ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY
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Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to FLORIAN LUCA, SCHOOL OF MATHEMATICS, UNIVERSITY OF THE WITWATERSRAND, PRIVATE BAG X3, WITS 2050, JOHANNESBURG, SOUTH AFRICA or by e-mail at the address florian.luca@wits.ac.za as files of the type tex, dvi, ps, doc, html, pdf, etc. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions sent by regular mail should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-809 Proposed by Hideyuki Ohtsuka, Saitama, Japan.
Prove that
\[
\left(1 - \alpha \frac{1}{L_2}\right)\left(1 - \beta \frac{1}{L_2^2}\right)\left(1 - \alpha \frac{1}{L_2^3}\right)\left(1 - \beta \frac{1}{L_2^4}\right) \cdots = \frac{7\sqrt{5} - 5}{22}.
\]

H-810 Proposed by Ángel Plaza, Gran Canaria, Spain.
Prove that
\[
\sum_{n=3}^{\infty} \frac{1}{L_n^k - 25} = \frac{5}{63} - \frac{1}{6\sqrt{5}}.
\]

H-811 Proposed by Ángel Plaza, Gran Canaria, Spain.
For any positive integer \( k \) let \( \{ F_{k,n}\}_{n \geq 0} \) be defined by \( F_{k,n+2} = kF_{k,n+1} + F_{k,n} \) for \( n \geq 0 \) with \( F_{k,0} = 0 \), \( F_{k,1} = 1 \). Prove that
\[
\sum_{n=0}^{\infty} \frac{1}{1 + F_{k,2n+1}} = \frac{\sqrt{k^2 + 4}}{2k}.
\]

H-812 Proposed by Hideyuki Ohtsuka, Saitama, Japan.
Prove that
\[
\sum_{i+j = F_{n+1}} \left( \begin{array}{c} F_{n+1} \\ i \end{array} \right) \left( \begin{array}{c} F_n \\ i \end{array} \right) \left( \begin{array}{c} L_n \\ j \end{array} \right) = \sum_{i+j = F_{n+1}} \left( \begin{array}{c} F_n \\ i \end{array} \right) \left( \begin{array}{c} L_n \\ i \end{array} \right) \left( 2j \right) \left( F_{n+1} \right).
\]
SOLUTIONS

A Series Related to the Sum of the Reciprocals of the Fibonacci Numbers

H-775 Proposed by H. Ohtsuka, Saitama, Japan.
(Vol. 53, No. 3, August 2015)

Let \( c \) be any real number \( c \neq 2, -L_2 \) for \( n \geq 0 \). Let

\[
\gamma_c = \sqrt{5} \prod_{n=1}^{\infty} \left( 1 + \frac{c}{L_{2n}} \right)^{-1}.
\]

Prove that

\[
\sum_{k=1}^{\infty} \frac{1}{(L_2 + c)(L_4 + c) \cdots (L_{2k} + c)} = \frac{\gamma_c + c - 3}{c^2 - c - 2}.
\]

Solution by the proposer.

Let \( P_n = (L_2 + c)(L_4 + c) \cdots (L_{2n} + c) \). For \( n \geq 1 \), we show that

\[
(c^2 - c - 2) \sum_{k=1}^{n} \frac{1}{P_k} = \frac{L_{2n+1} - c}{P_n} + c - 3.
\]

(1)

The proof of (1) is by mathematical induction on \( n \). For \( n = 1 \), we have

\[
LHS = \frac{c^2 - c - 2}{P_1} = \frac{c^2 - c - 2}{3 + c} = \frac{7 - c}{3 + c} + c - 3 = \frac{L_4 - c}{P_1} + c - 3 = RHS.
\]

We assume that (1) holds for \( n \). For \( n + 1 \), we have

\[
(c^2 - c - 2) \sum_{k=1}^{n+1} \frac{1}{P_k} = (c^2 - c - 2) \left( \frac{1}{P_{n+1}} + \sum_{k=1}^{n} \frac{1}{P_k} \right)
\]

\[
= \frac{c^2 - c - 2}{P_{n+1}} + \frac{L_{2n+1} - c}{P_n} + c - 3
\]

\[
= \frac{c^2 - c - 2 + (L_{2n+1} - c)(L_{2n+1} + c)}{P_{n+1}} + c - 3
\]

\[
= \frac{L_{2n+1} - 2 - c}{P_{n+1}} + c - 3
\]

\[
= \frac{L_{2n+2} - c}{P_{n+1}} + c - 3,
\]

since \( L_m^2 - 2(-1)^m = L_{2m} \). Thus, (1) holds for \( n + 1 \). Therefore (1) is proved. We have

\[
P_n = \prod_{k=1}^{n} (L_{2k} + c) = \prod_{k=1}^{n} L_{2k} \prod_{k=1}^{n} \left( 1 + \frac{c}{L_{2k}} \right).
\]

