ADVANCED PROBLEMS AND SOLUTIONS

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Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to FLORIAN LUCA, SCHOOL OF MATHEMATICS, UNIVERSITY OF THE WITWATERSRAND, PRIVATE BAG 3, WITS 2050, JOHANNESBURG, SOUTH AFRICA or by e-mail at the address florian.luca@wits.ac.za as files of the type tex, dvi, ps, doc, html, pdf, etc. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions sent by regular mail should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-873 Proposed by Robert Frontczak, Stuttgart, Germany

Let $(T_n)_{n\geq 0}$ be the Tribonacci sequence defined by $T_{n+3} = T_{n+2} + T_{n+1} + T_n$ for all $n \geq 0$ with $T_0 = 0$, $T_1 = T_2 = 1$. Prove the following identities valid for all $n \geq 2$:

(i)

$$T_n = (-1)^{n+1} F_n + 2(-1)^n F_{n-1} + \sum_{k=0}^{n-2} (-1)^{k+1} F_k (2T_{n-k} + T_{n-2-k}).$$

(ii)

$$\sum_{1\leq i<j\leq n} (F_j - F_i)(T_{n-j} - T_{n-i}) = n(T_{n+2} - F_{n+2}) - \frac{1}{2}(F_{n+2} - 1)(T_{n+1} + T_{n-1} - 1).$$

(iii)

$$\sum_{1\leq i<j\leq n} (L_j - L_i)(T_{n-j} - T_{n-i}) = n(2T_{n+3} - T_{n+2} - 2T_n - L_{n+2}) - \frac{1}{2}(L_{n+2} - 3)(T_{n+1} + T_{n-1} - 1).$$

H-874 Proposed by Robert Frontczak, Stuttgart, Germany

Let C_n be the nth Catalan number; i.e., $C_n = \frac{1}{n+1}\binom{2n}{n}$, and α be the golden section. Prove that

$$\sum_{n=1}^{\infty} \frac{F_{2n}}{n(n+1)C_n} = \alpha^{-2} \sum_{n=1}^{\infty} \frac{L_{2n}}{n(n+1)C_n} = 2\pi\sqrt{\frac{\alpha}{25\sqrt{5}}}.$$

H-875 Proposed by D. M. Bătineţu-Giurgiu, Bucharest, Romania, and Neculai Stanciu, Buzău, Romania

Let ABC be a triangle with a, b, c the lengths of the sides, R the length of the circumradius, r the length of the inradius, and s the semiperimeter. Prove that

$$\left(\frac{F_n^2a^2+F_{n+1}^2b^2}{c}\right)^2+\left(\frac{F_n^2b^2+F_{n+1}^2c^2}{a}\right)^2+\left(\frac{F_n^2c^2+F_{n+1}^2a^2}{b}\right)^2\geq 2F_{2n+1}^2(s^2-r^2-4Rr)$$

holds for all $n\geq 0$.

H-876 Proposed by I. V. Fedak, Ivano-Frankivsk, Ukraine

For all positive integers n, prove that

$$F_{n+2}\geq\sqrt{\frac{F_nF_{n+1}+1}{n+1}}+n\sqrt[n+1]{F_1F_2\cdots F_n};\qquad L_{n+2}\geq\sqrt{\frac{L_nL_{n+1}+1}{n+3}}+(n+2)\sqrt[n+3]{L_1L_2\cdots L_n}.$$

H-877 Proposed by Hideyuki Ohtsuka, Saitama, Japan

Given an even integer r and an integer $n\geq 0$, prove that

$$\sum_{k=0}^{n}\binom{2n-k}{n}L_r^kL_{r(k+1)}=L_r^{2n+1}.$$

SOLUTIONS

A series with k-Fibonacci hyperbolic tangent terms

H-839 Proposed by Sergio Falcón and Ángel Plaza, Gran Canaria, Spain (Vol. 57, No. 2, May 2019)

