ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY
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PROBLEMS PROPOSED IN THIS ISSUE

H-765 Proposed by D. M. Bătinețu-Giurgiu, Bucharest and Neculai Stanciu, Buzău, Romania.

Prove that for positive integer \( n \) and \( m > 0 \) we have:

(i) \[
\frac{L^4_n}{L_nL_{n+1}} + \frac{L^4_{n+1}}{L_{n+1}L_{n+3}} + \frac{L^4_n + L^4_n}{L_{n+3}L_n} \geq \frac{2}{3}L^2_{n+4};
\]

(ii) \[
\left( \sum_{k=1}^{n} \frac{F^2m+4}{F^2m} \right) \left( \sum_{k=1}^{n} \frac{1}{F^2m} \right) \geq F^2mF^2_{n+1};
\]

(iii) \[
\left( \sum_{k=1}^{n} \frac{L^2m+4}{L^2m} \right) \left( \sum_{k=1}^{n} \frac{1}{L^2m} \right) \geq (L_nL_{n+1} - 1)^2;
\]

(iv) \[
\left( \sum_{k=1}^{n} \frac{F^m+2}{F^m} \right) \left( \sum_{k=1}^{n} \frac{1}{F^m} \right) \geq (F_{n+2} - 1)^2;
\]

(v) \[
1 + \frac{n}{F^{m+1}} \geq F_{n+2} \quad \text{and} \quad 3 + \frac{n}{L^{m+1}} \geq L_{n+2}.
\]

H-766 Proposed by H. Ohtsuka, Saitama, Japan.

Let \( n = m + 2 \). For \( m \geq 1 \), prove that

\[
\sum_{h=1}^{m} \sum_{i=1}^{h} \sum_{j=1}^{i} \sum_{k=1}^{j} F^4_k = \frac{4F^4_n + n^4 - 5n^2}{100}.
\]

H-767 Proposed by H. Ohtsuka, Saitama, Japan.

Prove that

\[
\lim_{n \to \infty} \sqrt{F^2_2 + \sqrt{F^2_4 + \sqrt{F^2_8 + \cdots}}} = 3.
\]
H-768 Proposed by H. Ohtsuka, Saitama, Japan.

Let \( \binom{n}{k}_F \) denote the Fibonomial coefficient. For \( n \geq 1 \), prove that

\[
\begin{align*}
(i) \quad & \sum_{k=0}^{n} F_{2(n-k)} \binom{2n}{k}_F = \frac{F_{2n+1}(F_{2n+2}+1)}{F_{2n+3}} - \frac{F_{n+1}F_{n+3}}{F_{2n+3}} \binom{2n}{n}_F; \\
(ii) \quad & \sum_{k=0}^{n} F_{2(n-k)} \binom{2n}{k}_F = \frac{F_{2n+1}^2}{F_{2n+2}} - \frac{F_{n+1}^2}{F_{n+1}} \binom{2n}{n}_F.
\end{align*}
\]

SOLUTIONS

H-734 Proposed by H. Ohtsuka, Saitama, Japan.

(Vol. 51, No. 1, February 2013)

For \( n \geq 3 \) find closed form expressions for

\[
\left[ \left( 1 - \prod_{k=n}^{\infty} \left( 1 - \frac{1}{F_k} \right) \right)^{-1} \right] \quad \text{and} \quad \left[ \left( 1 - \prod_{k=n}^{\infty} \left( 1 - \frac{1}{F_k^2} \right) \right)^{-1} \right].
\]

Here, \( \lfloor x \rfloor \) be the largest integer less than or equal to \( x \).

Solution by the proposer.

We need the following lemma.

