Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to FLORIAN LUCA, SCHOOL OF MATHEMATICS, UNIVERSITY OF THE WITWATERSRAND, PRIVATE BAG 3, WITS 2050, JOHANNESBURG, SOUTH AFRICA or by e-mail at the address florian.luca@wits.ac.za as files of the type tex, dvi, ps, doc, html, pdf, etc. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions sent by regular mail should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-863 Proposed by Kenneth B. Davenport, Dallas, PA
Show that
\[ \sum_{n \geq 1} \frac{\zeta(2n+1) - 1}{2n+1} = 1 - \gamma - \frac{\ln 2}{2} \quad \text{and} \quad \sum_{n \geq 1} \frac{\zeta(2n) - 1}{n(n+1)} = \ln(2\pi) - \frac{3}{2}, \]
where \( \zeta(n) \) is the Riemann zeta function.

H-864 Proposed by Hideyuki Ohtsuka, Saitama, Japan
The Pell numbers \( \{P_n\}_{n \geq 0} \) satisfy \( P_0 = 0, P_1 = 1, \) and \( P_n = 2P_{n-1} + P_{n-2} \) for \( n \geq 2. \) Prove that
\[ \sum_{n=1}^{\infty} \tan^{-1} \frac{1}{\sqrt{2P_n}} \tan^{-1} \frac{1}{\sqrt{2P_{n+1}}} = \frac{\pi}{4} \tan^{-1} \frac{1}{2\sqrt{2}}. \]

H-865 Proposed by D. M. Bătinețu-Giurgiu, Bucharest, Romania, and Neculai Stanciu, Buzău, Romania
Let \( \{x_n\}_{n \geq 0} \) be the sequence given by \( x_0 = 0, x_1 = 1, \) and
\[ x_{n+2} = (2n+5)x_{n+1} - (n^2 + 4n + 3)x_n \quad \text{for} \quad n \geq 0. \]
Find
\[ \lim_{n \to \infty} \left( \frac{n+1}{\sqrt{F_{n+1}L_{n+1}x_{n+1}}} - \frac{n}{\sqrt{F_nL_nx_n}} \right). \]

H-866 Proposed by Ángel Plaza, Gran Canaria, Spain
Let \( a_n \) denote the \( n \)th number in the sequence given by \( a_{n+1} = a_n + a_{n-1} \) for \( n \geq 1 \) with initial values \( a_0 = a - 1 \) and \( a_1 = 1 \) with some \( a \geq 1. \) Prove that
\[ \sum_{k=1}^{n} \frac{2(a_{k+1} - a_k)}{a_{k+1} + a_k} < \ln a_{n+1} < \sum_{k=1}^{n} \frac{a_{k+1}^2 - a_k^2}{2a_{k+1}a_k}. \]
**THE FIBONACCI QUARTERLY**

**H-867** Proposed by Hideyuki Ohtsuka, Saitama, Japan

Let \( a, b, c, d \) be even positive integers with \( a + b = c + d \). Prove that

\[
\sum_{k=1}^{a} \frac{L_b}{F_k L_{k+b}} + \sum_{k=1}^{b} \frac{L_a}{L_k F_{k+a}} = \sum_{k=1}^{c} \frac{L_d}{F_k L_{k+d}} + \sum_{k=1}^{d} \frac{L_c}{L_k F_{k+c}}.
\]

**SOLUTIONS**

**A sum of arctangents**

**H-829** Proposed by Ángel Plaza and Francisco Perdomo, Gran Canaria, Spain (Vol. 56, No. 4, November 2018)

For any positive integer \( k \), let \( \{F_{k,n}\}_{n\geq 0} \) be the sequence defined by \( F_{k,0} = 0 \), \( F_{k,1} = 1 \), and \( F_{k,n+1} = kF_{k,n} + F_{k,n-1} \) for \( n \geq 1 \). Find the limit

\[
\lim_{k \to \infty} \frac{k + \sqrt{k^2 + 4}}{2} \sum_{n=1}^{\infty} \arctan \left( \frac{kF_{k,n+1}^2}{1 + F_{k,n}F_{k,n+1}^2F_{k,n+2}} \right).
\]

