 2\left(G_{k+2}^2 - G_{k+1}G_{k+2} - G_{k+1}^2\right) - 2\left(G_2^2 - G_1G_2 - G_1^2\right)\\
&= 2D_{G_{k+1},G_{k+2}} - 2D_{G_1,G_2}\\
&= 2\left((-1)^{k+1}\cdot\mu\right) - 2\left((-1)^1\cdot\mu\right) && \text{by Lemma 3.10}\\
&= 2\left((-1)^{k+1}+1\right)\cdot\mu\\
&= \begin{cases} 0, & \text{if } k \text{ is even},\\ 4\mu, & \text{if } k \text{ is odd}.\end{cases}
\end{aligned}$$

Combining the latter equality with Equation (3.4), the result follows. ☐

We are now ready to state and prove the main theorem of this subsection.

Theorem 3.11. *For all $k \ge 1$, we have*

$$\mathcal{G}^2_{G_0,G_1}(k) = \begin{cases} \gcd\left(G_{k+1}^2 - G_1^2,\ G_{k+2}^2 - G_2^2\right), & \text{if } k \text{ is even},\\ \gcd\left(2\mu,\ G_{k+1}^2 - G_1^2,\ G_{k+2}^2 - G_2^2\right), & \text{if } k \text{ is odd},\end{cases}$$

where $\mu = G_1^2 - G_0G_1 - G_0^2$.

Proof. For ease of notation, set $M_i := G_{k+i}^2 - G_i^2$. Hence it suffices to show that if k is even (respectively, odd), then $\mathcal{G}^2_{G_0,G_1}(k) = \gcd(M_1, M_2)$ (respectively, $\mathcal{G}^2_{G_0,G_1}(k) = \gcd(2\mu, M_1, M_2)$). To that end, observe the sequence of equalities

$$\begin{aligned}
G_kG_{k+1} - G_0G_1 &= G_k(G_{k+2} - G_k) - G_0(G_2 - G_0)\\
&= G_kG_{k+2} - G_k^2 - G_0G_2 + G_0^2\\
&= (G_{k+1}^2 + (-1)^{k+1}\mu) - G_k^2 - (G_1^2 + (-1)^1\mu) + G_0^2 && \text{by Lemma 3.8}\\
&= (G_{k+1}^2 - G_1^2) - (G_k^2 - G_0^2) + \left((-1)^{k+1}+1\right)\cdot\mu\\
&= M_1 - M_0 + \left((-1)^{k+1}+1\right)\cdot\mu.
\end{aligned}$$

So it follows that

$$G_kG_{k+1} - G_0G_1 = \begin{cases} M_1 - M_0, & \text{if } k \text{ is even}, \quad (3.5\text{a})\\ M_1 - M_0 + 2\mu, & \text{if } k \text{ is odd}. \quad (3.5\text{b})\end{cases}$$

By Theorem 3.4, if k is even, then we have

$$\begin{aligned}
\mathcal{G}^2_{G_0,G_1}(k) &= \gcd\left(G_kG_{k+1} - G_0G_1,\ G_{k+1}^2 - G_1^2,\ G_{k+2}^2 - G_2^2\right)\\
&= \gcd\left(G_kG_{k+1} - G_0G_1, M_1, M_2\right)
\end{aligned}$$

$$\begin{aligned}
&= \gcd(M_1 - M_0, M_1, M_2) && \text{by Equation (3.5a)}\\
&= \gcd(M_0, M_1, M_2)\\
&= \gcd(3M_1 - M_2, M_1, M_2) && \text{by Lemma 3.10}\\
&= \gcd(M_1, M_2),
\end{aligned}$$

as desired. On the other hand, if k is odd, then we have

$$\begin{aligned}
\mathcal{G}^2_{G_0,G_1}(k) &= \gcd\left(G_kG_{k+1} - G_0G_1,\ G^2_{k+1} - G^2_1,\ G^2_{k+2} - G^2_2\right)\\
&= \gcd\left(G_kG_{k+1} - G_0G_1, M_1, M_2\right)\\
&= \gcd(M_1 - M_0 + 2\mu, M_1, M_2) && \text{by Equation (3.5b)}\\
&= \gcd(-M_0 + 2\mu, M_1, M_2)\\
&= \gcd(-3M_1 + M_2 - 2\mu, M_1, M_2) && \text{by Lemma 3.10}\\
&= \gcd(-2\mu, M_1, M_2)\\
&= \gcd(2\mu, M_1, M_2),
\end{aligned}$$