For a positive integer k, the k-Fibonacci hyperbolic sine and cosine functions are defined respectively by

$$sF_kh(x)=\frac{\sigma_k^x-\sigma_k^{-x}}{\sigma_k+\sigma_k^{-1}},\quad cF_kh(x)=\frac{\sigma_k^x+\sigma_k^{-x}}{\sigma_k+\sigma_k^{-1}},$$

where $\sigma_k=(k+\sqrt{k^2+4})/2$. If the k-Fibonacci hyperbolic tangent and cotangent are respectively $tF_kh(x)=\dfrac{sF_kh(x)}{cF_kh(x)}$ and $ctF_kh(x)=(tF_kh(x))^{-1}$, find a closed form expression for the following sum

$$\sum_{r=1}^{\infty}\frac{1}{2^r}tF_kh\left(\frac{x}{2^r}\right).$$

Solution by the proposers

If $x=0$, then the sum is 0. If $a\neq 0$, then from the identity $tF_kha=2ctF_kh(2a)-ctF_kha$, we obtain

$$tF_kh\left(\frac{x}{2^r}\right)=2ctF_kh\left(\frac{x}{2^{r-1}}\right)-ctF_kh\left(\frac{x}{2^r}\right).$$

Hence,

$$\begin{aligned}\sum_{r=1}^{N}\frac{1}{2^r}tF_kh\left(\frac{x}{2^r}\right)&=\sum_{r=1}^{N}\frac{1}{2^r}\left(2ctF_kh\left(\frac{x}{2^{r-1}}\right)-ctF_kh\left(\frac{x}{2^r}\right)\right)\\&=ctF_kh(x)-\frac{1}{2^N}ctF_kh\left(\frac{x}{2^N}\right)\end{aligned}$$

by telescoping. Now

$$\frac{1}{2^N} ctF_k h\left(\frac{x}{2^N}\right) = \frac{cF_k h(x/2^N)}{x} \cdot \frac{x/2^N}{sF_k h(x/2^N)} \to \frac{cF_k h(0)}{x} \cdot \frac{1}{cF_k h(0)\ln \sigma_k} = \frac{1}{x \ln \sigma_k}$$

as $N \to \infty$, because

$$sF_k h(y) = \frac{2}{\sigma_k + \sigma_k^{-1}} \sinh(y \ln \sigma_k) = cF_k h(0) \sinh(y \ln \sigma_k)$$

(with $y = x/2^N$). Therefore,

$$\sum_{r=1}^{\infty} \frac{1}{2^r} tF_k h\left(\frac{x}{2^r}\right) = ctF_k h(x) - \frac{1}{x \ln \sigma_k}.$$

Also solved by Brian Bradie, Irina Dobrovolska and Dmitriy Shtefan (jointly), Dmitry Fleischman, and David Terr.

A multiple of 150

H-840 Proposed by Arkady Alt, San Jose, California (Vol. 57, No. 2, May 2019)

Prove that $(n-1)(n+1)(2nF_{n+1} - (n+6)F_n)$ is divisible by 150 for all $n \geq 1$.

Solution by Hideyuki Ohtsuka, Saitama, Japan

We have

$$4F_{n-2} + 3F_{n-3} = 3F_{n-1} + F_{n-2} = F_n + 2F_{n-1} = 2F_{n+1} - F_n \tag{1}$$

and

$$2F_{n-2} + F_{n-3} = F_{n-1} + F_{n-2} = F_n. \tag{2}$$

We have

$$\begin{aligned} & (4n^3 - 12n^2 - 4n + 12)F_{n-2} + (3n^3 - 6n^2 - 3n + 6)F_{n-3} \\ = \;& (n-1)(n+1)((4F_{n-2} + 3F_{n-3})n - 6(2F_{n-2} + F_{n-3})) \\ = \;& (n-1)(n+1)((2F_{n+1} - F_n)n - 6F_n) \quad \text{(by 1 and 2)} \\ = \;& (n-1)(n+1)(2nF_{n+1} - (n+6)F_n). \end{aligned}$$

By the above identity and the congruence

$$(4n^3 - 12n^2 - 4n + 12)F_{n-2} + (3n^3 - 6n^2 - 3n + 6)F_{n-3} \equiv 0 \pmod{150}$$

