ADVANCED PROBLEMS AND SOLUTIONS

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PROBLEMS PROPOSED IN THIS ISSUE

H-765 Proposed by D. M. Bătineţu-Giurgiu, Bucharest and Neculai Stanciu, Buzău, Romania.

Prove that for positive integer n and $m > 0$ we have:

(i) $$\frac{L_n^4 + L_{n+1}^4}{L_n L_{n+1}} + \frac{L_{n+1}^4 + L_{n+3}^4}{L_{n+1} L_{n+3}} + \frac{L_{n+3}^4 + L_n^4}{L_{n+3} L_n} \geq \frac{2}{3} L_{n+4}^2;$$

(ii) $$\left(\sum_{k=1}^{n} F_k^{2m+4}\right)\left(\sum_{k=1}^{n} \frac{1}{F_k^{2m}}\right) \geq F_n^2 F_{n+1}^2;$$

(iii) $$\left(\sum_{k=1}^{n} L_k^{2m+4}\right)\left(\sum_{k=1}^{n} \frac{1}{L_k^{2m}}\right) \geq (L_n L_{n+1} - 1)^2;$$

(iv) $$\left(\sum_{k=1}^{n} F_k^{m+2}\right)\left(\sum_{k=1}^{n} \frac{1}{F_k^{m}}\right) \geq (F_{n+2} - 1)^2;$$

(v) $$1 + \sum_{k=1}^{n} \frac{F_k^{m+1}}{F_{n-k+1}^m} \geq F_{n+2} \quad \text{and} \quad 3 + \sum_{k=1}^{n} \frac{L_k^{m+1}}{L_{n-k+1}^m} \geq L_{n+2}.$$

H-766 Proposed by H. Ohtsuka, Saitama, Japan.

Let $n = m + 2$. For $m \geq 1$, prove that

$$\sum_{h=1}^{m}\sum_{i=1}^{h}\sum_{j=1}^{i}\sum_{k=1}^{j} F_k^4 = \frac{4F_n^4 + n^4 - 5n^2}{100}.$$

H-767 Proposed by H. Ohtsuka, Saitama, Japan.

Prove that

$$\lim_{n\to\infty} \sqrt{F_2^2 + \sqrt{F_4^2 + \sqrt{F_8^2 + \sqrt{\cdots + \sqrt{F_{2^n}^2}}}}} = 3.$$

H-768 **Proposed by H. Ohtsuka, Saitama, Japan.**

Let $\binom{n}{k}_F$ denote the Fibonomial coefficient. For $n \geq 1$, prove that

(i) $\displaystyle\sum_{k=0}^{n} F_{2(n-k)} \binom{2n}{k}_F^{-1} = \frac{F_{2n+1}(F_{2n+2}+1)}{F_{2n+3}} - \frac{F_{n+1}F_{n+3}}{F_{2n+3}} \binom{2n}{n}_F^{-1};$

(ii) $\displaystyle\sum_{k=0}^{n} F_{2(n-k)} \binom{2n}{k}_F^{-2} = \frac{F_{2n+1}^2}{F_{2n+2}} - \frac{F_{n+1}}{L_{n+1}} \binom{2n}{n}_F^{-2}.$

SOLUTIONS

Integer Parts of Reciprocals of Tails of Infinite Products with Fibonacci Numbers

H-734 **Proposed by H. Ohtsuka, Saitama, Japan.**
(Vol. 51, No. 1, February 2013)

For $n \geq 3$ find closed form expressions for

$$\left\lfloor \left(1 - \prod_{k=n}^{\infty}\left(1 - \frac{1}{F_k}\right)\right)^{-1} \right\rfloor \quad \text{and} \quad \left\lfloor \left(1 - \prod_{k=n}^{\infty}\left(1 - \frac{1}{F_k^2}\right)\right)^{-1} \right\rfloor.$$

Here, $\lfloor x \rfloor$ be the largest integer less than or equal to x.

Solution by the proposer.

We need the following lemma.