as desired. □

4. Closed forms in the generalized Fibonacci setting

In this section, we prove the following closed forms on the GCD of the sum of k consecutive squares of Gibonacci numbers, noting $g_k := \gcd\left(G^2_{k+1} - G^2_1,\ G^2_{k+2} - G^2_2\right)$:

k	$\mathcal{G}^2_{G_0,G_1}(k)$	Proof in this paper
k even and $5 \nmid \mu$	F_k	Theorem 4.7
k even and $5 \mid \mu$	$5F_k$	Theorem 4.7
$k \equiv 3 \pmod 6$	$\gcd(2\mu, g_k)$	Theorem 3.11
$k \equiv 1, 5 \pmod 6$	$\gcd(2\mu, g_k)$	Theorem 3.11

4.1. Closed form for $\mathcal{G}^2_{G_0,G_1}(k)$ when k is even.

Lemma 4.1. *For all $i \in \mathbb{Z}$, we have*

$$G_i = G_0F_{i-1} + G_1F_i.$$

Proof. This follows from Lemma 3.3, if we set $m := i$ and $n := 0$. □

The following well-known identity is a generalization of Cassini's Identity (recall Lemma 3.8) in the Fibonacci setting and will be useful in proving Lemma 4.3.

Lemma 4.2 (Catalan's Identity)**.** *For $n, r \in \mathbb{Z}$ with $n \geq r$, we have*

$$F^2_n - F_{n-r}F_{n+r} = (-1)^{n-r}F^2_r.$$

Proof. See Theorem 5.11 of Koshy [6, p. 106]. □

The next lemma reveals the interesting fact that if k is even, then F_k divides the difference of the squares of any two Fibonacci numbers that are k terms apart. We use this fact to prove the subsequent result, Lemma 4.6, which states that if k is even, then F_k also divides the difference of the squares of any two Gibonacci numbers that are k terms apart.

Lemma 4.3. *For $k \geq 0$ with k even and $\ell \in \mathbb{Z}$, we have*

$$F^2_{k+\ell} - F^2_\ell = F_kF_{k+2\ell}.$$

Proof. By Catalan's Identity of Lemma 4.2, if we set $r := \ell$ and $n := k+\ell$, then it follows that

$$F_{k+\ell}^2 - F_{(k+\ell)-\ell}F_{(k+\ell)+\ell} = (-1)^{(k+\ell)-\ell}F_\ell^2.$$

Noting that k is even, the latter equality reduces to $F_{k+\ell}^2 - F_kF_{k+2\ell} = F_\ell^2$, and the result follows. □

Corollary 4.4. *For $k \geq 0$ with k even and $\ell \in \mathbb{Z}$, we have*

$$F_{k+\ell-1}^2 - F_{\ell-1}^2 = F_kF_{k+2\ell-2}.$$

Proof. This follows immediately from Lemma 4.3 if we replace ℓ with $\ell - 1$. □

Lemma 4.5. *For $k, \ell \geq 0$ with k even, we have*

$$F_{k+\ell-1}F_{k+\ell} - F_{\ell-1}F_\ell = F_kF_{k+2\ell-1}.$$

Proof. By Identity (20a) of Vajda [12, p. 28], for all $a, b, c \geq 0$, we have the following identity:

$$F_{a+b}F_{a+c} - (-1)^aF_bF_c = F_aF_{a+b+c}$$

Setting $a := k$, $b := \ell - 1$, and $c := \ell$, and noting that k is even, the result follows. □

Lemma 4.6. *For $k, \ell \geq 0$ with k even, we have*

$$G_{k+\ell}^2 - G_\ell^2 = F_k \cdot \left(G_0^2F_{k+2\ell-2} + 2G_0G_1F_{k+2\ell-1} + G_1^2F_{k+2\ell}\right).$$