Lemma 1. For \( n \geq 3 \), we have

\[
\begin{align*}
(1) \quad & \frac{F_{n-2} - 1}{F_{n-2}} < \frac{F_{n-1} - 1}{F_{n-1}} \times \frac{F_{n-1} - 1}{F_{n-1}}; \\
(2) \quad & \frac{F_{n-1} + 1}{F_{n-2}} > \frac{F_{n-1} + 1}{F_{n-1}}; \\
(3) \quad & \frac{F_nF_{n-1} - 1}{F_nF_{n-1}} < \frac{F_n^2 - 1}{F_n^2} \times \frac{F_{n+1}F_n - 1}{F_{n+1}F_n} \quad \text{(if \( n \) is odd)}; \\
(4) \quad & \frac{F_nF_{n-1} - 2}{F_nF_{n-1}} > \frac{F_n^2 - 1}{F_n^2} \times \frac{F_{n+1}F_n - 2}{F_{n+1}F_n}; \\
(5) \quad & \frac{F_nF_{n-1} - 1}{F_nF_{n-1}} < \frac{F_n^2 - 1}{F_n^2} \times \frac{F_{n+1}F_n - 1}{F_{n+1}F_n}; \\
(6) \quad & \frac{F_nF_{n-1} - 1}{F_nF_{n-1}} > \frac{F_n^2 - 1}{F_n^2} \times \frac{F_{n+1}F_n - 1}{F_{n+1}F_n} \quad \text{(if \( n \) is even)}.
\end{align*}
\]

Proof. We will only prove (1) since all other verifications are similar. We have

\[
\begin{align*}
F_{n-2}(F_n - 1)(F_{n-1} - 1) - F_nF_{n-1}(F_{n-2} - 1) &= F_{n-2} + F_{n-1}F_n - F_{n-1}F_{n-2} - F_nF_{n-2} \\
&= F_{n-2} + F_{n-1}F_n - F_{n-1}F_{n-2} - F_nF_{n-2} \\
&= F_{n-2} + F_{n-1}F_n - F_{n-1}F_{n-2} = F_{n-2} + (-1)^n \geq 0.
\end{align*}
\]

Therefore, we obtain the desired inequality (1). \(\square\)
(i) Using Lemma 1 (1), we have
\[
\frac{F_{n-2} - 1}{F_{n-2}} \leq \frac{F_{n-1} - 1}{F_n} \times \frac{F_{n-1} - 1}{F_{n-1}} \leq \frac{F_{n-1} - 1}{F_{n+1}} \times \frac{F_{n+1} - 1}{F_{n+2}} \times \frac{F_{n+1} - 1}{F_{n+1}} \leq \cdots \leq \prod_{k=n}^{\infty} \frac{F_k - 1}{F_k}.
\]
Using Lemma 1 (2), we have
\[
\frac{F_{n-2}}{F_{n-2} + 1} > \frac{F_{n-1}}{F_n} > \frac{F_{n-1}}{F_{n-1} + 1} > \frac{F_{n-1} - 1}{F_{n+1}} \times \frac{F_{n+1} - 1}{F_{n+2}} \times \frac{F_{n+1} - 1}{F_{n+1} + 1} \cdots > \prod_{k=n}^{\infty} \frac{F_k - 1}{F_k}.
\]
Therefore,
\[
1 - \frac{1}{F_{n-2}} \leq \prod_{k=n}^{\infty} \left(1 - \frac{1}{F_k}\right) < 1 - \frac{1}{F_{n-2} + 1}.
\]
That is,
\[
F_{n-2} \leq \left(1 - \prod_{k=n}^{\infty} \left(1 - \frac{1}{F_k}\right)\right)^{-1} < F_{n-2} + 1.
\]
Thus, for \(n \geq 3\), we obtain
\[
\left\lfloor \left(1 - \prod_{k=n}^{\infty} \left(1 - \frac{1}{F_k}\right)\right)^{-1} \right\rfloor = F_{n-2}.
\]

(ii) Case 1. \(n \geq 3\) is odd. Using Lemma 1 (3) and (4), we obtain the following inequality in the same manner as (i):
\[
\frac{F_n F_{n-1} - 1}{F_n F_{n-1}} < \prod_{k=n}^{\infty} \frac{F_k^2 - 1}{F_k} < \frac{F_n F_{n-1} - 1}{F_n F_{n-1} + 1}.
\]
Therefore,
\[
1 - \frac{1}{F_n F_{n-1}} < \prod_{k=n}^{\infty} \left(1 - \frac{1}{F_k^2}\right) < 1 - \frac{1}{F_n F_{n-1} + 1}.
\]
That is,
\[
F_n F_{n-1} < \left(1 - \prod_{k=n}^{\infty} \left(1 - \frac{1}{F_k^2}\right)\right)^{-1} < F_n F_{n-1} + 1.
\]
Case 2. \(n \geq 4\) is even. Using Lemma 1 (5) and (6), we obtain the following inequality in the same manner as (i):
\[
\frac{F_n F_{n-1} - 2}{F_n F_{n-1} - 1} < \prod_{k=n}^{\infty} \frac{F_k^2 - 1}{F_k} < \frac{F_n F_{n-1} - 1}{F_n F_{n-1}}.
\]
Therefore,
\[
1 - \frac{1}{F_n F_{n-1} - 1} < \prod_{k=n}^{\infty} \left(1 - \frac{1}{F_k^2}\right) < 1 - \frac{1}{F_n F_{n-1}}.
\]
That is,