Lemma 1. *For $n \geq 3$, we have*

(1) $\dfrac{F_{n-2}-1}{F_{n-2}} < \dfrac{F_n - 1}{F_n} \times \dfrac{F_{n-1}-1}{F_{n-1}};$

(2) $\dfrac{F_{n-2}}{F_{n-2}+1} > \dfrac{F_n - 1}{F_n} \times \dfrac{F_{n-1}}{F_{n-1}+1};$

(3) $\dfrac{F_nF_{n-1}-1}{F_nF_{n-1}} < \dfrac{F_n^2 - 1}{F_n^2} \times \dfrac{F_{n+1}F_n - 1}{F_{n+1}F_n}$ *(if n is odd);*

(4) $\dfrac{F_nF_{n-1}}{F_nF_{n-1}+1} > \dfrac{F_n^2 - 1}{F_n^2} \times \dfrac{F_{n+1}F_n}{F_{n+1}F_n + 1};$

(5) $\dfrac{F_nF_{n-1}-2}{F_nF_{n-1}-1} < \dfrac{F_n^2 - 1}{F_n^2} \times \dfrac{F_{n+1}F_n - 2}{F_{n+1}F_n - 1};$

(6) $\dfrac{F_nF_{n-1}-1}{F_nF_{n-1}} > \dfrac{F_n^2 - 1}{F_n^2} \times \dfrac{F_{n+1}F_n - 1}{F_{n+1}F_n}$ *(if n is even).*

Proof. We will only prove (1) since all other verifications are similar. We have

$$\begin{aligned} &F_{n-2}(F_n - 1)(F_{n-1} - 1) - F_nF_{n-1}(F_{n-2} - 1) \\ &= F_{n-2} + F_{n-1}F_n - F_{n-1}F_{n-2} - F_nF_{n-2} \\ &= F_{n-2} + F_{n-1}^2 - F_nF_{n-2} = F_{n-2} + (-1)^n \geq 0. \end{aligned}$$

Therefore, we obtain the desired inequality (1). □

(i) Using Lemma 1 (1), we have

$$\frac{F_{n-2}-1}{F_{n-2}} \le \frac{F_n-1}{F_n} \times \frac{F_{n-1}-1}{F_{n-1}} \le \frac{F_n-1}{F_n} \times \frac{F_{n+1}-1}{F_{n+1}} \times \frac{F_n-1}{F_n}$$
$$\le \frac{F_n-1}{F_n} \times \frac{F_{n+1}-1}{F_{n+1}} \times \frac{F_{n+2}-1}{F_{n+2}} \times \frac{F_{n+1}-1}{F_{n+1}} \le \cdots \le \prod_{k=n}^{\infty} \frac{F_k-1}{F_k}.$$

Using Lemma 1 (2), we have

$$\frac{F_{n-2}}{F_{n-2}+1} > \frac{F_n-1}{F_n} \times \frac{F_{n-1}}{F_{n-1}+1} > \frac{F_n-1}{F_n} \times \frac{F_{n+1}-1}{F_{n+1}} \times \frac{F_n}{F_n+1}$$
$$> \frac{F_n-1}{F_n} \times \frac{F_{n+1}-1}{F_{n+1}} \times \frac{F_{n+2}-1}{F_{n+2}} \times \frac{F_{n+1}}{F_{n+1}+1} > \cdots > \prod_{k=n}^{\infty} \frac{F_k-1}{F_k}.$$

Therefore,

$$1-\frac{1}{F_{n-2}} \le \prod_{k=n}^{\infty}\left(1-\frac{1}{F_k}\right) < 1-\frac{1}{F_{n-2}+1}.$$

That is,

$$F_{n-2} \le \left(1-\prod_{k=n}^{\infty}\left(1-\frac{1}{F_k}\right)\right)^{-1} < F_{n-2}+1.$$

Thus, for $n \ge 3$, we obtain

$$\left\lfloor \left(1-\prod_{k=n}^{\infty}\left(1-\frac{1}{F_k}\right)\right)^{-1} \right\rfloor = F_{n-2}.$$

(ii) Case 1. $n \ge 3$ is odd. Using Lemma 1 (3) and (4), we obtain the following inequality in the same manner as (i):

$$\frac{F_nF_{n-1}-1}{F_nF_{n-1}} < \prod_{k=n}^{\infty} \frac{F_k^2-1}{F_k^2} < \frac{F_nF_{n-1}}{F_nF_{n-1}+1}.$$

Therefore,

$$1-\frac{1}{F_nF_{n-1}} < \prod_{k=n}^{\infty}\left(1-\frac{1}{F_k^2}\right) < 1-\frac{1}{F_nF_{n-1}+1}.$$