Proof. For ease of notation, set $\gamma_{k,\ell} := G_0^2F_{k+2\ell-2} + 2G_0G_1F_{k+2\ell-1} + G_1^2F_{k+2\ell}$. Observe that

$$\begin{aligned}
&G_{k+\ell}^2 - G_\ell^2 \\
&= (G_0F_{k+\ell-1} + G_1F_{k+\ell})^2 - (G_0F_{\ell-1} + G_1F_\ell)^2 && \text{by Lemma 4.1} \\
&= \left(G_0^2F_{k+\ell-1}^2 + 2G_0G_1F_{k+\ell-1}F_{k+\ell} + G_1^2F_{k+\ell}^2\right) \\
&\qquad - \left(G_0^2F_{\ell-1}^2 + 2G_0G_1F_{\ell-1}F_\ell + G_1^2F_\ell^2\right) \\
&= G_0^2\left(F_{k+\ell-1}^2 - F_{\ell-1}^2\right) + G_1^2\left(F_{k+\ell}^2 - F_\ell^2\right) + 2G_0G_1\left(F_{k+\ell-1}F_{k+\ell} - F_{\ell-1}F_\ell\right) \\
&= G_0^2(F_kF_{k+2\ell-2}) + G_1^2(F_kF_{k+2\ell}) + 2G_0G_1\left(F_{k+\ell-1}F_{k+\ell} - F_{\ell-1}F_\ell\right) && \text{by Lemma 4.3} \\
&= G_0^2(F_kF_{k+2\ell-2}) + G_1^2(F_kF_{k+2\ell}) + 2G_0G_1(F_kF_{k+2\ell-1}) && \text{by Lemma 4.5} \\
&= F_k \cdot \left(G_0^2F_{k+2\ell-2} + 2G_0G_1F_{k+2\ell-1} + G_1^2F_{k+2\ell}\right) \\
&= F_k \cdot \gamma_{k,\ell},
\end{aligned}$$

and the result follows. □

We are now ready to state and prove the main theorem of this subsection.

Theorem 4.7. *For all $k \geq 1$ with k even, we have*

$$\mathcal{G}_{G_0,G_1}^2(k) = \begin{cases} F_k, & \text{if } 5 \text{ does not divide } \mu, \\ 5F_k, & \text{if } 5 \text{ divides } \mu, \end{cases}$$

where $\mu = G_1^2 - G_0G_1 - G_0^2$.

Proof. Suppose that $k \geq 1$ is even. Recalling that $\gamma_{k,\ell} := G_0^2F_{k+2\ell-2}+2G_0G_1F_{k+2\ell-1}+G_1^2F_{k+2\ell}$ from Lemma 4.6, we have the sequence of equalities

$$\begin{aligned}
\mathcal{G}_{G_0,G_1}^2(k) &= \gcd\left(G_{k+1}^2 - G_1^2,\ G_{k+2}^2 - G_2^2\right) && \text{by Theorem 3.11} \\
&= \gcd(F_k \cdot \gamma_{k,1},\ F_k \cdot \gamma_{k,2}) && \text{by Lemma 4.6} \\
&= F_k \cdot \gcd(\gamma_{k,1},\ \gamma_{k,2}).
\end{aligned}$$

It suffices to show that $\gcd(\gamma_{k,1},\ \gamma_{k,2}) = \gcd(\mu, 5)$. Setting $\beta_n := G_0^2F_n+2G_0G_1F_{n+1}+G_1^2F_{n+2}$, we see that $\gamma_{k,1} = \beta_k$ and $\gamma_{k,2} = \beta_{k+2}$. Moreover, the sequence $(\beta_n)_{n\geq 0}$ is itself a generalized Fibonacci sequence since the recursion $\beta_{n+2} = \beta_n + \beta_{n+1}$ holds for all $n \geq 0$ as follows:

$$\begin{aligned}\beta_{n+2} &= G_0^2F_{n+2} + 2G_0G_1F_{n+3} + G_1^2F_{n+4}\\ &= G_0^2(F_n + F_{n+1}) + 2G_0G_1(F_{n+1} + F_{n+2}) + G_1^2(F_{n+2} + F_{n+3})\\ &= \left(G_0^2F_n + 2G_0G_1F_{n+1} + G_1^2F_{n+2}\right) + \left(G_0^2F_{n+1} + 2G_0G_1F_{n+2} + G_1^2F_{n+3}\right)\\ &= \beta_n + \beta_{n+1}.\end{aligned}$$

Hence $\gcd(\beta_n, \beta_{n+1}) = \gcd(\beta_0, \beta_1)$ for all $n \geq 0$ by Lemma 3.5. Thus we have

$$\gcd(\gamma_{k,1},\ \gamma_{k,2}) = \gcd(\beta_k, \beta_{k+2}) = \gcd(\beta_k, \beta_k + \beta_{k+1}) = \gcd(\beta_k, \beta_{k+1}) = \gcd(\beta_0, \beta_1).$$