Hence, using \( F_m L_m = F_{2m} \), we have

\[
F_2 L_2 L_4 L_8 \cdots L_{2n} = F_4 L_4 L_8 \cdots L_{2n} = \cdots = F_{2n} L_{2n} = F_{2n+1}.
\]

Thus,

\[
P_n = \prod_{k=1}^{n} \left( 1 + \frac{c}{L_{2k}} \right).
\]

(2)
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Therefore, by (1) and (2), we have
\[
\sum_{k=1}^{n} \frac{1}{P_k} = \frac{1}{c^2 - c - 2} \left[ \frac{L_{2n+1} - c}{F_{2n+1}} \prod_{k=1}^{n} \left( 1 + \frac{c}{L_{2k}} \right)^{-1} + c - 3 \right] \to \frac{\gamma c + c - 3}{c^2 - c - 2}
\]
as \(n \to \infty\) since \(L_m/F_m \to \sqrt{5}\) as \(m \to \infty\).

Note: If \(c = 0\), we then have
\[
\sum_{k=1}^{\infty} \frac{1}{L_2 L_4 \cdots L_{2k}} = \frac{\gamma_0 - 3}{2} \quad \text{i.e.,} \quad \sum_{k=1}^{\infty} \frac{1}{F_{2k+1}} = \frac{3 - \sqrt{5}}{2}.
\]

From the above identity, we obtain the well-known identity
\[
\sum_{k=0}^{\infty} \frac{1}{F_{2k}} = \frac{7 - \sqrt{5}}{2}.
\]

Also solved by Dmitry Fleischman.

A Series of Inverse Tangents of Reciprocals of Lucas Numbers

H-776 Proposed by H. Ohtsuka, Saitama, Japan.
(Vol. 53, No. 3, August 2015)

Determine

(i) \(\sum_{n=0}^{\infty} (-1)^n \tan^{-1} \frac{1}{L_{3^n}}\) \quad and \quad (ii) \(\sum_{n=1}^{\infty} \tan^{-1} \frac{1}{F_{2n}} \tan^{-1} \frac{1}{L_{2^n}}\).

Solution by the proposer.

(i) For \(n \geq 0\), we have
\[
\tan^{-1} \frac{1}{\alpha^{3^n}} + \tan^{-1} \frac{1}{\alpha^{3^{n+1}}} = \tan^{-1} \left( \frac{1}{\alpha^{3^n}} + \frac{1}{\alpha^{3^{n+1}}} \right) = \tan^{-1} \left( \frac{\alpha^{3^{n+1}} + \alpha^{3^n}}{\alpha^{3^{n+1}} - 1} \right) = \tan^{-1} \left( \frac{\alpha^{2 \cdot 3^n} (\alpha^{3^n} + \alpha^{-3^n})}{\alpha^{2 \cdot 3^n} (\alpha^{3^n} + \alpha^{-3^n}) (\alpha^{3^n} - \alpha^{-3^n})} \right) = \tan^{-1} \left( \frac{1}{\alpha^{3^n} + \beta^{3^n}} \right) = \tan^{-1} \frac{1}{L_{3^n}}.
\]