That is,

$$F_nF_{n-1} < \left(1-\prod_{k=n}^{\infty}\left(1-\frac{1}{F_k^2}\right)\right)^{-1} < F_nF_{n-1}+1.$$

Case 2. $n \ge 4$ is even. Using Lemma 1 (5) and (6), we obtain the following inequality in the same manner as (i):

$$\frac{F_nF_{n-1}-2}{F_nF_{n-1}-1} < \prod_{k=n}^{\infty} \frac{F_k^2-1}{F_k^2} < \frac{F_nF_{n-1}-1}{F_nF_{n-1}}.$$

Therefore,

$$1-\frac{1}{F_nF_{n-1}-1} < \prod_{k=n}^{\infty}\left(1-\frac{1}{F_k^2}\right) < 1-\frac{1}{F_nF_{n-1}}.$$

That is,

$$F_nF_{n-1}-1<\left(1-\prod_{k=n}^{\infty}\left(1-\frac{1}{F_k^2}\right)\right)^{-1}<F_nF_{n-1}.$$

Therefore, we obtain

$$\left\lfloor\left(1-\prod_{k=n}^{\infty}\left(1-\frac{1}{F_k^2}\right)\right)^{-1}\right\rfloor=\begin{cases}F_nF_{n-1} & \text{if } n\equiv 1 \pmod 2,\quad n\geq 3;\\ F_nF_{n-1}-1 & \text{if } n\equiv 1 \pmod 2,\quad n\geq 4.\end{cases}$$

Proposer's note: For $m\geq 2$ and $n\geq 2$, we obtain the following identity in the same manner:

$$\left\lfloor\left(1-\prod_{k=n}^{\infty}\left(1-\frac{1}{F_{mk}}\right)\right)^{-1}\right\rfloor=F_{mn}-F_{m(n-1)}.$$

Also solved by Paul S. Bruckman.

On a Power Series with Binomial Coefficients

H-735 Proposed by Paul S. Bruckman, BC.
(Vol. 51, No. 2, May 2013)

Let $F_m(x)=\sum_{n=0}^{\infty}\binom{2n+m}{n}x^n$, where m is any real number and $|x|<1/4$. Also let $\theta(x)=(1-4x)^{1/2}$. For brevity, write $F_m=F_m(x)$, $\theta=\theta(x)$. Prove the following:

(a) $F_0=\frac{1}{\theta}$, $F_1=\frac{(1-\theta)}{2x\theta}$;

(b) for all real m, $\frac{F_m}{F_0}=\left(\frac{F_1}{F_0}\right)^m$;

(c) for all real m, $\sum_{k=0}^{n}\binom{2k+m}{k}\binom{2n-2k-m}{n-k}=4^n,\quad n=0,1,2,\ldots.$

Solution by Ángel Plaza, Gran Canaria, Spain.

(a) $F_0=\frac{1}{\theta}$ is $\sum_{n=0}^{\infty}\binom{2n}{n}x^n=\frac{1}{\sqrt{1-4x}}$, which is given as identity (2.5.1) in [1].

$F_1=\frac{(1-\theta)}{2x\theta}$ is equivalent to $\sum_{n=0}^{\infty}\binom{2n+1}{n}x^n=\frac{1-\sqrt{1-4x}}{2x\sqrt{1-4x}}$. Then

$$\begin{aligned}RHS&=\frac{1}{2x}\left(\frac{1}{\sqrt{1-4x}}-1\right)=\frac{1}{2x}\sum_{n=1}^{\infty}\binom{2n}{n}x^n=\frac{1}{2}\sum_{n=1}^{\infty}\binom{2n}{n}x^{n-1}\\&=\frac{1}{2}\sum_{n=0}^{\infty}\binom{2n+2}{n+1}x^n=\sum_{n=0}^{\infty}\frac{\binom{2n+1}{n+1}+\binom{2n+1}{n}}{2}x^n\\&=\sum_{n=0}^{\infty}\binom{2n+1}{n}x^n=LHS.\end{aligned}$$

(b) By (a), we have to show that for all m, $F_m = F_0 \left(\frac{F_1}{F_0}\right)^m$, where $F_0 = \frac{1}{\sqrt{1-4x}}$, $\frac{F_1}{F_0} = \frac{1-\sqrt{1-4x}}{2x}$. That is

$$\sum_{n=0}^{\infty} \binom{2n+m}{n} x^n = \frac{1}{\sqrt{1-4x}} \left(\frac{1-\sqrt{1-4x}}{2x}\right)^m,$$

which is identity (2.5.15) in [1].

