ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY
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Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to FLORIAN LUCA, SCHOOL OF MATHEMATICS, UNIVERSITY OF THE WITWATERSRAND, PRIVATE BAG X3, WITS 2050, JOHANNESBURG, SOUTH AFRICA or by e-mail at the address florian.luca@wits.ac.za as files of the type tex, dvi, ps, doc, html, pdf, etc. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions sent by regular mail should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-821 Proposed by Hideyuki Ohtsuka, Saitama, Japan

Prove that

$$\frac{\pi^2}{8} = \sum_{n=1}^{\infty} \tan^{-1}\frac{1}{F_n}\tan^{-1}\frac{1}{F_{n+1}}.$$

H-822 Proposed by D. M. Bătineţu-Giurgiu, Bucharest and Neculai Stanciu, Buzău, Romania

Prove the following inequalities:

(a) $\dfrac{F_nF_{n+2}^2}{F_{n+3}} + \dfrac{F_{n+1}F_{n+3}^2}{F_n+F_{n+2}} + (F_n+F_{n+2})^2 \geq 2\sqrt{6}\sqrt{F_nF_{n+1}}F_{n+2}$;

(b) $F_{n+2}^2 + (F_n+F_{n+2})^2 + F_{n+3}^2 > 4\sqrt{6}\sqrt{F_nF_{n+1}}F_{n+2}$;

(c) $L_{n+2}^2 + (L_n+L_{n+2})^2 + L_{n+3}^2 > 4\sqrt{6}\sqrt{L_nL_{n+1}}L_{n+2}$;

(d) $\sqrt{2}\sqrt{1+F_n^4} + \sum_{k=1}^{n-1}\sqrt{(F_k^4+1)(F_{k+1}^4+1)} > 2F_nF_{n+1}$ for $n > 1$.

H-823 Proposed by Hideyuki Ohtsuka, Saitama, Japan

Given an integer $r \geq 2$, define the sequence $\{G_n\}_{n\geq -r+1}$ by

$$G_n = G_{n-1} + G_{n-2} + \cdots + G_{n-r} \quad \text{for} \quad n \geq 1$$

with arbitrary $G_0, G_{-1}, G_{-2}, \ldots, G_{-r+1}$. For an integer $n \geq 1$, prove that

(i) $\displaystyle\sum_{k=1}^{n} G_kG_{k+r} = \sum_{k=1}^{r}\frac{k(r-k-1)+r+1}{2(r-1)}\sum_{i=1}^{k}(G_{n+i-k}G_{n+i} - G_{i-k}G_i)$;

(ii) $\displaystyle\sum_{k=1}^{n} G_kG_{k+r+1} = \sum_{k=1}^{r}\frac{k(r-k-1)+2r}{2(r-1)}\sum_{i=1}^{k}(G_{n+i-k}G_{n+i} - G_{i-k}G_i)$.

H-824 Proposed by Hideyuki Ohtsuka, Saitama, Japan

Define the generalized Fibonomial coefficient $\binom{n}{k}_{F;r}$ by

$$\binom{n}{k}_{F;r} = \frac{F_{rn}F_{r(n-1)}F_{r(n-2)}\cdots F_{r(n-k+1)}}{F_{rk}F_{r(k-1)}F_{r(k-2)}\cdots F_r} \quad \text{for} \quad 0 < k \le n,$$

with $\binom{n}{0}_{F;r} = 1$ and $\binom{n}{k}_{F;r} = 0$ (otherwise). For positive integers n, r, and s, find closed form expressions for the sums

(i) $\displaystyle\sum_{i+j=2s-1} (-1)^{(r+1)i} \binom{n-1}{i}_{F;r} \binom{n+1}{j}_{F;r}$;

(ii) $\displaystyle\sum_{i+j=2s} (-1)^{i} \binom{n-1}{i}_{F;r} \binom{n+1}{j}_{F;r}$.

SOLUTIONS

Closed form for the sum of a series

H-787 Proposed by Hideyuki Ohtsuka, Saitama, Japan
(Vol. 54, No. 2, May 2016)

Prove that

$$\sum_{n=1}^{\infty} \frac{1}{\alpha^{2F_{n+1}} F_{2F_n} F_{2F_{n+2}}} = \frac{7-3\sqrt{5}}{2}.$$

Solution by Brian Bradie

Note

$$\begin{aligned} \alpha^{2F_{n+1}} F_{2F_n} F_{2F_{n+2}} &= \frac{1}{5}\alpha^{2F_{n+1}} \left(\alpha^{2F_n} - \frac{1}{\alpha^{2F_n}}\right)\left(\alpha^{2F_{n+2}} - \frac{1}{\alpha^{2F_{n+2}}}\right) \\ &= \frac{\alpha^{2F_{n+1}}}{5\alpha^{2F_n}\alpha^{2F_{n+2}}}(\alpha^{4F_n}-1)(\alpha^{4F_{n+2}}-1) \\ &= \frac{1}{5\alpha^{4F_n}}(\alpha^{4F_n}-1)(\alpha^{4F_{n+2}}-1). \end{aligned}$$