\[ F_n F_{n-1} - 1 < \left( 1 - \prod_{k=n}^{\infty} \left( 1 - \frac{1}{F_k^2} \right) \right)^{-1} < F_n F_{n-1}. \]

Therefore, we obtain

\[ \left( 1 - \prod_{k=n}^{\infty} \left( 1 - \frac{1}{F_k^2} \right) \right)^{-1} = \begin{cases} F_n F_{n-1} & \text{if } n \equiv 1 \pmod{2}, \; n \geq 3; \\ F_n F_{n-1} - 1 & \text{if } n \equiv 1 \pmod{2}, \; n \geq 4. \end{cases} \]

**Proposer’s note:** For \( m \geq 2 \) and \( n \geq 2 \), we obtain the following identity in the same manner:

\[ \left( 1 - \prod_{k=n}^{\infty} \left( 1 - \frac{1}{F_{mk}} \right) \right)^{-1} = F_{mn} - F_{m(n-1)}. \]

Also solved by Paul S. Bruckman.

**On a Power Series with Binomial Coefficients**

**H-735** Proposed by Paul S. Bruckman, BC.

(Vol. 51, No. 2, May 2013)

Let \( F_m(x) = \sum_{n=0}^{\infty} \binom{2n + m}{n} x^n \), where \( m \) is any real number and \( |x| < 1/4 \). Also let 

\( \theta(x) = (1 - 4x)^{1/2} \). For brevity, write \( F_m = F_m(x) \), \( \theta = \theta(x) \). Prove the following:

(a) \( F_0 = \frac{1}{\theta}, \; F_1 = \frac{(1-\theta)}{2x\theta} \);

(b) for all real \( m \), \( F_m^m = \left( \frac{F_1}{F_0} \right)^m \);

(c) for all real \( m \), \( \sum_{k=0}^{n} \binom{2k + m}{k} \binom{2n - 2k - m}{n - k} = 4^n, \; n = 0, 1, 2, \ldots. \)

Solution by Ángel Plaza, Gran Canaria, Spain.

(a) \( F_0 = \frac{1}{\theta} \) is \( \sum_{n=0}^{\infty} \binom{2n}{n} x^n = \frac{1}{\sqrt{1 - 4x}} \), which is given as identity (2.5.1) in [1].

\( F_1 = \frac{(1-\theta)}{2x\theta} \) is equivalent to \( \sum_{n=0}^{\infty} \binom{2n + 1}{n} x^n = \frac{1 - \sqrt{1 - 4x}}{2x\sqrt{1 - 4x}} \). Then

\[ RH\ S = \frac{1}{2x} \left( \frac{1}{\sqrt{1 - 4x}} - 1 \right) = \frac{1}{2x} \sum_{n=1}^{\infty} \binom{2n}{n} x^n = \frac{1}{2} \sum_{n=1}^{\infty} \binom{2n}{n} x^{n-1} \]

\[ = \frac{1}{2} \sum_{n=0}^{\infty} \binom{2n + 2}{n + 1} x^n = \sum_{n=0}^{\infty} \binom{2n + 1}{n + 1} x^n + \frac{1}{2} \sum_{n=0}^{\infty} \binom{2n + 1}{n} x^n \]

\[ = \sum_{n=0}^{\infty} \binom{2n + 1}{n} x^n = LHS. \]
(b) By (a), we have to show that for all \(m\), \(F_m = F_0 \left( \frac{F_1}{F_0} \right)^m\), where \(F_0 = \frac{1}{\sqrt{1 - 4x}}\) and \(F_1 = 1 - \sqrt{1 - 4x}\). That is
\[
\sum_{n=0}^{\infty} \binom{2n + m}{n} x^n = \frac{1}{\sqrt{1 - 4x}} \left( \frac{1 - \sqrt{1 - 4x}}{2x} \right)^m,
\]
which is identity (2.5.15) in [1].