(see [1]), we have

$$(n-1)(n+1)(2nF_{n+1} - (n+6)F_n) \equiv 0 \pmod{150}.$$

Editor's Note: By the Binet formula, the expression $(n-1)(n+1)(2nF_{n+1} - (n+6)F_n)$ is of the form $P(n)\alpha^n + Q(n)\beta^n$, where $P(x),\ Q(x) \in \mathbb{R}[x]$ are polynomials of degree 3. In particular, the sequence whose nth term is the above expression is linearly recurrent of order 6 and characteristic polynomial $(x^2 - x - 1)^3$. Hence, it suffices to check that the desired divisibility holds for the first six values of n, namely $n = 0, 1, 2, 3, 4, 5$ because then it will hold for all $n \geq 0$ by induction.

Reference

[1] W. Zhang, *Some identities involving the Fibonacci numbers*, The Fibonacci Quarterly, **35.3** (1997), 225–229.

Also solved by Brian Bradie, Kenneth B. Davenport, Dmitry Fleischman, Raphael Schumacher, Albert Stadler, and the proposer.

A general inequality with Lucas numbers

H-841 Proposed by Hideyuki Ohtsuka, Saitama, Japan (Vol. 57, No. 2, May 2019)

For any integer $n \geq 2$, prove that

$$\sum_{j=1}^{n} L_{a_j} < \frac{L_{n+a_n}}{L_n - 1}$$

for any integer sequence $\{a_m\}_{m\geq 1}$ with $a_1 \geq 1$ and $a_{m+1} \geq a_m + 2m + 1$ for all $m \geq 1$.

Solution by the proposer

First, we will prove the following lemma.

Lemma. *We have*

(1) $L_s L_t \leq L_{s+t} + L_{s-t}$ *for* $s \geq t$;
(2) $L_{n+1} - 1 \leq 2(L_n - 1)$ *for* $n \geq 2$;
(3) $2L_{a_n+n} < L_{a_{n+1}} - L_{a_{n+1}-n-1}$ *for* $n \geq 2$.

Proof of Lemma. (1) By (17a) in [1], we have

$$L_s L_t = L_{s+t} + (-1)^t L_{s-t} \leq L_{s+t} + L_{s-t}.$$

(2) We have

$$RHS - LHS = 2L_n - L_{n+1} - 1 = L_{n-2} - 1 \geq 0.$$

(3) By $L_p - L_{p-r} \geq L_q - L_{q-r}$ for $p \geq q > r > 0$, we have

$$L_{a_{n+1}} - L_{a_{n+1}-n-1} \geq L_{a_n+2n+1} - L_{a_n+n} \geq L_{a_n+n+3} - L_{a_n+n} \geq 2L_{a_n+n+1} > 2L_{a_n+n}.$$

□

The proof of the desired inequality is by mathematical induction on n. For $n = 2$, the inequality holds because

$$2(RHS - LHS) = L_{a_2+2} - 2L_{a_1} - 2L_{a_2} = L_{a_2-1} - 2L_{a_1} \geq L_{a_1+2} - 2L_{a_1} = L_{a_1-1} > 0.$$

We assume the inequality holds for $n \geq 2$. For $n + 1$ we have

$$\begin{aligned}(L_{n+1} - 1)\sum_{j=1}^{n+1} L_{a_j} &= (L_{n+1} - 1)\left(L_{a_{n+1}} + \sum_{j=1}^{n} L_{a_j}\right)\\ &< L_{a_{n+1}}L_{n+1} - L_{a_{n+1}} + (L_{n+1} - 1) \times \frac{L_{a_n+n}}{L_n - 1}\\ &\leq L_{a_{n+1}+n+1} + L_{a_{n+1}-n-1} - L_{a_{n+1}} + 2L_{a_n+n} \quad \text{(by (1) and (2))}\\ &< L_{a_{n+1}+n+1} \quad \text{(by (3))}.\end{aligned}$$

Thus, the inequality holds for $n + 1$. Hence, the desired inequality is proved.