Using the above identity, we have
\[
\sum_{n=0}^{m} (-1)^n \tan^{-1} \frac{1}{L_{3^n}} = \sum_{n=0}^{m} \left[ (-1)^n \tan^{-1} \frac{1}{\alpha^{3^n}} - (-1)^{n+1} \tan^{-1} \frac{1}{\alpha^{3^{n+1}}} \right] = \tan^{-1} \frac{1}{\alpha} - (-1)^{m+1} \tan^{-1} \frac{1}{\alpha^{3^{m+1}}} \to \tan^{-1} \frac{1}{\alpha}
\]
as \(m \to \infty\).
(ii) We have
\[
\tan^{-1} \frac{1}{\alpha^{2n-1}} + \tan^{-1} \frac{1}{\alpha^{2n+1}} = \tan^{-1} \left( \frac{1}{\alpha^{2n-1}} + \frac{1}{\alpha^{2n+1}} \right) = \tan^{-1} \left( \frac{\alpha + \alpha^{-1}}{\alpha^{2n} - \alpha^{-2n}} \right) = \tan^{-1} \frac{\sqrt{5}}{\alpha^{2n} - \beta^{2n}} = \tan^{-1} \frac{1}{F_{2n}},
\]
and similarly
\[
\tan^{-1} \frac{1}{\alpha^{2n-1}} - \tan^{-1} \frac{1}{\alpha^{2n+1}} = \tan^{-1} \left( \frac{1}{\alpha^{2n-1}} - \frac{1}{\alpha^{2n+1}} \right) = \tan^{-1} \left( \frac{\alpha - \alpha^{-1}}{\alpha^{2n} + \alpha^{-2n}} \right) = \tan^{-1} \frac{1}{\alpha^{2n} + \beta^{2n}} = \tan^{-1} \frac{1}{L_{2n}}.
\]

Using the above identities, we have
\[
\sum_{n=1}^{m} \tan^{-1} \frac{1}{F_{2n}} \tan^{-1} \frac{1}{L_{2n}} = \sum_{n=1}^{m} \left[ \left( \tan^{-1} \frac{1}{\alpha^{2n-1}} \right)^2 - \left( \tan^{-1} \frac{1}{\alpha^{2n+1}} \right)^2 \right] = \left( \tan^{-1} \frac{1}{\alpha} \right)^2 - \left( \tan^{-1} \frac{1}{\alpha^{2n+1}} \right)^2 \rightarrow \left( \tan^{-1} \frac{1}{\alpha} \right)^2
\]
as \(m \to \infty\).

Also solved by Kenneth B. Davenport, Dmitry Fleischman, and David Terr.

Sums of Products of Binomial Coefficients


For any nonnegative integers \(n, m, l\) prove that
\[
\sum_{k=0}^{n} \binom{n}{k}^2 \sum_{i \geq 0} \binom{2k}{i} \binom{2n-2k}{m-i} (-1)^{m-i} = \begin{cases} \binom{2l}{i} \binom{2n-2l}{n-i} & \text{if } m = 2l; \\ 0 & \text{if } m = 2l+1. \end{cases}
\]

Solution by the proposers.

For any nonnegative integer \(n\) and formal power series \(f(x) = \sum_{k=0}^{\infty} a_k x^k\), let \([x^n]f(x)\) denote \(a_n\). Let \(a(n, m)\) be the left-hand side of the identity to be proved. Then we have
\[
a(n, m) = [z^n] \sum_{k=0}^{n} \binom{n}{k}^2 (1 + z)^{2k} (1 - z)^{2n-2k},
\]
since we obtain
\[
(1 + z)^{2k} (1 - z)^{2n-2k} = \sum_{m,i} \binom{2k}{i} \binom{2n-2k}{m-i} (-1)^{m-i} z^m \quad \text{for} \quad n \geq k \geq 0
\]

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using the binomial theorem. We have

\[ a(n, m) = [z^m t^n] \{(1 + z)^2 t^2 + 2(1 + z^2) t + (1 - z)^2\}^n, \]

since

\[ [t^n] \{(1 + z)^2 t + (1 - z)^2\}^n (1 + t)^n = \sum_{k=0}^{n} \binom{n}{k} (n - k) (1 + z)^{2k} (1 - z)^{2n-2k}. \]

Using the trinomial theorem,

\[ a(n, m) = [z^m t^n] \sum_{k_0+k_1+k_2=n} \binom{n}{k_0, k_1, k_2} (1 - z)^{2k_0} (2 + 2z^2)^{k_1} (1 + z)^{2k_2} t^{k_1+2k_2}, \]

where the sum runs over all nonnegative integers \(k_0, k_1, k_2\) satisfying \(k_0 + k_1 + k_2 = n\). Emphasizing the coefficient of \(t^n\), we have

\[ a(n, m) = [z^m] \sum_k \binom{n}{k, n-2k, k} 2^{n-2k} (1 + z^2)^{n-k} (1 - z^2)^{2k}. \]