Then,

$$\begin{aligned} \frac{1}{\alpha^{2F_{n+1}} F_{2F_n} F_{2F_{n+2}}} &= \frac{5\alpha^{4F_n}}{(\alpha^{4F_n}-1)(\alpha^{4F_{n+2}}-1)} \\ &= \frac{5}{(\alpha^{4F_n}-1)(\alpha^{4F_{n+1}}-1)} - \frac{5}{(\alpha^{4F_{n+1}}-1)(\alpha^{4F_{n+2}}-1)}, \end{aligned}$$

so that

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{\alpha^{2F_{n+1}} F_{2F_n} F_{2F_{n+2}}} &= \frac{5}{(\alpha^{4F_1}-1)(\alpha^{4F_2}-1)} - \lim_{n\to\infty} \frac{5}{(\alpha^{4F_{n+1}}-1)(\alpha^{4F_{n+2}}-1)} \\ &= \frac{5}{(\alpha^4-1)^2} - 0 = \frac{7-3\sqrt{5}}{2}. \end{aligned}$$

Also solved by Dmitry Fleischman and the proposer.

The limit of a parametric nested radical

H-788 Proposed by Hideyuki Ohtsuka, Saitama, Japan (Vol. 54, No. 2, May 2016)

Given $c > 0$, determine

$$\lim_{n\to\infty}\sqrt{cF_2^2+\sqrt{cF_4^2+\sqrt{cF_8^2+\sqrt{\cdots+\sqrt{cF_{2^n}^2}}}}}.$$

Solution by the proposer

We use the identities

(i) $F_{2m} = F_m L_m$ (see [1] (13));
(ii) $L_m^2 + 5F_m^2 = 2L_{2m}$ (see [1] (22));
(iii) $L_{2m} = L_m^2 - 2(-1)^m$ (see [1] (17c)).

Let

$$f_n := \frac{L_{2^n} + F_{2^n}\sqrt{4c+5}}{2}.$$

We have

$$\begin{aligned}
4f_n^2 &= (L_{2^n} + F_{2^n}\sqrt{4c+5})^2 = L_{2^n}^2 + 2L_{2^n}F_{2^n}\sqrt{4c+5} + F_{2^n}^2(4c+5)\\
&= 4cF_{2^n}^2 + 2L_{2^{n+1}} + 2F_{2^{n+1}}\sqrt{4c+5} \quad \text{(by (i) and (ii))}\\
&= 4(cF_{2^n}^2 + f_{n+1}).
\end{aligned}$$

Thus, we have

$$f_n = \sqrt{cF_{2^n}^2 + f_{n+1}}.$$

Using this identity repeatedly, we have

$$\begin{aligned}
f_1 &= \sqrt{cF_{2^1}^2 + f_2} = \sqrt{cF_{2^1}^2 + \sqrt{cF_{2^2} + f_3}} = \cdots\\
&= \sqrt{cF_2^2+\sqrt{cF_4^2+\sqrt{\cdots+\sqrt{cF_{2^{n-1}}^2+\sqrt{cF_{2^n}^2+f_{n+1}}}}}}. \qquad (1)
\end{aligned}$$

By (1), we have

$$f_1 > \sqrt{cF_2^2+\sqrt{cF_4^2+\sqrt{\cdots+\sqrt{cF_{2^{n-1}}^2+\sqrt{cF_{2^n}^2}}}}}. \qquad (2)$$

Given any $\varepsilon \in (0, f_1)$, let $s = 1 - \varepsilon/f_1$. We have

$$\begin{aligned}
f_1 - \varepsilon &= sf_1 \\
&= s\sqrt{cF_2^2 + \sqrt{cF_4^2 + \sqrt{\cdots + \sqrt{cF_{2^{n-1}}^2 + \sqrt{cF_{2^n}^2 + f_{n+1}}}}}} \quad \text{(by (1))} \\
&= \sqrt{s^2cF_2^2 + \sqrt{s^4cF_4^2 + \sqrt{\cdots + \sqrt{s^{2^{n-1}}cF_{2^{n-1}}^2 + \sqrt{s^{2^n}(cF_{2^n}^2 + f_{n+1})}}}}} \\
&< \sqrt{cF_2^2 + \sqrt{cF_4^2 + \sqrt{\cdots + \sqrt{cF_{2^{n-1}}^2 + \sqrt{s^{2^n}(cF_{2^n}^2 + f_{n+1})}}}}},
\end{aligned}$$