(c) Let $A(x)$ be the generating function of the LHS. That is

$$\begin{aligned} A(x) &= \sum_{n\ge 0} x^n \sum_{k=0}^{n} \binom{2k+m}{k}\binom{2n-2k-m}{n-k} \\ &= \sum_{k\ge 0} \binom{2k+m}{k} x^k \sum_{n-k\ge 0} \binom{2n-2k-m}{n-k} x^{n-k} \\ &= \frac{1}{\sqrt{1-4x}} \left(\frac{1-\sqrt{1-4x}}{2x}\right)^m \frac{1}{\sqrt{1-4x}} \left(\frac{1-\sqrt{1-4x}}{2x}\right)^{-m}, \\ &= \frac{1}{1-4x}, \end{aligned}$$

which is precisely the generating function of the RHS, 4^n. Note that we have used the identity (2.5.15) in [1].

References

[1] H. S. Wilf, *Generatingfunctionology*, 2nd. edition, (1992).

Also solved by Kenneth B. Davenport and the proposer.

On the Sum of the Cubes of the Tribonacci Numbers

H-736 Proposed by Hideyuki Ohtsuka, Saitama, Japan.
(Vol. 51, No. 2, May 2013)

The Tribonacci numbers T_n satisfy $T_0 = 0$, $T_1 = T_2 = 1$, $T_{n+3} = T_{n+2} + T_{n+1} + T_n$ for $n \ge 0$. Find an explicit formula for the sum $\sum_{k=1}^{n} T_k^3$.

Solution by the proposer.

Let $S_n = \sum_{k=1}^{n} T_k^3$. We need the following lemma.

Lemma 2. *We have*

(i) $\sum_{k=1}^{n}(T_k^2 T_{k+1} + T_k T_{k+1}^2) = T_n T_{n+1} T_{n+2}$;

(ii) $\sum_{k=1}^{n}(T_{k+1}^2 T_{k+2} + T_{k+1} T_{k+2}^2) = T_{n+1} T_{n+2} T_{n+3} - 2$;

(iii) $\sum_{k=1}^{n} T_k^2 T_{k+2} = S_n + T_n T_{n+1} T_{n+2} - T_n T_{n+1}^2$;

(iv) $-6\sum_{k=1}^{n} T_{k+1}T_{k+2}^2 = 2S_n + A_n,$

where

$$A_n = -T_{n+2}^3 - T_n^3 - 3T_nT_{n+1}^2 - 3T_n^2T_{n+1} - 3T_{n+1}T_{n+2}^2 - 3T_{n+1}^2T_{n+2} + 7.$$

Proof. (i) We have

$$\sum_{k=1}^{n}(T_k^2T_{k+1} + T_kT_{k+1}^2) = \sum_{k=1}^{n} T_kT_{k+1}(T_k + T_{k+1}) = \sum_{k=1}^{n} T_kT_{k+1}(T_{k+2} - T_{k-1})$$
$$= \sum_{k=1}^{n}(T_kT_{k+1}T_{k+2} - T_{k-1}T_kT_{k+1}) = T_nT_{n+1}T_{n+2}.$$

(ii) We have

$$\sum_{k=1}^{n}(T_{k+1}^2T_{k+2} + T_{k+1}T_{k+2}^2) = \sum_{k=2}^{n}(T_k^2T_{k+1} + T_kT_{k+1}^2) = T_{n+1}T_{n+2}T_{n+3} - 2, \text{ (by (i))}.$$

(iii) We have

$$\sum_{k=1}^{n} T_k^2T_{k+2} = \sum_{k=1}^{n} T_k^2(T_{k+1} + T_k + T_{k-1}) = \sum_{k=1}^{n} T_k^3 + \sum_{k=1}^{n}(T_k^2T_{k+1} + T_{k-1}T_k^2)$$
$$= S_n + \sum_{k=1}^{n}(T_k^2T_{k+1} + T_kT_{k+1}^2) - T_nT_{n+1}^2 = S_n + T_nT_{n+1}T_{n+2} - T_nT_{n+1}^2, \text{ (by (i))}.$$