Since $\beta_0 = 2G_0G_1 + G_1^2$ and $\beta_1 = G_0^2 + 2G_0G_1 + 2G_1^2$, it follows that

$$\begin{aligned}\gcd(\gamma_{k,1},\ \gamma_{k,2}) = \gcd(\beta_0, \beta_1) &= \gcd\left(2G_0G_1 + G_1^2,\ G_0^2 + 2G_0G_1 + 2G_1^2\right)\\ &= \gcd\left(2G_0G_1 + G_1^2,\ G_0^2 + G_1^2 + (2G_0G_1 + G_1^2)\right)\\ &= \gcd\left(2G_0G_1 + G_1^2,\ G_0^2 + G_1^2\right).\end{aligned}$$

We now show that $\gcd\left(2G_0G_1 + G_1^2,\ G_0^2 + G_1^2\right) = \gcd(\mu, 5)$ by proving the following two claims:

$$\gcd\left(2G_0G_1 + G_1^2,\ G_0^2 + G_1^2\right) \geq \gcd(\mu, 5), \text{ and} \tag{4.1}$$

$$\gcd\left(2G_0G_1 + G_1^2,\ G_0^2 + G_1^2\right) \leq \gcd(\mu, 5). \tag{4.2}$$

To that end, suppose that for some prime p and $j \geq 1$, we have p^j divides $\gcd(\mu, 5)$. Then p^j divides 5, in particular, and hence $p = 5$ and $j = 1$, and so 5 divides μ. Observe that

$$(2G_0 + G_1)^2 = 4G_0^2 + 4G_0G_1 + G_1^2 \equiv -G_0^2 - G_0G_1 + G_1^2 \equiv \mu \pmod 5. \tag{4.3}$$

But since 5 divides μ, we deduce that $(2G_0+G_1)^2 \equiv 0 \pmod 5$, and hence 5 divides $2G_0+G_1$. This yields the congruence $2G_0G_1 + G_1^2 = (2G_0 + G_1)G_1 \equiv 0 \pmod 5$, and so 5 divides $2G_0G_1 + G_1^2$. Moreover, observe that

$$\begin{aligned}G_0^2 + G_1^2 = 5G_0^2 - 4G_0^2 + G_1^2 &\equiv -4G_0^2 + G_1^2 \pmod 5\\ &\equiv -(4G_0^2 - G_1^2) \pmod 5\\ &\equiv -(2G_0 + G_1)(2G_0 - G_1) \pmod 5\\ &\equiv 0 \pmod 5,\end{aligned}$$

since 5 divides $2G_0 + G_1$. Hence 5 divides $G_0^2 + G_1^2$. Thus p^j divides both $2G_0G_1 + G_1^2$ and $G_0^2+G_1^2$, and we conclude p^j divides $\gcd\left(2G_0G_1 + G_1^2,\ G_0^2 + G_1^2\right)$, which proves Inequality (4.1) holds.

To prove Inequality (4.2) holds, suppose that for some prime p and $j \geq 1$, we have p^j divides $\gcd\left(2G_0G_1 + G_1^2,\ G_0^2 + G_1^2\right)$. Then p^j divides both $2G_0G_1 + G_1^2$ and $G_0^2 + G_1^2$. The prime p cannot divide G_1 for otherwise it would also divide G_0, contradicting $\gcd(G_0, G_1) = 1$ (recall Convention 2.2). Since p^j divides $2G_0G_1 + G_1^2$ and $2G_0G_1 + G_1^2 = (2G_0 + G_1)G_1$, then p^j must divide $2G_0 + G_1$ since p does not divide G_1. Thus we have

$$\begin{aligned}5G_0^2 &= \left(4G_0^2 - G_1^2\right) + \left(G_0^2 + G_1^2\right)\\ &= (2G_0 + G_1)(2G_0 - G_1) + \left(G_0^2 + G_1^2\right)\\ &\equiv 0 \pmod{p^j},\end{aligned}$$

and hence p^j divides $5G_0^2$. The prime p cannot divide G_0 for otherwise it would also divide G_1, contradicting $\gcd(G_0, G_1) = 1$. Thus, p^j dividing $5G_0^2$ implies that p^j divides 5, and hence $p = 5$ and $j = 1$. Since we proved earlier that p^j divides $2G_0 + G_1$, it follows that 5 divides $2G_0 + G_1$. In particular, due to Congruence (4.3), we have 5 divides μ. We conclude p^j divides $\gcd(\mu, 5)$, which proves Inequality (4.2) holds. Therefore $\mathcal{G}^2_{G_0,G_1}(k) = F_k \cdot \gcd(\mu, 5)$, as desired. □

4.2. **Remarks on the closed form for $\mathcal{G}^2_{G_0,G_1}(k)$ when k is odd.** When k is odd, Theorem 3.11 yields the following closed form for the value $\mathcal{G}^2_{G_0,G_1}(k)$:

$$\mathcal{G}^2_{G_0,G_1}(k) = \gcd(2\mu, g_k),$$

where $g_k := \gcd\left(G^2_{k+1} - G_1^2,\ G^2_{k+2} - G_2^2\right)$. We immediately arrive at the following theorem as a consequence.