Example. If $a_n = n^2$, then for $n \geq 2$, we have

$$\sum_{j=1}^{n} L_{j^2} < \frac{L_{n(n+1)}}{L_n - 1}.$$

Reference

[1] S. Vajda, *Fibonacci and Lucas Numbers and the Golden Section*, Dover 2000.

Also solved by Dmitry Fleischman.

A closed form expression for a sum of products of Fibonacci numbers

H-842 Proposed by Hideyuki Ohtsuka, Saitama, Japan (Vol. 57, No. 3, August 2019)

Given an integer $n \geq 0$, find a closed for expression for the sum

$$\sum_{\substack{a+b+c=n \\ a,b,c\geq 0}} F_{a+b}F_{b+c}F_{c+a}.$$

Solution by Brian Bradie, Newport News, VA

First

$$\sum_{\substack{a+b+c=n \\ a,b,c\geq 0}} F_{a+b}F_{b+c}F_{c+a} = \sum_{a=0}^{n}\sum_{b=0}^{n-a} F_{a+b}F_{n-a}F_{n-b}.$$

Next,

$$F_{a+b}F_{n-b} = \frac{1}{5}(L_{n+a} - (-1)^{a+b}L_{n-a-2b}),$$

and

$$\begin{aligned} F_{a+b}F_{n-a}F_{n-b} &= \frac{1}{5}(F_{2n} + (-1)^{n+a}F_{-2a} - (-1)^{a+b}F_{2(n-a-b)} - (-1)^{n-b}F_{2b}) \\ &= \frac{1}{5}(F_{2n} - (-1)^{n-a}F_{2a} - (-1)^{a+b}F_{2(n-a-b)} - (-1)^{n-b}F_{2b}). \end{aligned}$$

Now,

$$\begin{aligned} \sum_{a=0}^{n}\sum_{b=0}^{n-a} F_{2n} &= \frac{(n+1)(n+2)}{2}F_{2n}, \\ \sum_{a=0}^{n}\sum_{b=0}^{n-a} (-1)^{n-a}F_{2a} &= (-1)^n \sum_{b=0}^{n}\sum_{a=0}^{n-b}(-1)^a F_{2a}, \quad \text{and} \\ \sum_{a=0}^{n}\sum_{b=0}^{n-a}(-1)^{a+b}F_{2(n-a-b)} &= \sum_{j=0}^{n}(-1)^j(j+1)F_{2(n-j)} = \sum_{j=0}^{n}(-1)^{n-j}(n-j+1)F_{2j} \\ &= (-1)^n \sum_{j=0}^{n}\sum_{a=0}^{n-j}(-1)^j F_{2j} = (-1)^n \sum_{a=0}^{n}\sum_{j=0}^{n-a}(-1)^j F_{2j}. \end{aligned}$$

Thus,

$$\sum_{\substack{a+b+c=n\\a,b,c\geq 0}} F_{a+b}F_{b+c}F_{c+a} = \frac{1}{5}\left(\frac{(n+1)(n+2)}{2}F_{2n} - 3(-1)^n\sum_{a=0}^{n}\sum_{b=0}^{n-a}(-1)^b F_{2b}\right).$$

Rearranging the Fibonacci number recurrence relation as

$$(-1)^k F_{2k} = (-1)^k F_{2k+1} - (-1)^k F_{2k-1} \quad \text{and} \quad (-1)^k F_{2k-1} = (-1)^k F_{2k} - (-1)^k F_{2k-2},$$

and then summing from $k = 1$ through $k = n$ yields

$$(-1)^k F_{2k} = -2\sum_{k=1}^{n}(-1)^k F_{2k-1} + (-1)^n F_{2n+1} - 1$$

and

$$\sum_{k=1}^{n}(-1)^k F_{2k-1} = 2\sum_{k=1}^{n}(-1)^k F_{2k} - (-1)^n F_{2n}.$$

It follows that

$$\sum_{k=1}^{n}(-1)^k F_{2k} = \frac{(-1)^n F_{2n+1} + 2(-1)^n F_{2n} - 1}{5} = \frac{(-1)^n(F_{2n+2} + F_{2n}) - 1}{5}.$$