Solution by Albert Stadler, Herrliberg, Switzerland

We note that

\[
\arctan \frac{1}{F_{k,n}F_{k,n+1}} - \arctan \frac{1}{F_{k,n+1}F_{k,n+2}} = \arctan \frac{1}{F_{k,n}F_{k,n+1}} - \arctan \frac{1}{1 + F_{k,n}F_{k,n+1}^2F_{k,n+2}}.
\]

\[
= \arctan \frac{1}{F_{k,n+1}(F_{k,n+2} - F_{k,n})} = \arctan \frac{kF_{k,n+1}^2}{1 + F_{k,n}F_{k,n+1}^2F_{k,n+2}}.
\]

So,

\[
\sum_{n=1}^{\infty} \arctan \frac{kF_{k,n+1}^2}{1 + F_{k,n}F_{k,n+1}^2F_{k,n+2}} = \arctan \frac{1}{F_{k,1}F_{k,2}} = \arctan \frac{1}{k},
\]

and

\[
\lim_{k \to \infty} \frac{k + \sqrt{k^2 + 4}}{2} \sum_{n=1}^{\infty} \arctan \frac{kF_{k,n+1}^2}{1 + F_{k,n}F_{k,n+1}^2F_{k,n+2}} = \lim_{k \to \infty} \frac{k + \sqrt{k^2 + 4}}{2} \arctan \frac{1}{k} = 1.
\]

Also solved by Brian Bradie, Dmitry Fleischman, Robert Frontczak, and the proposers.
A sum divisible by four consecutive Fibonacci numbers

H-830 Proposed by Hideyuki Ohtsuka, Saitama, Japan
(Vol. 56, No. 4, November 2018)

For an integer \( n \geq 1 \), prove that
\[
12 \sum_{k=1}^{n} (F_k F_{k+1} F_{k+2})^2 \equiv 0 \pmod{F_n F_{n+1} F_{n+2} F_{n+3}}.
\]

Solution by the proposer

Using \( F_{a+b} F_{a+c} = F_a F_{a+b+c} + (-1)^a F_b F_c \) (see [3] (20a)), we have
\[
F_k F_{k+2} = F_{k-1} F_{k+3} + (-1)^{k-1} F_1 F_3 = F_{k-1} F_{k+2} - 2(-1)^k. \tag{1}
\]
We have
\[
\sum_{k=1}^{n} (F_k F_{k+1} F_{k+2})^2 = \sum_{k=1}^{n} (F_k F_{k+1}^2 F_{k+2}) \times (F_k F_{k+2})
\]
\[= \sum_{k=1}^{n} F_k F_{k+1}^2 F_{k+2}(F_{k-1} F_{k+3} - 2(-1)^k) \quad \text{by (1)}
\]
\[= \sum_{k=1}^{n} F_{k-1} F_k F_{k+1}^2 F_{k+2} F_{k+3} - 2 \sum_{k=1}^{n} (-1)^k F_k F_{k+1}^2 F_{k+2}. \tag{2}
\]
From identity (2.1) in [1], we have
\[
\sum_{k=1}^{n} F_{k-1} F_k F_{k+1}^2 F_{k+2} F_{k+3} = \frac{1}{4} F_{n-1} F_n F_{n+1} F_{n+2} F_{n+3} F_{n+4}. \tag{3}
\]
From identity (2.17) in [2], we have
\[
\sum_{k=1}^{n} (-1)^k F_k F_{k+1}^2 F_{k+2} = \frac{1}{3} (-1)^n F_n F_{n+1} F_{n+2} F_{n+3}. \tag{4}
\]
By (2), (3), and (4), we have
\[
12 \sum_{k=1}^{n} (F_k F_{k+1} F_{k+2})^2 = F_n F_{n+1} F_{n+2} F_{n+3} (3F_{n-1} F_{n+4} - 8(-1)^n)
\]
\[\equiv 0 \pmod{F_n F_{n+1} F_{n+2} F_{n+3}}.
\]


Also solved by Kenneth B. Davenport and Raphael Schumacher.
Let $P_j(x) = 2^{-j}((x - \sqrt{x^2 - 4})^j + (x + \sqrt{x^2 - 4})^j)$, where $j$ and $x$ are nonnegative integers. Let $N = k2^m + 1$ with $k$ odd, $k < 2^m$, and $m > 2$. Let $S_0 = P_k(F_n)$ and $S_i = S_{i-1}^2 - 2$ for $i \geq 1$. Prove the following statement: If there exists $F_n$ for which $S_{m-2} \equiv 0 \pmod{N}$, then $N$ is prime.