Theorem 4.8. *For all $k \geq 1$ with k odd, we have $\mathcal{G}^2_{G_0,G_1}(k)$ divides the value $|2\mu|$ where $\mu = G_1^2 - G_0G_1 - G_0^2$.*

Proof. Given any odd integer $k \geq 1$, Theorem 3.11 implies that $\mathcal{G}^2_{G_0,G_1}(k) = \gcd(2\mu, g_k)$, where $g_k := \gcd\left(G^2_{k+1} - G_1^2,\ G^2_{k+2} - G_2^2\right)$. Hence $\mathcal{G}^2_{G_0,G_1}(k)$ divides $|2\mu|$, and the result follows. □

Remark 4.9. It is worthy to note that Theorem 4.8 does not necessarily hold for even k values. For example, consider the Fibonacci setting with $k = 4$. Then $\mathcal{F}^2(k) = F_4 = 3$ by Theorem 5.2. In the Fibonacci setting, $\mu = 1$, however 3 does not divide $|2\mu|$, which equals 2.

Remark 4.10. It is also worthy to note that the sequence $(\mathcal{G}^2_{G_0,G_1}(k))_{k\geq 1}$ can attain all positive divisors of $|2\mu|$ as values. For example, consider the Gibonacci sequence with initial values $G_0 = 3$ and $G_1 = 1$. Then $\mu = 1^2 - 1 \cdot 3 - 3^2 = -11$ and hence $|2\mu| = 22$. Using the closed form $\mathcal{G}^2_{G_0,G_1}(k) = \gcd(2\mu, g_k)$ implied by Theorem 3.11, we leave it to the reader to verify the following:

$$\mathcal{G}^2_{G_0,G_1}(k) = \begin{cases} 1, & \text{if } k = 7, \\ 2, & \text{if } k = 3, \\ 11, & \text{if } k = 5, \\ 22, & \text{if } k = 15. \end{cases}$$

The following theorem provides a sufficiency condition for when the value $\mathcal{G}^2_{G_0,G_1}(k)$ attains its maximal value $|2\mu|$ in the case that k is odd.

Theorem 4.11. *Let $k \geq 1$ be odd, and set $\ell_k := \gcd(G_{k+1} - G_1, G_{k+2} - G_2)$. If 2μ divides ℓ_k, then $\mathcal{G}^2_{G_0,G_1}(k\ell) = |2\mu|$ for all odd integers $\ell \geq 1$.*

Proof. Suppose that $k \geq 1$ is odd, and set $\ell_k := \gcd(G_{k+1} - G_1, G_{k+2} - G_2)$. Assume that 2μ divides ℓ_k. Then 2μ divides both $G_{k+1} - G_1$ and $G_{k+2} - G_2$. By a simple inductive argument, the latter implies that 2μ divides $G_{k+n} - G_n$ for all $n \geq 1$. Hence, it follows that 2μ divides $G_{ki+1} - G_{k(i-1)+1}$ and also $G_{ki+2} - G_{k(i-1)+2}$ for all $1 \leq i \leq \ell$. Moreover, since $\sum_{i=1}^{\ell}\left(G_{ki+1} - G_{k(i-1)+1}\right) = G_{k\ell+1} - G_1$ and $\sum_{i=1}^{\ell}\left(G_{ki+2} - G_{k(i-1)+2}\right) = G_{k\ell+2} - G_2$, then we conclude that 2μ divides both $G_{k\ell+1} - G_1$ and $G_{k\ell+2} - G_2$. Recalling that Theorem 3.11 implies the formula $\mathcal{G}^2_{G_0,G_1}(k) = \gcd(2\mu, g_k)$, where $g_k := \gcd\left(G^2_{k+1} - G_1^2,\ G^2_{k+2} - G_2^2\right)$, we have

$$g_{k\ell} = \gcd\left(G^2_{k\ell+1} - G_1^2,\ G^2_{k\ell+2} - G_2^2\right)$$

$$= \gcd\big((G_{k\ell+1} - G_1)(G_{k\ell+1} + G_1),\ (G_{k\ell+2} - G_2)(G_{k\ell+2} + G_2)\big),$$

and hence 2μ divides $g_{k\ell}$. Thus $\mathcal{G}^2_{G_0,G_1}(k\ell) = \gcd(2\mu, g_{k\ell}) = |2\mu|$ for all odd integers $\ell \geq 1$. □