Finally,

$$\begin{aligned}(-1)^n\sum_{a=0}^{n}\sum_{b=0}^{n-a}(-1)^b F_{2b} &= \frac{(-1)^n}{5}\sum_{a=0}^{n}((-1)^{n-a}(F_{2n-2a+2} + F_{2n-2a}) - 1)\\ &= \frac{(-1)^n}{5}\sum_{a=0}^{n}((-1)^a(F_{2n+2} + F_{2n}) - 1)\\ &= \frac{(-1)^n}{5}((-1)^n F_{2n+2} - (n+1))\\ &= \frac{F_{2n+2} - (-1)^n(n+1)}{5},\end{aligned}$$

and

$$\sum_{\substack{a+b+c=n\\a,b,c\geq 0}} F_{a+b}F_{b+c}F_{c+a} = \frac{1}{5}\left(\frac{(n+1)(n+2)}{2}F_{2n} - \frac{3}{5}(F_{2n+2} - (-1)^n(n+1))\right).$$

Also solved by Jason L. Smith, Raphael Schumacher, and the proposer.

Some divisibilities with Fibonacci numbers

H-843 Proposed by Hideyuki Ohtsuka, Saitama, Japan
(Vol. 57, No. 3, August 2019)

If integers a and b have the same parity with $a > b > 0$ and c is odd, show that

$$(F_a - F_b) \mid (F_{ac} - F_{bc}) \qquad \text{and} \qquad (L_a - L_b) \mid (L_{ac} - L_{bc}).$$

Solution by Raphael Schumacher, ETH Zurich, Switzerland

Because (see Formula (37) in [1] and Formula (7) in [2])

$$F_{-ca} = (-1)^{ca+1} F_{ca} \quad \text{and} \quad L_{-ca} = (-1)^{ca} L_{ca},$$

we only have to consider positive $c \in 2\mathbb{N}_0 + 1$ to prove it for all $c \in 2\mathbb{Z} + 1$.
We have for $k \in 2\mathbb{N}_0 + 1$ that (see Formula (47) in [1])

$$F_{kn} = \sum_{i=0}^{\frac{k-1}{2}} (-1)^{in} \frac{k}{k-i} \binom{k-i}{i} 5^{\frac{k-1}{2}-i} F_n^{k-2i}.$$

Because

$$\frac{k}{k-i}\binom{k-i}{i} = \frac{k-i+i}{k-i}\binom{k-i}{i} = \binom{k-i}{i} + \frac{i}{k-i}\binom{k-i}{i} = \binom{k-i}{i} + \binom{k-i-1}{i-1},$$

which implies that

$$\frac{k}{k-i}\binom{k-i}{i} \in \mathbb{N}_0 \quad \text{for all} \quad i = 0, 1, 2, \ldots, \frac{k-1}{2},$$

we deduce with $k := c$, $n := a$, and $n := b$ that

$$\begin{aligned}
\frac{F_{ca} - F_{cb}}{F_a - F_b} &= \frac{1}{F_a - F_b} \left(\sum_{i=0}^{\frac{c-1}{2}} (-1)^{ia} \frac{c}{c-i}\binom{c-i}{i} 5^{\frac{c-1}{2}-i} F_a^{c-2i} - \sum_{i=0}^{\frac{c-1}{2}} (-1)^{ib} \frac{c}{c-i}\binom{c-i}{i} 5^{\frac{c-1}{2}-i} F_b^{c-2i} \right) \\
&= \frac{1}{F_a - F_b} \left(\sum_{i=0}^{\frac{c-1}{2}} (-1)^{ia} \frac{c}{c-i}\binom{c-i}{i} 5^{\frac{c-1}{2}-i} F_a^{c-2i} - \sum_{i=0}^{\frac{c-1}{2}} (-1)^{ia} \frac{c}{c-i}\binom{c-i}{i} 5^{\frac{c-1}{2}-i} F_b^{c-2i} \right) \\
&= \frac{1}{F_a - F_b} \sum_{i=0}^{\frac{c-1}{2}} (-1)^{ia} \frac{c}{c-i}\binom{c-i}{i} 5^{\frac{c-1}{2}-i} \left(F_a^{c-2i} - F_b^{c-2i} \right) \\
&= \sum_{i=0}^{\frac{c-1}{2}} (-1)^{ia} \frac{c}{c-i}\binom{c-i}{i} 5^{\frac{c-1}{2}-i} \underbrace{\frac{F_a^{c-2i} - F_b^{c-2i}}{F_a - F_b}}_{\in \mathbb{N}_0} \in \mathbb{N}_0.
\end{aligned}$$