(iv) We have

$$0 = \sum_{k=1}^{n}\left((T_k + T_{k-1})^3 - (T_{k+2} - T_{k+1})^3\right)$$
$$= 3\sum_{k=1}^{n}(T_{k+2}^2T_{k+1} + T_k^2T_{k-1}) - 3\sum_{k=1}^{n}(T_{k+2}T_{k+1}^2 - T_kT_{k-1}^2) + \sum_{k=1}^{n}(T_k^3 + T_{k-1}^3 - T_{k+2}^3 + T_{k+1}^3)$$
$$= 6\sum_{k=1}^{n} T_{k+2}^2T_{k+1} + 2S_n + A_n.$$

□

Let $x = T_k,\ y = T_{k+1},\ z = T_{k+2}$. We have

$$x^3 + y^3 + z^3 + 3x^2y + 3xy^2 + 3x^2z + 3xz^2 + 3y^2z + 3yz^2 + 6xyz = (x + y + z)^3; \qquad (1)$$

$$x^3 + 2y^3 + z^3 + 2x^2y + 2xy^2 + x^2z - xz^2 - 2yz^2 - 2xyz = 1 \qquad (2)$$

(see [1]). Multiplying (2) by 3 and adding the resulting identity to (1), we get

$$4x^3 + 7y^3 + 4z^3 + 9x^2y + 9xy^2 + 6x^2z + 3y^2z - 3yz^2 = T_{k+3}^3 + 3.$$

From the above identity, we have

$$\sum_{k=1}^{n}(4T_k^3 + 7T_{k+1}^3 + 4T_{k+2}^3 - T_{k+3}^3) + 9\sum_{k=1}^{n}(T_k^2T_{k+1} + T_kT_{k+1}^2)$$
$$+ 6\sum_{k=1}^{n}T_k^2T_{k+2} + 3\sum_{k=1}^{n}(T_{k+1}^2T_{k+2} + T_{k+1}T_{k+2}^2) - 6\sum_{k=1}^{n}T_{k+1}T_{k+2}^2 = 3n.$$

Using Lemma 2 (i), (ii), (iii) and (iv), we have

$$\sum_{k=1}^{n}(12T_k^3 + 7T_{k+1}^3 + 4T_{k+2}^3 - T_{k+3}^3)$$
$$+ 15T_nT_{n+1}T_{n+2} + 3T_{n+1}T_{n+2}T_{n+3} - 6T_nT_{n+1}^2 + A_n - 6 = 3n. \quad (3)$$

Here,

$$\sum_{k=1}^{n}(12T_k^3 + 7T_{k+1}^3 + 4T_{k+2}^3 - T_{k+3}^3) = 22S_n + 10T_{n+1}^3 + 3T_{n+2}^3 - T_{n+3}^3 - 5.$$

Therefore, (3) is

$$22S_n = T_{n+3}^3 - 2T_{n+2}^3 - 10T_{n+1}^3 + T_n^3 + 9T_nT_{n+1}^2 + 3T_n^2T_{n+1}$$
$$- 15T_nT_{n+1}T_{n+2} - 3T_{n+1}T_{n+2}(T_{n+3} - T_{n+2} - T_{n+1}) + 3n + 4.$$

Since

$$- - 3T_{n+1}T_{n+2}(T_{n+3} - T_{n+2} - T_{n+1}) = -3T_nT_{n+1}T_{n+2},$$

we obtain

$$S_n = \frac{1}{22}(T_{n+3}^3 - 2T_{n+2}^3 - 10T_{n+1}^3 + T_n^3 + 9T_nT_{n+1}^2 + 3T_n^2T_{n+1} - 18T_nT_{n+1}T_{n+2} + 3n + 4).$$

References

[1] M. Elia, *Derived sequences, the tribonacci recurrence and cubic forms*, The Fibonacci Quarterly **39.2** (2001), 107–115.

A Lucas Type Congruence with Fibonomials

H-737 Proposed by Hideyuki Ohtsuka, Saitama, Japan.
(Vol. 51, No. 2, May 2013)

Let $\binom{n}{k}_F$ denote the Fibonomial coefficient. For an odd prime p and a positive integer n, prove that

$$\binom{np-1}{p-1}_F \equiv (-1)^{\frac{(n-1)(p-1)}{2}} \pmod{F_p^2L_p}.$$

Solution by Christian Ballot, Caen, France.