Example 4.12. *Consider the Gibonacci sequence $(G_n)_{n\geq 0}$ with initial values $G_0 = 2$ and $G_1 = 7$. Then $\mu = 7^2 - 2 \cdot 7 - 2^2 = 31$. Also observe that for $k = 15$, we have*

$$\ell_{15} = \gcd(G_{16} - G_1, G_{17} - G_2) = \gcd(8122, 13144) = 62,$$

and indeed 2μ clearly divides ℓ_k since $2\mu = \ell_k$, in particular, in this case. Hence by Theorem 4.11, we know $\mathcal{G}^2_{G_0,G_1}(15\ell) = 62$ for all odd integers $\ell \geq 1$. We leave it to the reader to verify this.

5. Closed forms in the Fibonacci and Lucas settings

In this section, we prove the following closed forms on the GCD of the sum of k consecutive squares of Fibonacci and Lucas numbers:

k	$\mathcal{F}^2(k)$	$\mathcal{L}^2(k)$	Proof in this paper
k even	F_k	$5F_k$	Theorems 5.2 and 5.4, respectively
$k \equiv 3 \pmod 6$	2	2	Theorems 5.2 and 5.4, respectively
$k \equiv 1, 5 \pmod 6$	1	1	Theorems 5.2 and 5.4, respectively

Lemma 5.1. *For all $n \in \mathbb{Z}$, the value F_n is even if and only if 3 divides n.*

Proof. This follows from the fact that $F_3 = 2$ and the well-known identity, F_m divides F_n if and only if m divides n (see Corollary 10.2 of Koshy [6, p. 173]). □

Theorem 5.2. *For all $k \geq 1$, we have*

$$\mathcal{F}^2(k) = \begin{cases} F_k, & \text{if } k \text{ is even}, \\ 2, & \text{if } k \equiv 3 \pmod 6, \\ 1, & \text{if } k \equiv 1, 5 \pmod 6. \end{cases}$$

Proof. Since $\mu = F_1^2 - F_0F_1 - F_0^2 = 1$, Theorem 4.7 in the Fibonacci setting yields $\mathcal{F}^2(k) = F_k$ when k is even since 5 does not divide μ. On the other hand, when k is odd, Theorem 4.8 in the Fibonacci setting yields $\mathcal{F}^2(k)$ divides $|2\mu|$, and hence $\mathcal{F}^2(k) = 1$ or 2. Recall that by Theorem 3.4 in the Fibonacci setting, we have

$$\mathcal{F}^2(k) = \gcd\left(F_kF_{k+1},\ F^2_{k+1} - 1,\ F^2_{k+2} - 1\right),$$

and hence if $\mathcal{F}^2(k) = 2$, then 2 divides F_kF_{k+1} so either F_k or F_{k+1} is even. However, 2 dividing $F^2_{k+1} - 1$ implies F_{k+1} is odd, and thus F_k must be even. By Lemma 5.1, it follows that 3 divides k, and in particular $k \equiv 3 \pmod 6$ since k is odd. On the other hand, if $k \equiv 3 \pmod 6$, then 3 divides k and hence F_k is even by Lemma 5.1, and so both F_{k+1} and F_{k+2} are both odd implying $F^2_{k+1} - 1$ and $F^2_{k+2} - 1$ are both even. Thus $\mathcal{F}^2(k)$ is even; that is, $\mathcal{F}^2(k) = 2$ is forced. Thus if k is odd, then $\mathcal{F}^2(k) = 2$ if and only if $k \equiv 3 \pmod 6$. And consequently if k is odd, then $\mathcal{F}^2(k) = 1$ if and only if $k \equiv 1, 5 \pmod 6$. □

To prove the Lucas version of Theorem 5.2, we first give the Lucas version of the necessary and sufficient condition for L_n to be even (compare this with the Fibonacci version given in Lemma 5.1).