(c) Let \(A(x)\) be the generating function of the LHS. That is
\[
A(x) = \sum_{n \geq 0} x^n \sum_{k=0}^{n} \binom{2k + m}{k} \binom{2n - 2k - m}{n - k}
= \sum_{k \geq 0} \binom{2k + m}{k} x^k \sum_{n-k \geq 0} \binom{2n - 2k - m}{n - k} x^{n-k}
= \frac{1}{\sqrt{1 - 4x}} \left( \frac{1 - \sqrt{1 - 4x}}{2x} \right)^m \frac{1}{\sqrt{1 - 4x}} \left( \frac{1 - \sqrt{1 - 4x}}{2x} \right)^{-m},
\]
which is precisely the generating function of the RHS, \(4^n\). Note that we have used the identity (2.5.15) in [1].

References


Also solved by Kenneth B. Davenport and the proposer.

On the Sum of the Cubes of the Tribonacci Numbers

H-736 Proposed by Hideyuki Ohtsuka, Saitama, Japan.
(Vol. 51, No. 2, May 2013)

The Tribonacci numbers \(T_n\) satisfy \(T_0 = 0\), \(T_1 = T_2 = 1\), \(T_{n+3} = T_{n+2} + T_{n+1} + T_n\) for \(n \geq 0\). Find an explicit formula for the sum \(\sum_{k=1}^{n} T_k^3\).

Solution by the proposer.

Let \(S_n = \sum_{k=1}^{n} T_k^3\). We need the following lemma.

Lemma 2. We have

(i) \(\sum_{k=1}^{n} (T_k^2 T_{k+1} + T_k T_{k+1}^2) = T_n T_{n+1} T_{n+2}\);

(ii) \(\sum_{k=1}^{n} (T_{k+1}^2 T_{k+2} + T_{k+1} T_{k+2}^2) = T_{n+1} T_{n+2} T_{n+3} - 2\);

(iii) \(\sum_{k=1}^{n} T_k^2 T_{k+2} = S_n + T_n T_{n+1} T_{n+2} - T_n^2 T_{n+1}\);
(iv) \(-6 \sum_{k=1}^{n} T_{k+1}T_{k+2} = 2S_n + A_n,\)

where

\[ A_n = -T_{n+2}^3 - T_n^3 - 3T_nT_{n+1}^2 - 3T_nT_{n+1}^2 - 3T_{n+1}T_{n+2}^2 + 7. \]

Proof. (i) We have

\[ \sum_{k=1}^{n} (T_{k+1}^2 + T_k^2) = \sum_{k=1}^{n} T_k(T_{k+1} + T_{k+1}) = \sum_{k=1}^{n} T_k(T_{k+2} - T_{k-1}) \]
\[ = \sum_{k=1}^{n} (T_{k+1}T_{k+2} - T_{k-1}T_{k+1}) = T_nT_{n+1}T_{n+2}. \]

(ii) We have

\[ \sum_{k=1}^{n} (T_{k+1}^2 + T_k^2) = \sum_{k=2}^{n} (T_k^2 + T_{k+1}^2) = T_{n+1}T_{n+2}T_{n+3} - 2, \] (by (i)).

(iii) We have

\[ \sum_{k=1}^{n} T_k^2T_{k+2} = \sum_{k=1}^{n} T_k(T_{k+1}^2 + T_{k+1}^2) = \sum_{k=1}^{n} T_k^3 + \sum_{k=1}^{n} (T_k^2T_{k+1}^2 + T_{k+1}^2T_k^2) \]
\[ = S_n + \sum_{k=1}^{n} (T_k^2T_{k+1}^2 + T_{k+1}^2T_k^2) - T_nT_{n+1}^2 = S_n + T_nT_{n+1}T_{n+2} - T_nT_{n+1}^2, \] (by (i)).