Therefore, we have that $(F_a - F_b) \,|\, (F_{ac} - F_{bc})$, because $\frac{a^n - b^n}{a-b} \in \mathbb{N}_0$ for all integers $a, b, n \in \mathbb{N}_0$.

Similarly, we have that (see Formula (16) in [2])

$$L_{kn} = \sum_{i=0}^{\frac{k-1}{2}} (-1)^{i(n+1)} \frac{k}{k-i}\binom{k-i}{i} L_n^{k-2i}.$$

Because we have again that

$$\frac{k}{k-i}\binom{k-i}{i} \in \mathbb{N}_0 \quad \text{for all} \quad i = 0, 1, 2, \ldots, \frac{k-1}{2},$$

we deduce with $k := c$, $n := a$, and $n := b$ as above that

$$\begin{aligned}\frac{L_{ca}-L_{cb}}{L_a-L_b} &= \frac{1}{L_a-L_b}\left(\sum_{i=0}^{\frac{c-1}{2}}(-1)^{i(a+1)}\frac{c}{c-i}\binom{c-i}{i}L_a^{c-2i} - \sum_{i=0}^{\frac{c-1}{2}}(-1)^{i(b+1)}\frac{c}{c-i}\binom{c-i}{i}L_b^{c-2i}\right)\\ &= \frac{1}{L_a-L_b}\left(\sum_{i=0}^{\frac{c-1}{2}}(-1)^{i(a+1)}\frac{c}{c-i}\binom{c-i}{i}L_a^{c-2i} - \sum_{i=0}^{\frac{c-1}{2}}(-1)^{i(a+1)}\frac{c}{c-i}\binom{c-i}{i}L_b^{c-2i}\right)\\ &= \frac{1}{L_a-L_b}\sum_{i=0}^{\frac{c-1}{2}}(-1)^{i(a+1)}\frac{c}{c-i}\binom{c-i}{i}\left(L_a^{c-2i}-L_b^{c-2i}\right)\\ &= \sum_{i=0}^{\frac{c-1}{2}}(-1)^{i(a+1)}\frac{c}{c-i}\binom{c-i}{i}\underbrace{\frac{L_a^{c-2i}-L_b^{c-2i}}{L_a-L_b}}_{\in\mathbb{N}_0} \in \mathbb{N}_0.\end{aligned}$$

Editor's Note: By the Binet formula, for fixed a, b of the same parity, the sequence of nth term $F_{a(2n+1)} - F_{b(2n+1)}$ is linearly recurrent of order 4 of roots α^{2a}, β^{2a}, α^{2b}, β^{2b}. Hence, to show the required divisibility it suffices to show that it holds for four consecutive ns. We choose these ns to be −1, 0, 1, 2. When $n = -1$, we have $F_{-a} - F_{-b} = (-1)^{a-1}(F_a - F_b)$. When $n = 0$, the divisibility is clear. When $n = 1,\ 2$, we do the same calculation as in Schumacher's solution but only for these two particular values of n. For example, for $n = 1$, we need to expand $F_{3a} - F_{3b}$ and we use that $F_{3a} = 5F_a^3 + 3(-1)^a F_a$ and that the same holds for a replaced by b. Similar remarks apply to the problem involving Lucas numbers.

References

[1] http://mathworld.wolfram.com/FibonacciNumber.html.
[2] http://mathworld.wolfram.com/LucasNumber.html.

Also solved by the proposer.