Lemma 5.3. *For all $n \in \mathbb{Z}$, the value L_n is even if and only if 3 divides n.*

Proof. See Identity (23.2) in Theorem 23.1 of Koshy [6, p. 462]. □

Theorem 5.4. *For all $k \geq 1$, we have*

$$\mathcal{L}^2(k) = \begin{cases} 5F_k, & \text{if } k \text{ is even,} \\ 2, & \text{if } k \equiv 3 \pmod 6, \\ 1, & \text{if } k \equiv 1, 5 \pmod 6. \end{cases}$$

Proof. Since $\mu = L_1^2 - L_0L_1 - L_0^2 = -5$, Theorem 4.7 in the Lucas setting yields $\mathcal{L}^2(k) = 5F_k$ when k is even since 5 divides μ. On the other hand, when k is odd, Theorem 4.8 in the Lucas setting yields $\mathcal{L}^2(k)$ divides $|2\mu|$, and hence $\mathcal{L}^2(k) = 1, 2, 5,$ or 10. Recall that by Theorem 3.4 in the Lucas setting, we have

$$\mathcal{L}^2(k) = \gcd\left(L_kL_{k+1} - 2,\ L_{k+1}^2 - 1,\ L_{k+2}^2 - 9\right).$$

We will first rule out the possibility that $\mathcal{L}^2(k) = 5$ or 10. Suppose by way of contradiction that 5 divides $\mathcal{L}^2(k)$. Then 5 divides both $L_{k+1}^2 - 1$ and $L_{k+2}^2 - 9$, and hence we have the congruences

$$L_{k+1}^2 \equiv 1 \pmod 5 \tag{5.1}$$

$$L_{k+2}^2 \equiv 4 \pmod 5. \tag{5.2}$$

It is readily verified that $\pi_L(5) = 4$, where $\pi_L(5)$ is the Pisano period of the Lucas sequence modulo 5 (recall Definition 2.5). This length 4 period repeats the sequence terms $(L_n \pmod 5))_{n=0}^3 = (2, 1, 3, 4)$. Squaring this sequence we get $(L_n^2 \pmod 5))_{n=0}^3 = (4, 1, 4, 1)$, a repeating sequence of length 2. More precisely, $L_n^2 \equiv 4 \pmod 5$ if and only if n is even, and hence Congruences (5.1) and (5.2) force k to be even, which is a contradiction. Therefore, $\mathcal{L}^2(k) \neq 5$ or 10, and thus $\mathcal{L}^2(k) = 1$ or 2.

If $\mathcal{L}^2(k) = 2$, then 2 divides $L_kL_{k+1} - 2$, $L_{k+1}^2 - 1$, and $L_{k+2}^2 - 9$. Since 2 divides $L_kL_{k+1} - 2$, then either L_k or L_{k+1} is even. However, 2 dividing $L_{k+1}^2 - 1$ implies L_{k+1} is odd, and thus L_k must be even. By Lemma 5.3, it follows that 3 divides k, and in particular $k \equiv 3 \pmod 6$ since k is odd. On the other hand, if $k \equiv 3 \pmod 6$, then 3 divides k and hence F_k is even by Lemma 5.3, and so L_{k+1} and L_{k+2} are both odd implying $L_{k+1}^2 - 1$ and $L_{k+2}^2 - 1$ are both even. Thus $\mathcal{L}^2(k)$ is even; that is, $\mathcal{L}^2(k) = 2$ is forced. Thus if k is odd, then $\mathcal{L}^2(k) = 2$ if and only if $k \equiv 3 \pmod 6$. And consequently if k is odd, then $\mathcal{L}^2(k) = 1$ if and only if $k \equiv 1, 5 \pmod 6$. □

6. Open questions

6.1. Extension to higher powers.

Question 6.1. For k even, the formulas for the GCD of all sums of k consecutive Gibonacci numbers and the GCD of all sums of k consecutive squares of Gibonacci numbers, respectively are

$$\mathcal{G}_{G_0,G_1}(k) = \gcd(G_{k+1} - G_1,\ G_{k+2} - G_2)$$
$$\mathcal{G}^2_{G_0,G_1}(k) = \gcd\left(G_{k+1}^2 - G_1^2,\ G_{k+2}^2 - G_2^2\right).$$

The first formula holds from Guyer-Mbirika [4, Theorem 15], while the second formula holds from Theorem 3.11 in this current paper. For certain even values k and $n \geq 3$, it seems reasonable that the following may hold:

$$\mathcal{G}^n_{G_0,G_1}(k) = \gcd\left(G_{k+1}^n - G_1^n,\ G_{k+2}^n - G_2^n\right).$$