(iv) We have

\[ 0 = \sum_{k=1}^{n} ((T_k + T_{k-1})^3 - (T_{k+2} - T_{k+1})^3) \]
\[ = 3 \sum_{k=1}^{n} (T_k^2T_{k+1}^2 + T_k^2T_{k-1}^2) - 3 \sum_{k=1}^{n} (T_{k+2}^2T_{k+1}^2 - T_{k+1}^2T_{k-1}^2) + \sum_{k=1}^{n} (T_k^3 + T_{k-1}^3 - T_{k+2}^3 + T_{k+1}^3) \]
\[ = 6 \sum_{k=1}^{n} T_k^2T_{k+1}^2 + 2S_n + A_n. \]

Let \(x = T_k, y = T_{k+1}, z = T_{k+2}.\) We have

\[ x^3 + y^3 + z^3 + 3x^2y + 3xy^2 + 3xz^2 + 3x^2z + 3y^2z + 3yz^2 + 6xyz = (x + y + z)^3; \] (1)
\[ x^3 + 2y^3 + z^3 + 2x^2y + 2xy^2 + 2x^2z - xz^2 - 2y^2z - 2xyz = 1 \] (2)

(see [1]). Multiplying (2) by 3 and adding the resulting identity to (1), we get

\[ 4x^3 + 7y^3 + 4z^3 + 9x^2y + 9xy^2 + 6x^2z + 3y^2z - 3yz^2 = T_{k+3}^3 + 3. \]
THE FIBONACCI QUARTERLY

From the above identity, we have
\[
\sum_{k=1}^{n} \left( 4T_{3k}^3 + 7T_{3k+1}^3 + 4T_{3k+2}^3 - T_{3k+3}^3 \right) + 9 \sum_{k=1}^{n} \left( T_{2k}^2 T_{k+1}^2 + T_k^2 T_{k+1}^2 \right)
\]
\[+ 6 \sum_{k=1}^{n} T_k^2 T_{k+2} + 3 \sum_{k=1}^{T_{k+1}} \left( T_{k+1}^2 T_{k+2} + T_{k+1}^2 T_{k+2} \right) - 6 \sum_{k=1}^{T_{k+1}} T_{k+1}^2 T_{k+2} = 3n.\]

Using Lemma 2 (i), (ii), (iii) and (iv), we have
\[
\sum_{k=1}^{n} \left( 12T_{3k}^3 + 7T_{3k+1}^3 + 4T_{3k+2}^3 - T_{3k+3}^3 \right)
\]
\[+ 15T_n T_{n+1} T_{n+2} + 3T_{n+1} T_{n+2} T_{n+3} - 6T_n T_{n+1}^2 + A_n - 6 = 3n. \tag{3}\]

Here,
\[
\sum_{k=1}^{n} \left( 12T_{3k}^3 + 7T_{3k+1}^3 + 4T_{3k+2}^3 - T_{3k+3}^3 \right) = 22S_n + 10T_{n+1}^3 + 3T_{n+2}^3 - T_{n+3}^3 - 5.
\]

Therefore, (3) is
\[
22S_n = T_{n+3}^3 - 2T_{n+2}^3 - 10T_{n+1}^3 + T_n^3 + 9T_n T_{n+1}^2 + 3T_{n+1}^2 T_n + 15T_n T_{n+1} T_{n+2} - 3T_{n+1} T_{n+2} (T_{n+3} - T_{n+2} - T_{n+1}) + 3n + 4.
\]

Since
\[- 3T_{n+1} T_{n+2} (T_{n+3} - T_{n+2} - T_{n+1}) = -3T_n T_{n+1} T_{n+2},\]

we obtain
\[
S_n = \frac{1}{22} (T_{n+3}^3 - 2T_{n+2}^3 - 10T_{n+1}^3 + T_n^3 + 9T_n T_{n+1}^2 + 3T_{n+1}^2 T_n - 18T_n T_{n+1} T_{n+2} + 3n + 4).
\]

References


A Lucas Type Congruence with Fibonomials

H-737 Proposed by Hideyuki Ohtsuka, Saitama, Japan.

Let \( \binom{n}{k}_F \) denote the Fibonomial coefficient. For an odd prime \( p \) and a positive integer \( n \), prove that
\[
\binom{np - 1}{p - 1}_F \equiv (-1)^{(n-1)(p-1)/2} \pmod{F_p^2 L_p}.
\]

Solution by Christian Ballot, Caen, France.