where the last inequality holds because $0 < s < 1$. Here, we have

$$\begin{aligned}
\frac{f_{n+1}}{F_{2^n}^2} &= \frac{L_{2^n}^2 - 2 + F_{2^n}L_{2^n}\sqrt{4c+5}}{2F_{2^n}^2} \quad \text{(by (i) and (iii))} \\
&= \frac{1}{2}\left(\frac{L_{2^n}}{F_{2^n}}\right)^2 - \frac{1}{F_{2^n}^2} + \frac{L_{2^n}}{F_{2^n}} \times \frac{\sqrt{4c+5}}{2} \to \frac{5 + \sqrt{5(4c+5)}}{2} \quad (n \to \infty)
\end{aligned}$$

since $\lim_{m\to\infty} L_m/F_m = \sqrt{5}$. Therefore,

$$\lim_{n\to\infty} \frac{cF_{2^n}^2}{cF_{2^n}^2 + f_{n+1}} = \lim_{n\to\infty} \frac{1}{1 + f_{n+1}/(cF_{2^n}^2)} = \frac{2c}{2c + 5 + \sqrt{5(4c+5)}} > 0.$$

Since $s \in (0, 1)$, there exists $N > 1$ such that for all $n > N$, we have

$$s^{2^n} < \frac{cF_{2^n}^2}{cF_{2^n}^2 + f_{n+1}}.$$

Using the above inequality, we have for $n > N$,

$$f_1 - \varepsilon < \sqrt{cF_2^2 + \sqrt{cF_4^2 + \sqrt{\cdots + \sqrt{cF_{2^{n-1}}^2 + \sqrt{cF_{2^n}^2}}}}}. \tag{3}$$

By (2) and (3), we obtain

$$\lim_{n\to\infty} \sqrt{cF_2^2 + \sqrt{cF_4^2 + \sqrt{\cdots + \sqrt{cF_{2^{n-1}}^2 + \sqrt{cF_{2^n}^2}}}}} = f_1 = \frac{3 + \sqrt{4c+5}}{2}.$$

[1] S. Vajda, *Fibonacci and Lucas Numbers and the Golden Section*, Dover 2008.

Partially solved by Dmitry Fleischman.

Some cyclic inequalities

H-789 Proposed by D. M. Bătineţu-Giurgiu, Bucharest, and Neculai Stanciu, Buzău, Romania
(Vol. 54, No. 2, May 2016)

For any real numbers $x,\ y$, we denote $B(x,y) = \sqrt{\frac{x^2+xy+y^2}{3}}$. Prove that for $n \geq 1$, we have

(i) $\left(\dfrac{L_{n+2}-3}{n}\right)^2 \leq \dfrac{1}{n} \displaystyle\sum_{n \text{ cyclic}} B^2(L) \leq \dfrac{L_n L_{n+1}-2}{n}$;

(ii) $\left(\dfrac{F_{n+2}-1}{n}\right)^2 \leq \dfrac{1}{n} \displaystyle\sum_{n \text{ cyclic}} B^2(F) \leq \dfrac{F_n F_{n+1}}{n}$,

where for a sequence $X := \{X_m\}_{m \geq 1}$, we use

$$\sum_{n \text{ cyclic}} B^2(X) = B^2(X_1, X_2) + B^2(X_2, X_3) + \cdots + B^2(X_{n-1}, X_n) + B^2(X_n, X_1).$$

Solution by Ángel Plaza

Here we prove (i) since (ii) is analogous. We use that $L_{n+2} - 3 = \displaystyle\sum_{k=1}^{n} L_k$ and also that $\displaystyle\sum_{k=1}^{n} L_k^2 = L_n L_{n+1} - 2$. Then, (i) reads as

(i) $\left(\dfrac{\sum_{k=1}^n L_k}{n}\right)^2 \leq \dfrac{1}{n} \displaystyle\sum_{k=1 \text{ cyclic}}^{n} \dfrac{L_k^2 + L_k L_{k+1} + L_{k+1}^2}{3} \leq \dfrac{\sum_{k=1}^n L_k^2}{n}$.

We will prove the following more general inequalities, which apply to (i) and (ii):
If $x_k > 0$ for $k = 1, 2, \ldots, n$, then

$$\left(\frac{\sum_{k=1}^n x_k}{n}\right)^2 \leq \frac{1}{n} \sum_{k=1 \text{ cyclic}}^{n} \frac{x_k^2 + x_k x_{k+1} + x_{k+1}^2}{3} \leq \frac{\sum_{k=1}^n x_k^2}{n}. \tag{4}$$

The left inequality (4) may be written as:

$$\frac{\sum_{k=1}^n x_k}{n} \leq \sqrt{\frac{1}{n} \sum_{k=1 \text{ cyclic}}^{n} \frac{x_k^2 + x_k x_{k+1} + x_{k+1}^2}{3}}.$$

This last inequality follows from the following two arguments:

1) $\frac{x_k + x_{k+1}}{2} \leq \sqrt{\dfrac{x_k^2 + x_k x_{k+1} + x_{k+1}^2}{3}}$. This may be checked by simple computations.