Although this does not appear to be true even in the case of $n = 3$, data collected via `Mathematica` for this n value in the Fibonacci and Lucas settings supports the following conjectures for k even:

$$\mathcal{F}^3(k) = \begin{cases} \gcd\left(F_{k+1}^3 - 1,\ F_{k+2}^3 - 1\right), & \text{if 6 divides } k, \\ \frac{1}{2} \cdot \gcd\left(F_{k+1}^3 - 1,\ F_{k+2}^3 - 1\right), & \text{if 6 does not divide } k. \end{cases}$$

$$\mathcal{L}^3(k) = \begin{cases} \gcd\left(L_{k+1}^3 - 1,\ L_{k+2}^3 - 9\right), & \text{if 6 divides } k, \\ \frac{1}{2} \cdot \gcd\left(L_{k+1}^3 - 1,\ L_{k+2}^3 - 9\right), & \text{if 6 does not divide } k. \end{cases}$$

Can this conjecture not only be proved, but also extended to higher values $n \geq 4$?

Question 6.2. Fix $G_0, G_1 \in \mathbb{Z}$ and set $d := \gcd(G_0, G_1)$. Consider the two Gibonacci sequences $(G_n)_{n\geq 0}$ and $(G'_n)_{n\geq 0}$, where $(G'_n)_{n=0}^{\infty}$ is generated by the relatively prime initial conditions $G'_0 = \frac{G_0}{d}$ and $G'_1 = \frac{G_1}{d}$. Then the following identities holds:

$$\mathcal{G}_{G_0,G_1}(k) = d \cdot \mathcal{G}_{G'_0,G'_1}(k)$$
$$\mathcal{G}^2_{G_0,G_1}(k) = d^2 \cdot \mathcal{G}^2_{G'_0,G'_1}(k).$$

The first formula holds from Guyer-Mbirika [4, Theorem 19], while the second formula holds from Theorem 3.6 in this current paper. Will it be the case that $\mathcal{G}^n_{G_0,G_1}(k) = d^n \cdot \mathcal{G}^n_{G'_0,G'_1}(k)$ for all $n \geq 3$?

6.2. Periodicity of the sequences $(\mathcal{G}_{G_0,G_1}(k))_{k\geq 1}$ and $(\mathcal{G}^2_{G_0,G_1}(k))_{k\geq 1}$ when k is odd.

Example 6.3. *Consider the Gibonacci sequence with initial values $G_0 = -1$ and $G_1 = 3$. Then $\mu = 3^2 - (-1)(3) - (-1)^2 = 11$. Moreover for k odd, we have the following (formal proofs omitted, but easily verified using the formulas for $\mathcal{G}_{G_0,G_1}(k)$ in Guyer-Mbirika* [4, Theorem 15] *and $\mathcal{G}^2_{G_0,G_1}(k)$ in Theorem 3.4 in this current paper, respectively):*

$$\mathcal{G}_{G_0,G_1}(k) = \begin{cases} 2, & \textit{if } k \equiv 3 \pmod 6, \\ 1, & \textit{if } k \equiv 1, 5 \pmod 6, \end{cases}$$

and

$$\mathcal{G}^2_{G_0,G_1}(k) = \begin{cases} 22, & \textit{if } k \equiv 15 \pmod{30}, \\ 11, & \textit{if } k \equiv 5, 25 \pmod{30}, \\ 2, & \textit{if } k \equiv 3, 9, 21, 27 \pmod{30}, \\ 1, & \textit{if } k \equiv 1, 7, 11, 13, 17, 19, 23, 29 \pmod{30}. \end{cases}$$

Hence we have the periodic relationships in the sequences $(\mathcal{G}_{G_0,G_1}(k))_{k\geq 1}$ and $(\mathcal{G}^2_{G_0,G_1}(k))_{k\geq 1}$:

$$\mathcal{G}_{G_0,G_1}(k+6) = \mathcal{G}_{G_0,G_1}(k) \ \ \textit{and} \ \ \mathcal{G}^2_{G_0,G_1}(k+30) = \mathcal{G}^2_{G_0,G_1}(k),$$

for all odd k.

Question 6.4. Under what conditions on the initial values G_0 and G_1 are $\mathcal{G}_{G_0,G_1}(k)$ and $\mathcal{G}^2_{G_0,G_1}(k)$ periodic on odd k values?

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