With \( m := n - 1 \), define the rational polynomial
\[
P(x) := \prod_{i=1}^{p-1} \left( x + \frac{F_{mp} L_i}{L_{mp} F_i} \right).
\]
Expanding $P(x)$ yields

$$P(x) = x^{p-1} + \frac{F_{mp}}{L_{mp}} \sum_{i=1}^{p-1} \frac{L_i}{F_i} x^{i-2} + \frac{F_{mp}}{L_{mp}^2} \sum_{0<i<j<p} \frac{L_iL_j}{F_iF_j} x^{i+j-3} + \cdots + \frac{F_{mp}^{p-1}}{L_{mp}^p} \prod_{i=1}^{p-1} \frac{L_i}{F_i}.$$ 

All coefficients, except that of $x^{p-1}$, are $0 \pmod{F_p^2}$. Indeed, $F_{mp}^k$ is divisible by $F_p^2$ for $k \geq 2$ and $F_p$ is prime to $L_{mp} \prod_{i=1}^{p-1} F_i$. Moreover, $S := \sum_{i=1}^{p-1} \frac{F_{p-i}L_i + F_p}{F_{p-i}} \equiv 0 \pmod{F_p}$ because

$$2S = \sum_{i=1}^{p-1} \left( \frac{L_i}{F_i} + \frac{L_{p-i}}{F_{p-i}} \right) = \sum_{i=1}^{p-1} \frac{F_{p-i}L_i + F_p}{F_iF_p} = \sum_{i=1}^{p-1} \frac{2F_p}{F_iF_{p-i}}.$$ 

All forthcoming sums and products are for indices $i$ running from 1 to $p-1$. As $2F_{i+j} = F_iL_j + F_jL_i$ we find that

$$2^{p-1} \prod F_{mp+i} = \prod 2F_{mp+i} = \prod (F_{mp}L_i + L_{mp}F_i) = L_{mp}^{p-1}(1) \prod F_i.$$ 

Therefore,

$$\left( \frac{np-1}{p-1} \right)_F = \prod \frac{F_{mp+i}}{F_i} = \left( \frac{L_{mp}}{2} \right)^{p-1}(1).$$ 

Since $L_k^2 - 5F_k^2 = 4(-1)^k$, we see that $(L_{mp}/2)^{p-1} \equiv (-1)^{m} \left( \frac{p-1}{2} \right) \pmod{F_p^2}$. To establish the congruences modulo $L_p$, note that $L_p$ divides $L_{mp}$ if and only if $m$ is odd and $L_p$ divides $F_{mp}$ if $m$ is even. Thus, all coefficients of $(L_{mp}/2)^{p-1} P(x)$ are $0 \pmod{L_p}$ except possibly and respectively the constant term $(L_{mp}/2)^{p-1} \prod L_i/F_i$, if $m$ is odd, and the leading term $(L_{mp}/2)^{p-1}$, if $m$ is even. If $m$ is odd, then, as \(2(-1)^iL_{p-i} = L_pL_i - 5F_pF_i, \) we find that

$$\prod \frac{L_i}{F_i} = \prod \frac{L_{p-i}}{F_i} = (2^{p-1}(-1)^{\sum i})^{-1} \prod \frac{2(-1)^iL_{p-i}}{F_i} = 2^{-p+1}(-1)^{\frac{p-1}{2}} \prod (-5F_p) \pmod{L_p}.$$ 

Hence,

$$(-1)^{\frac{p-1}{2}}(L_{mp}/2)^{p-1} \prod \frac{L_i}{F_i} \equiv (-5F_{mp}^2/4)^{\frac{p-1}{2}}(-5F_p^2/4)^{\frac{p-1}{2}} \equiv ((-1)^{m})^{\frac{p-1}{2}}(-1)^{\frac{p-1}{2}} \pmod{L_p},$$

which yields the congruence. If $m$ is even, then a simple induction using the identity

$$L_{2k} = L_k^2 - 2(-1)^k$$

gives that $L_{mp} \equiv 2 \pmod{L_p}$. Thus, $(L_{mp}/2)^{p-1} \equiv 1 \pmod{L_p}$, which, as $F_p$ and $L_p$ are coprime, fully lands the H-737 problem.

Also solved by the proposer.

Errata: In problem H-763, in the denominator of the RHS of (i), “$(n + 2)$” should be “$(n + 1)$” and in the denominator RHS of (iv), “$n^2(n + 1)^2$” should be “$n^3(n + 1)^3$”.

Late Acknowledgement: Kenneth B. Davenport solved H-733.