No solution to this problem was received. The proposer pointed out [1], where some particular cases are treated (the cases $n = 4, 5, 6$ and $k$ and $m$ in various residue classes).


**Closed form expressions for sums with Fibonacci and Lucas numbers**

**H-832** Proposed by Hideyuki Ohtsuka, Saitama, Japan

(Vol. 56, No. 4, November 2018)

For positive integers $n$ and $r$, find a closed form expression for

(i) $\sum_{k=1}^{n} F_{rk}^3 L_{rk}$;

(ii) $\sum_{k=1}^{n} F_{2rk}^3 F_{2rk}$.

**Solution by the proposer**

We use Catalan’s identity

$$F_n^2 - (-1)^{n-m}F_m^2 = F_{n+m}F_{n-m}. \tag{5}$$

(i) We have

$$F_{2r} \sum_{k=1}^{n} F_{rk}^3 L_{rk} = \sum_{k=1}^{n} F_{rk}^2 (F_{2rk}F_{2r})$$

$$= \sum_{k=1}^{n} F_{rk}^2 (F_{r(k+1)}^2 - F_{r(k-1)}^2) \text{ by (5)}$$

$$= \sum_{k=1}^{n} (F_{rk}^2 F_{r(k+1)}^2 - F_{rk}^2 F_{r(k-1)}^2)$$

$$= \frac{F_{rn}^2 F_{r(n+1)}^2}{F_{2r}}.$$

Thus, we obtain

$$\sum_{k=1}^{n} F_{rk}^3 L_{rk} = \frac{F_{rn}^2 F_{r(n+1)}^2}{F_{2r}}.$$
(ii) We have
\[
\sum_{k=1}^{n} F_{k}^3 F_{2L_k} = \sum_{k=1}^{n} F_{k}^2 (F_{k}^2 F_{2L_k})
\]
\[
= \sum_{k=1}^{n} F_{k}^2 F_{k}^2 - F_{k}^2 - L_k
\]
\[
= \sum_{k=1}^{n} F_{k}^2 F_{k}^2 (F_{k}^2 F_{k}^2 - F_{k}^2 - L_k) \quad \text{(since } L_k = F_{k-1} + F_{k+1})
\]
\[
= \sum_{k=1}^{n} F_{k}^2 F_{k}^2 (F_{k}^2 + 1 - F_{k}^2 - 1) = 1
\]

Also solved by Brian Bradie, Dmitry Fleischman, Robert Frontczak, and Raphael Schumacher.

Closed form for a sum of Tribonacci Lucas numbers

**H-833** Proposed by Robert Frontczak, Stuttgart, Germany (Vol. 57, No. 1, February 2019)

The Tribonacci-Lucas numbers \(\{K_n\}_{n \geq 0}\) satisfy \(K_0 = 3, K_1 = 1, K_2 = 3,\) and \(K_n = K_{n-1} + K_{n-2} + K_{n-3}\) for \(n \geq 3.\) Prove that for any \(n \geq 1\)
\[
\sum_{j=1}^{n} K_{2j} K_{2j+1} = \frac{1}{4}((K_{2n} + K_{2n+1})^2 - 16).
\]

**Solution by Brian Bradie, Newport News, VA**

Observe
\[
(K_{2j} + K_{2j+1})^2 - (K_{2j-2} + K_{2j-1})^2 = (K_{2j} + K_{2j+1} + K_{2j-2} + K_{2j-1})
\times (K_{2j} + K_{2j+1} - K_{2j-2} - K_{2j-1})
\]
\[
= (2K_{2j+1})(2K_{2j}) = 4K_{2j} K_{2j+1}.
\]
Therefore,
\[
\sum_{j=1}^{n} K_{2j} K_{2j+1} = \frac{1}{4} \sum_{j=1}^{n} ((K_{2j} + K_{2j+1})^2 - (K_{2j-2} + K_{2j-1})^2)
\]
\[
= \frac{1}{4}((K_{2n} + K_{2n+1})^2 - (K_0 + K_1)^2)
\]
\[
= \frac{1}{4}((K_{2n} + K_{2n+1})^2 - 16).
\]

Also solved by Kenneth B. Davenport, Wei-Kai Lai and John Risher (jointly), Hideyuki Ohtsuka, Ángel Plaza, Raphael Schumacher, David Terr, and the proposer.

Late acknowledgement: Albert Stadler has solved Advanced Problem H-825.