2) By the AM-RSM inequality:

$$\frac{\sum_{k=1}^n x_k}{n} \leq \frac{\sum_{k=1 \text{ cyclic}}^n \sqrt{\frac{x_k^2 + x_k x_{k+1} + x_{k+1}^2}{3}}}{n} \leq \sqrt{\frac{1}{n} \sum_{k=1 \text{ cyclic}}^{n} \frac{x_k^2 + x_k x_{k+1} + x_{k+1}^2}{3}}.$$

□

The right inequality (4) may be written as

$$\sum_{k=1 \text{ cyclic}}^{n} \frac{x_k^2 + x_k x_{k+1} + x_{k+1}^2}{3} \quad \leq \quad \sum_{k=1}^{n} x_k^2, \qquad \text{or}$$

$$\frac{2\displaystyle\sum_{k=1}^{n} x_k^2 + \sum_{k=1 \text{ cyclic}}^{n} x_k x_{k+1}}{3} \quad \leq \quad \sum_{k=1}^{n} x_k^2,$$

which follows by the rearrangement inequality $\displaystyle\sum_{k=1 \text{ cyclic}}^{n} x_k x_{k+1} \leq \sum_{k=1}^{n} x_k^2$.

Also solved by Dmitry Fleischman and the proposers.

A series involving harmonic numbers and the zeta function at positive integers

H-790 Proposed by Ovidiu Furdui, Cluj-Napoca, Romania (Vol. 54, No. 2, May 2016)

Calculate

$$\sum_{n=2}^{\infty} \left(H_n - \gamma - \frac{\zeta(2)}{2} - \frac{\zeta(3)}{3} - \cdots - \frac{\zeta(n)}{n} \right),$$

where ζ denotes the Riemann zeta function and $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ is the nth harmonic number.

Solution by Ramya Dutta

Lemma:

$$\sum_{n=2}^{\infty} \frac{\zeta(n) - 1}{n} = 1 - \gamma.$$

Proof: Using $-\log(1-x) = \displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n}$, for $x \in (-1, 1)$, we have

$$\begin{aligned}
\sum_{n=2}^{\infty} \frac{\zeta(n) - 1}{n} = \sum_{n=2}^{\infty} \sum_{m=2}^{\infty} \frac{1}{nm^n} &= \sum_{m=2}^{\infty} \left(-\log\left(1 - \frac{1}{m}\right) - \frac{1}{m} \right) \\
&= \lim_{N\to\infty} \sum_{m=2}^{N} \left(-\log\left(1 - \frac{1}{m}\right) - \frac{1}{m} \right) \\
&= \lim_{N\to\infty} 1 - H_N - \log \prod_{m=2}^{N} \left(1 - \frac{1}{m}\right) \\
&= \lim_{N\to\infty} 1 - H_N + \log N = 1 - \gamma
\end{aligned}$$

since, $H_N = \displaystyle\sum_{n=1}^{N} \frac{1}{n} = \log N + \gamma + O\left(\frac{1}{N}\right)$. $\square$

Thus,

$$\begin{aligned}
\sum_{n=2}^{\infty}\left(H_n-\gamma-\sum_{j=2}^{n}\frac{\zeta(j)}{j}\right) &= \sum_{n=2}^{\infty}\left(\sum_{j=2}^{\infty}\frac{\zeta(j)-1}{j}-\sum_{j=2}^{n}\frac{\zeta(j)-1}{j}\right)\\
&= \sum_{n=2}^{\infty}\sum_{j=n}^{\infty}\frac{\zeta(j+1)-1}{j+1}\\
&= \sum_{j=2}^{\infty}\sum_{n=2}^{j}\frac{\zeta(j+1)-1}{j+1} \qquad \text{(interchanging order of summation)}\\
&= \sum_{j=1}^{\infty}\left(1-\frac{2}{j+1}\right)(\zeta(j+1)-1)\\
&= \sum_{j=1}^{\infty}(\zeta(j+1)-1)-2\sum_{j=1}^{\infty}\frac{\zeta(j+1)-1}{j+1}\\
&= -2+2\gamma+\sum_{j=1}^{\infty}\sum_{m=2}^{\infty}\frac{1}{m^{j+1}}\\
&= -2+2\gamma+\sum_{m=2}^{\infty}\frac{1}{m(m-1)} = 2\gamma-1.
\end{aligned}$$

Also solved by Dmitry Fleischman and the proposer.