With $m := n-1$, define the rational polynomial

$$P(x) := \prod_{i=1}^{p-1}\left(x + \frac{F_{mp}}{L_{mp}}\frac{L_i}{F_i}\right).$$

Expanding $P(x)$ yields

$$P(x) = x^{p-1} + \frac{F_{mp}}{L_{mp}} \sum_{i=1}^{p-1} \frac{L_i}{F_i} x^{p-2} + \frac{F_{mp}^2}{L_{mp}^2} \sum_{0<i<j<p} \frac{L_i L_j}{F_i F_j} x^{p-3} + \cdots + \frac{F_{mp}^{p-1}}{L_{mp}^{p-1}} \prod_{i=1}^{p-1} \frac{L_i}{F_i}.$$

All coefficients, except that of x^{p-1}, are 0 $\pmod{F_p^2}$. Indeed, F_{mp}^k is divisible by F_p^2 for $k \geq 2$ and F_p is prime to $L_{mp} \prod_{i=1}^{p-1} F_i$. Moreover, $S := \sum_{i=1}^{p-1} \frac{L_i}{F_i} \equiv 0 \pmod{F_p}$ because

$$2S = \sum_{i=1}^{p-1} \left(\frac{L_i}{F_i} + \frac{L_{p-i}}{F_{p-i}} \right) = \sum_{i=1}^{p-1} \frac{F_{p-i} L_i + F_i L_{p-i}}{F_i F_{p-i}} = \sum_{i=1}^{p-1} \frac{2F_p}{F_i F_{p-i}}.$$

All forthcoming sums and products are for indices i running from 1 to $p-1$. As

$$2F_{i+j} = F_i L_j + F_j L_i$$

we find that

$$2^{p-1} \prod F_{mp+i} = \prod 2F_{mp+i} = \prod (F_{mp} L_i + L_{mp} F_i) = L_{mp}^{p-1} P(1) \prod F_i.$$

Therefore,

$$\binom{np-1}{p-1}_F = \frac{\prod F_{mp+i}}{\prod F_i} = \left(\frac{L_{mp}}{2} \right)^{p-1} P(1).$$

Since $L_k^2 - 5F_k^2 = 4(-1)^k$, we see that $(L_{mp}/2)^{p-1} \equiv \left((-1)^m\right)^{\frac{p-1}{2}} \pmod{F_p^2}$. To establish the congruences modulo L_p, note that L_p divides L_{mp} if and only if m is odd and L_p divides F_{mp} if m is even. Thus, all coefficients of $(L_{mp}/2)^{p-1} P(x)$ are 0 $\pmod{L_p}$ except possibly and respectively the constant term $(F_{mp}/2)^{p-1} \prod L_i/F_i$, if m is odd, and the leading term $(L_{mp}/2)^{p-1}$, if m is even. If m is odd, then, as $2(-1)^i L_{p-i} = L_p L_i - 5F_p F_i$, we find that

$$\prod \frac{L_i}{F_i} = \prod \frac{L_{p-i}}{F_i} = (2^{p-1}(-1)^{\sum i})^{-1} \prod \frac{2(-1)^i L_{p-i}}{F_i} \equiv 2^{-p+1}(-1)^{\frac{p-1}{2}} \prod (-5F_p) \pmod{L_p}.$$

Hence,

$$(-1)^{\frac{p-1}{2}} (F_{mp}/2)^{p-1} \prod \frac{L_i}{F_i} \equiv (-5F_{mp}^2/4)^{\frac{p-1}{2}} (-5F_p^2/4)^{\frac{p-1}{2}} \equiv \left((-1)^m\right)^{\frac{p-1}{2}} (-1)^{\frac{p-1}{2}} \pmod{L_p},$$

which yields the congruence. If m is even, then a simple induction using the identity

$$L_{2k} = L_k^2 - 2(-1)^k$$

gives that $L_{mp} \equiv 2 \pmod{L_p}$. Thus, $(L_{mp}/2)^{p-1} \equiv 1 \pmod{L_p}$, which, as F_p and L_p are coprime, fully lands the H-737 problem.

Also solved by the proposer.

Errata: In problem H-763, in the denominator of the RHS of (i), "$(n+2)$"should be "$(n+1)$" and in the denominator RHS of (iv), "$n^2(n+1)^2$", should be "$n^3(n+1)^3$".

Late Acknowledgement: Kenneth B. Davenport solved H-733.