Since all terms among the sum are polynomials in \(z^2\), we have that \(a(n, m) = 0\) if \(m = 2l + 1\) for some integer \(l \geq 0\). So, suppose that \(m = 2l\). Letting \(y = z^2\), we have

\[ a(n, 2l) = [y^l] \sum_k \binom{n}{k, n-2k, k} 2^{n-2k} (1 + y)^{n-2k} (1 - y)^{2k} \]

\[ = [y^l] \sum_k \binom{n}{2k} \binom{2k}{k} 2^{n-2k} (1 + y)^{n-2k} (1 - y)^{2k} \]

\[ = [x^n y^l] \sum_k \frac{\{2x(1+y)\}^{2k}}{(1-2x(1+y))^{2k+1}} \binom{2k}{k} 2^{2k} (1 + y)^{n-2k} (1 - y)^{2k} \]

\[ = [x^n y^l] \frac{1}{1-2x(1+y)} \sum_k \left( \frac{2k}{k} \right) \left( \frac{x(1-y)}{1-2x(1+y)} \right)^{2k} \]

\[ = [x^n y^l] \frac{1}{1-2x(1+y)} \left( 1 - 4 \left( \frac{x(1-y)}{1-2x(1+y)} \right)^2 \right)^{-1/2} \]

\[ = [x^n y^l] \{(1 - 4x)(1 - 4xy)\}^{-1/2}. \]

The second equality and the fourth equality hold by

\[ \binom{n}{k, n-2k, k} = \binom{n}{2k} \binom{2k}{k} \]

and

\[ \sum_n \binom{n}{k} z^n = \frac{z^k}{(1 - z)^{k+1}}, \]

respectively. Since

\[ (1 - 4x)^{-1/2} = \sum_{k=0}^{\infty} \binom{-1/2}{k} (-4x)^k = \sum_{k=0}^{\infty} \binom{2k}{k} x^k, \]

we have

\[ (1 - 4x)^{-1/2}(1 - 4xy)^{-1/2} = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \binom{2k}{k} \binom{2l}{l} x^{k+l} y^l. \]
Hence,

\[
a(n, 2l) = [x^n y^n] \{(1 - 4x)(1 - 4xy)\}^{-1/2} = \binom{2l}{l} \binom{2n - 2l}{n - l}.
\]

Also solved by Dmitry Fleischman.

A Series with Reciprocals of Products of Fibonacci Numbers

H-778 Proposed by Hideyuki Ohtsuka, Saitama, Japan. (Vol. 53, No. 4, November 2015)

Prove that

\[
\sum_{n=1}^{\infty} \frac{1}{(-\sqrt{5})^n F_2 F_4 \cdots F_{2n}} = \frac{\sqrt{5} - 3}{2}.
\]

Solution by the proposer.

Let \( F'_n = \sqrt{5} F_n \). For \( n \geq 2 \), we have

\[
\frac{(-1)^n}{\alpha^{2n} F'_2 F'_4 \cdots F'_{2n-1}} - \frac{(-1)^{n+1}}{\alpha^{2n+1} F'_2 F'_4 \cdots F'_{2n}} = \frac{(-1)^n (\alpha^{2n} F'_2 + 1)}{\alpha^{2n+1} F'_2 F'_4 \cdots F'_{2n}}
\]

\[
= \frac{(-1)^n (\alpha^{2n} - \beta^{2n}) + 1}{\alpha^{2n+1} F'_2 F'_4 \cdots F'_{2n}}
\]

\[
= \frac{(-1)^n}{F'_2 F'_4 \cdots F'_{2n}}.
\]

Using the above identity, we have

\[
\sum_{n=1}^{m} \frac{1}{(-\sqrt{5})^n F_2 F_4 \cdots F_{2n}} = \sum_{n=1}^{m} \frac{(-1)^n}{F'_2 F'_4 \cdots F'_{2n}}
\]

\[
= -\frac{1}{F'_2} + \sum_{n=2}^{m} \left( \frac{(-1)^n}{\alpha^{2n} F'_2 F'_4 \cdots F'_{2n-1}} - \frac{(-1)^{n+1}}{\alpha^{2n+1} F'_2 F'_4 \cdots F'_{2n}} \right)
\]

\[
= -\frac{1}{F'_2} + \frac{1}{\alpha^4 F'_2} - \frac{(-1)^{m+1}}{\alpha^{2m+1} F'_2 F'_4 \cdots F'_{2m}} \to \frac{\sqrt{5} - 3}{2}
\]

as \( m \to \infty \).

Also solved by Kenneth B. Davenport and Dmitry Fleischman.